Aktuárské vědy

Ferdinand Šamonil

Remark to the summation-formulas of the Lubbock's type

Aktuárské vědy, Vol. 4 (1933), No. 3, 120–124

Persistent URL: http://dml.cz/dmlcz/144604

Terms of use:

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Remark to the summation-formulas of the Lubbock's type.

Ferdinand Šamonil.

In this article I give the simple derivation of the recurrence-formulas for the coefficients L, P, Q, which are contained in the known summation-formulas of the Lubbock's type.

The reader who requires the details about these formulas, will find it in excellent Steffensen's book „The Interpolation", 1927, Baltimore, U. S. A.

Let us pay our attention above all to the coefficients L, contained in the first Lubbocks formula, which are defined by equation

$$\Lambda_\nu(h) = \frac{1}{\nu!}\sum_{s=0}^{h-1}\left(\frac{s}{h}\right)^{(\nu)}$$

in which means:

$$x^{(\nu)} = x\,(x-1)\ldots(x-\nu+1).$$

In order to get for them the recurrence-formula we start from the Newton's interpolation-formula:

$$f\left(a+\frac{s}{h}\right) = \sum_{\nu=0}^{r}\frac{1}{\nu!}\left(\frac{s}{h}\right)^{(\nu)}\Delta^\nu f(a) + \frac{1}{(r+1)!}\left(\frac{s}{h}\right)^{(r+1)} f^{(r+1)}(\xi)$$

where

$$\Delta f(x) = f(x+1) - f(x);$$

h is the positive integer.

Summing successively with regard to s from $s=0$ to $s=h-1$, we obtain

$$\sum_{s=0}^{h-1} f\left(a+\frac{s}{h}\right) = \sum_{\nu=0}^{r}\Lambda_\nu\Delta^\nu f(a) + \Lambda_{r+1}\,f^{(r+1)}(\xi).$$

We put now in this equation

$$\left.\begin{aligned} f(x) &= x^{(r)} \\ a &= \frac{1}{h}\end{aligned}\right\} r > 1.$$

The remainder-term in this case is equal to zero and paying our attention to the known relation

$$\Delta x^{(n)} = n x^{(n-1)}$$

we get

$$\sum_{s=0}^{h}\left(\frac{s}{h}\right)^{(r)} = r!\,\Lambda_r = \sum_{\nu=0}^{r}\Lambda_\nu\left(\frac{1}{h}\right)^{(r-\nu)} r^{(\nu)}.$$

If we divide it by $r!$ we shall get the wanted formula in the form

$$0 = \binom{\frac{1}{h}}{r} A_0 + \binom{\frac{1}{h}}{r-1} A_1 + \dots + \binom{\frac{1}{h}}{1} A_{r-1}$$

valid for $r > 1$.

By other choice of $f(x)$ we can get the further interesting relations.

Putting for example

$$f(x) = x^{(r)},$$

$$a = \frac{1}{2h}$$

we get

$$A_r(2h) - 2A_r = A_0 \binom{\frac{1}{2h}}{r} + A_1 \binom{\frac{1}{2h}}{r-1} + \dots + A_{r-1} \binom{\frac{1}{2h}}{1}$$

valid for $r > 1$.

II.

If we want to get the similar formula for coefficients P, defined by equation

$$P_{2\nu} = \frac{1}{(2\nu)!} \sum_{-\frac{1}{2}(h-1)}^{\frac{1}{2}(h-1)} \left(\frac{s}{h}\right)^{[2\nu]}$$

where

$$x^{[\nu]} = x(x + \tfrac{1}{2}\nu - 1)(x + \tfrac{1}{2}\nu - 2) \dots (x - \tfrac{1}{2}\nu + 1)$$

we start from the Stirling's interpolation-formula, which is written in the following form

$$f\left(a + \frac{s}{h}\right) = \sum_{\nu=0}^{r} \frac{1}{(2\nu)!} \left(\frac{s}{h}\right)^{[2\nu]} \delta^{2\nu} f(a) +$$

$$+ \sum_{\nu=1}^{r} \frac{1}{(2\nu-1)!} \left(\frac{s}{h}\right)^{[2\nu]-1} \square\, \delta^{2\nu-1} f(a) + R,$$

$$\delta f(x) = f(x + \tfrac{1}{2}) - f(x - \tfrac{1}{2}), \quad \square\, f(x) = \frac{f(x + \frac{1}{2}) + f(x - \frac{1}{2})}{2}$$

and sum from $s = \frac{1}{2}(-h + 1)$ to $s = \frac{1}{2}(h - 1)$.

We obtain

$$\sum_{-\frac{1}{2}(h-1)}^{\frac{1}{2}(h-1)} f\left(a + \frac{s}{h}\right) = \sum_{\nu=0}^{r} P_{2\nu} \delta^{2\nu} f(a) + R \tag{1}$$

where the remainder-term R contains as factor the differential coefficient $f(x)$ of $2r+2$ order.

We put

$$\left.\begin{aligned} f(x) &= x^{[2r]} \\ a &= \frac{1}{h} \end{aligned}\right\} r > 0$$

and according to

$$\delta x^{[n]} = n x^{[n-1]}$$

we obtain

$$(2r)!\,P_{2r} + \left(\frac{1+h}{2h}\right)^{[2r]} - \left(\frac{1-h}{2h}\right)^{[2r]} = \sum_{\nu=0}^{r} P_{2\nu}\,(2r)^{(2\nu)}\left(\frac{1}{h}\right)^{[2r-2\nu]}.$$

If we divide by $2r!$ and write for abbreviation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n^{[k]}}{k!}$$

we get

$$P_0\begin{bmatrix} \frac{1}{h} \\ 2r \end{bmatrix} + P_2\begin{bmatrix} \frac{1}{h} \\ 2r-2 \end{bmatrix} + \ldots + P_{2r-2}\begin{bmatrix} \frac{1}{h} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2h} \\ 2r-1 \end{bmatrix}.$$

valid for $r > 0$.

From other similar relations which is possible to deduce in such a way, we present this one, which we get, putting

$$f(x) = x^{[2r+1]},$$

$$a = \frac{1}{2h}.$$

The sum on the left side of the equation (1) is equal to zero, because the terme $x^{[2r+1]}$ is an odd function.

After arrangement we find

$$P_0\begin{bmatrix} \frac{1}{2h} \\ 2r+1 \end{bmatrix} + P_2\begin{bmatrix} \frac{1}{2h} \\ 2r-1 \end{bmatrix} + \ldots \quad + P_{2r}\begin{bmatrix} \frac{1}{2h} \\ 1 \end{bmatrix} = 0.$$

This relation is identical with that one, which was deduced by Steffensen in the appendix about Calculus of Symbols to his book „The Interpolation".

III.

For the coefficients

$$Q_{2\nu} = \frac{1}{(2\nu)!}\sum_{\frac{1}{2}(-h+2)}^{\frac{1}{2}(h-2)}\left(\frac{s}{h}\right)^{[2\nu+1]-1}$$

where

$$x^{[k]-1} = \frac{x^{[k]}}{x}$$

we find from Bessel's interpolation-formula analogically

$$\sum_{-\frac{1}{2}(h-2)}^{\frac{1}{2}(h-2)} f\left(a + \tfrac{1}{2} + \frac{x}{h}\right) = \sum_{\nu=0}^{r} Q_{2\nu} \square\, \delta^{2\nu} f(a + \tfrac{1}{2}) + R.$$

The remainder-term contains as factor the differential coefficient of $2r + 2$ order.

By choice

$$\left.\begin{aligned} f(x) &= x^{[2r+1]-1} \\ a + \tfrac{1}{2} &= \frac{1}{h} \end{aligned}\right\}$$

If we use the relations

$$\delta x^{[k+1]-1} = k \,.\, x^{[k]-1}$$

$$\square\, x^{[k+1]-1} = x^{[k]}$$

we get after arrangement

$$Q_0 \begin{bmatrix} \frac{1}{h} \\ 2r \end{bmatrix} + Q_2 \begin{bmatrix} \frac{1}{h} \\ 2r-2 \end{bmatrix} + \ldots + Q_{2r-2} \begin{bmatrix} \frac{1}{h} \\ 2 \end{bmatrix} = (rh - 1) \begin{bmatrix} \frac{1}{h} \\ 2r \end{bmatrix}.$$

An other relation, we obtain for

$$f(x) = x^{[2r+2]-1}$$

$$\tfrac{1}{2} + a = \frac{1}{2h}\cdot$$

So we get

$$\sum_{x=-\frac{1}{2}(h-3)}^{\frac{1}{2}(h-1)} \left(\frac{x}{h}\right)^{[2r+2]-1} = \sum_{\nu=0}^{r} Q_{2\nu} \left(\frac{1}{2h}\right)^{[2r+1-2\nu]} (2r+1)^{(2\nu)}$$

divided by $2r + 1!$

$$\sum_{\nu=0}^{r} Q_{2\nu} \begin{bmatrix} \frac{1}{2h} \\ 2r+1-2\nu \end{bmatrix} = \frac{1}{(2r+1)!} \left(\frac{h-1}{2h}\right)^{[2r+2]-1}.$$

If we use the identical-equation, which is very easy to be proved by induction:

$$(2rh + h - 1) \begin{bmatrix} \frac{1}{2h} \\ 2r+1 \end{bmatrix} = \frac{1}{(2r+1)!} \left(\frac{h-1}{2h}\right)^{[2r+2]-1}$$

and substitute

$$Q_0 = h - 1$$

we have

$$Q_2 \begin{bmatrix} \frac{1}{2h} \\ 2r-1 \end{bmatrix} + Q_4 \begin{bmatrix} \frac{1}{2h} \\ 2r-3 \end{bmatrix} + \ldots + Q_{2r} \begin{bmatrix} \frac{1}{2h} \\ 1 \end{bmatrix} = 2hr \begin{bmatrix} \frac{1}{2h} \\ 2r+1 \end{bmatrix}.$$

Also this formula we find in Steffensen's book deduced by other way.

Das zweiändrige Wahrscheinlichkeitsgesetz der Abweichungen der Prämienreserve eines Bestandes von Versicherungen mit verschiedenen Auflösungsmöglichkeiten.

Von *Hans Koeppler*, Berlin.

Die folgenden Betrachtungen betreffen Versicherungen, deren einmalige Prämie nach der Formel

$$\mathfrak{A}_{x\overline{n}|} = \sum_{t=1}^{t=n} {}_{t-1|}p_x^{(1)}\, v^t S_t^{(1)} + \sum_{t=1}^{t=n} {}_{t-1|}p_x^{(2)}\, v^t S_t^{(2)} + {}_np_x^{(3)}\, v^n S_n^{(3)}$$

berechnet wird. $S_t^{(1)}$, $S_t^{(2)}$, $S_n^{(3)}$ bedeuten die versicherten Summen, von denen die mit $S_t^{(2)}$ bezeichneten Summen fortfallen, wenn der Versicherungsschutz nicht so umfangreich gewährt wird. Die Summe der in der einmaligen Prämie vorkommenden Wahrscheinlichkeiten muß der Bedingung

$$\sum_{t=1}^{t=n} {}_{t-1|}p_x^{(1)} + \sum_{t=1}^{t=n} {}_{t-1|}p_x^{(2)} + {}_np_x^{(3)} = 1$$

genügen. Es sei noch bemerkt, daß ${}_np_x^{(3)}$ die Wahrscheinlichkeit ist, den Ablauf der Versicherung zu erleben, ohne von einem der zu befürchtenden Ereignisse betroffen worden zu sein. Die jährliche Prämie wird nach der einfachen Formel

$$P_{x\overline{n}|} = \frac{\mathfrak{A}_{x\overline{n}|}}{a_{x\overline{n}|}}$$

berechnet, in welcher

$$a_{x\overline{n}|} = \sum_{t=0}^{t=n-1} {}_tp_x^{(3)}\, v^t$$

den Barwert der pränumerando zahlbaren Beharrungsrente bedeutet.