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Jiřina Frantíková

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advantages are so important, and the additional premium so small, that it should always be possible to persuade anybody who wants a reversionary annuity to choose a death annuity instead.

The cheapness of the death annuity in comparison with various other forms of insurance is due to its nature of a decreasing insurance. But the disadvantages of a decreasing insurance are not so great as it might seem, provided that it follows approximately what may be termed the „money value" of the life at any time. An old man does not as a rule want a large insurance, because an insurance decreasing at the proper rate will provide the same annual amount for his widow, while his children become independent as he grows old. Besides, a death annuity may, if desired, be modified in various ways; thus, for example, it may be stipulated that the sum insured shall only decrease to a certain point and thereafter remain constant. Several other questions of a practical nature arise which I do not propose to discuss here, such as the choice of mortality tables and the question of loadings. Policy loans can be admitted, with obvious precautions as to repayment, because the insurance is decreasing. Also the wording of the policy conditions requires careful consideration.

The construction of a set of tables of $a_{x|y}$ for the values of x and y which are required in practice is a somewhat heavy piece of work which I am not prepared to tackle single-handed. I shall be satisfied if I have succeeded in convincing the actuarial world that the new form of insurance suggested above is a useful one, and worth the further consideration of the profession.

Some approximate formulas.

By *Jiřina Frantíková* (Prague).

The following note explains a method of obtaining approximate formulas of some actuarial values. We shall use the mean value theorem of a definite integral:

If the functions $\varphi(x)$, $\varphi(x) . \psi(x)$ are integrable over the range (a, b), $\varphi(x)$ does not change the sign over (a, b) and never is equal to zéro, then the following equation is valid

$$\int_a^b \psi(x) \,.\, \varphi(x)\, \mathrm{d}x = \psi(c) \,.\int_a^b \varphi(x)\, \mathrm{d}x \tag{1}$$

where

$$c = a + (b - a)\,\Theta,\; 0 < \Theta < 1.$$

In actuarial mathematics we have very often to calculate the integral of the following form

$$\int_a^b \psi(x) \,.\, \varphi(x)\, \mathrm{d}x.$$

It is possible to suppose the functions $\psi(x)$ and $\varphi(x)$ are of such a sort that we may use the mean value theorem. It happens that we can write the actuarial value in two alternative forms and then it is not difficult to determine the value c.

Evans[1]) has applied the mean value theorem for calculating of life annuities of a higher order, which it is possible to write in the following two ways:

$$\begin{aligned}\overset{<(k)}{\bar{a}_x} &= \frac{1}{k!}\int_0^\infty t^k \,.\, {}_tp_x \,.\, v^t\, \mathrm{d}t = \\ &= \frac{1}{(k-1)!}\int_0^\infty t^{k-1} \,.\, {}_tp_x \,.\, v^t \,.\, \bar{a}_{x+t} \,.\, \mathrm{d}t.\end{aligned} \tag{2}$$

Supposing that the non-negative function

$$\frac{t^k}{k!} \,.\, {}_tp_x \,.\, v^t \tag{S_1}$$

is integrable on the range $(0, \infty)$, and the function

$$\bar{a}_{x+t} \tag{S_2}$$

is nearly linear and integrable, we can apply the mean value theorem on both relations (2) and get

$$\overset{<}{\bar{a}}{}_x^{(k)} = \frac{1}{k}\, n_k \,.\, \overset{<}{\bar{a}}{}_x^{(k-1)} = \bar{a}_{x+n_k} \,.\, \overset{<}{\bar{a}}{}_x^{(k-1)} \tag{3}$$

from where n_k is determined by the following equation

$$n_k = k \,.\, \bar{a}_{x+n_k}. \tag{3'}$$

This derivation is, clearly, valid for $k = 1$.

Supposing that

$$\overset{<}{\bar{a}}{}_x^{(k-1)} = \frac{1}{(k-1)!}\, n_{k-1} \ldots n_1 \,.\, \bar{a}_x \tag{4}$$

then the above equations (3') und (4) give

$$\overset{<}{\bar{a}}{}_x^{(k)} = \frac{1}{k!}\, n_k \ldots n_1 \,.\, \bar{a}_x. \tag{5}$$

[1]) A. W. Evans. A Method of Approximating to Increasing Annuities Journal of Institut of Actuaries. Vol. LX. 1929.

So the inductive proof of the approximate formula (5) is given. The calculation of the life annuities by means of the above formula gives quite good results. The advantage of this formula is the simplicity in comparision with the laborious task of calculation of commutation columns of a higher order.

For the life assurance we can proceed in a similar way. Writing the values of life assurance of order k, similarly as above, in two forms

$$\overset{<}{\bar{A}}{}_x^{(k)} = \frac{1}{k!}\int_0^\infty t^k \cdot {}_tp_x \cdot \mu_{x+t} \cdot v^t \,\mathrm{d}t = \tag{6}$$
$$= \frac{1}{(k-1)!}\int_0^\infty t^{k-1} \cdot {}_tp_x \cdot \bar{A}_{x+t}\, v^t \,\mathrm{d}t$$

and supposing, analogically, that the non-negative function

$$\frac{1}{k!}\, t^k \cdot {}_tp_x \cdot \mu_{x+t} \qquad (S_3)$$

is integrable and the function

$$\bar{A}_{x+t} \qquad (S_4)$$

is nearly linear and also integrable, we use on the above equations (6) the mean value theorem.

If we employ the well-known relation

$$\overset{<}{\bar{A}}{}_x^{(k)} = \overset{<}{\bar{a}}{}_x^{(k-1)} - \delta\, \overset{<}{\bar{a}}{}_x^{(k)},$$

it is possible to write

$$\overset{<}{\bar{A}}{}_x^{(k)} = \frac{1}{k} \cdot n_k \cdot \overset{<}{\bar{A}}{}_x^{(k-1)} =$$
$$= \frac{1}{\delta}\, \bar{A}_{x+n_k} \left(1 - \bar{A}_x - \ldots - \delta^{k-1} \cdot \overset{<}{\bar{A}}{}_x^{(k-1)}\right)$$

and n_k is determined by

$$\frac{1}{k}\, n_k \cdot \overset{<}{\bar{A}}{}_x^{(k-1)} = \frac{1}{\delta^k}\, \bar{A}_{x+n_k} \left(1 - \bar{A}_x - \ldots - \delta^{k-1} \cdot \overset{<}{\bar{A}}{}_x^{(k-1)}\right). \tag{7}$$

The life assurance of the order k can be written as follows:

$$\overset{<}{\bar{A}}{}_x^{(k)} = \frac{1}{k!}\, n_1 \ldots n_k \cdot \bar{A}_x. \tag{8}$$

The proof is the same as for life annuities.

Let us deduce an approximate formula for contingent invalidity annuities. In this case it is necessary to make more suppositions.

We denote:

p_x^{aa} the probability of an active being alive on the end of the year as active;

p_x^{i} the probability of an invalid being alive on the end of the year;

ν_x the force of invalidity.

The value of contingent invalidity annuity can be written as:

$$\bar{a}_x^{ai} = \int_0^\infty \int_0^\infty {}_tp_x^{aa}\ \nu_{x+t}\ {}_\tau p_{x+t}^i\ v^{t+\tau}\, dt\, d\tau.$$

We write the value of an increasing contingent invalidity annuity of the order k, likewise, as in ordinary life annuity, in the following ways:

$$\overset{<}{\bar{a}}{}_x^{ai(k)} = \frac{1}{k!}\int_0^\infty \int_0^\infty {}_tp_x^{aa}\ \nu_{x+t}\ {}_\tau p_{x+t}^i (t+\tau)^k\ v^{t+\tau}\, dt\, d\tau = \tag{9}$$

$$= \frac{1}{(k-1)!}\int_0^\infty \int_0^\infty {}_tp_x^{aa}\ \nu_{x+t}\ {}_\tau p_{x+t}^i\ v^{t+\tau}\ (t+\tau)^{k-1}\ v^{t+\tau}\ (t+\bar{a}_{x+t+\tau}^i)\, dt\, d\tau$$

If we wish to use the mean value theorem it is necessary to suppose, that the non-negative function

$$ {}_tp_x^{aa}\ \nu_{x+t}\ {}_\tau p_{x+t}^i\ v^{t+\tau}\ \frac{(t+\tau)^k}{k!} \tag{S_5}$$

is integrable and the functions

$$\bar{a}_{x+\tau+t}^i,\ \bar{a}_{x+t}^{aa} \tag{S_6}$$

are nearly linear and integrable. We get from the first relation (9):

$$\overset{<}{\bar{a}}{}_x^{ai(k)} = \frac{1}{k}\, m_k \,.\, \overset{<}{\bar{a}}{}_x^{ai(k-1)}.$$

After the arrangement of the second relation from (9) we use the mean value theorem and get

$$\overset{<}{\bar{a}}{}_x^{ai(k)} = \frac{1}{(k-1)!}\int_0^\infty t\,\frac{D_{x+t}^{aa}}{D_x^{aa}}\,\nu_{x+t}\int_0^\infty {}_\tau p_{x+t}^i\,(t+\tau)^{k-1}\,v^\tau\, dt\, d\tau +$$

$$+\ \bar{a}_{x+m_k}^i \,.\, \overset{<}{\bar{a}}{}_x^{ai(k-1)}.$$

If we pay our attention that the integral in the above relation may be written as follows:

$$\frac{1}{(k-1)!}\int_0^\infty \frac{D_{x+t}^{aa}}{D_x^{aa}}\,\bar{a}_{x+t}^{aa}\,\nu_{x+t}\int_0^\infty {}_\tau p_{x+t}^i\,(t+\tau)^{k-1}\,v^\tau\, dt\, d\tau$$

we can use the mean value theorem and get finally

$$\overset{<}{\bar{a}}{}_x^{ai(k)} = l_1 \overset{<}{\bar{a}}{}_x^{ai(k-1)} + \bar{a}^i_{x+m_k} \overset{<}{\bar{a}}{}_x^{ai(k-1)}$$

where

$$l_1 = \bar{a}^{aa}_{x+l_1}$$

$$\frac{1}{k} m_k = l_1 + \bar{a}^i_{x+m_k}$$

Analogically as formerly, we have the approximate formula:

$$\overset{<}{\bar{a}}{}_x^{ai(k)} = \frac{1}{k!} m_1 m_2 \ldots m_k \,.\, \bar{a}^{ai}_x. \tag{10}$$

In order to use the above formula it is necessary to have tabulated the values of annuities acording to the order of active and invalid persons.

The calculation of the non-continuous values is possible owing to the analogous theorem for sums to the mean value theorem of a definite integral. The results are, of course, not so precise. Evans uses for life annuities a correction term. But it is not necessary to introduce a correction owing to the experiences on which the tables are based.

Zwei versicherungsmathematische Integralgleichungen.

Von *Hans Koeppler*, Berlin.

Man kann zwei ganz einfache Integralgleichungen aufstellen, welche von der Anzahl der versicherten Leistungen, beziehungsweise von den verschiedenen statistischen Auflösungsmöglichkeiten eines Versicherungsvertrages vollständig unabhängig sind.

Bei den folgenden Betrachtungen wollen wir von einer erweiterten Versicherungsform ausgehen, deren einmalige Prämie nach der Formel

$$\mathfrak{A}_{x\overline{n|}} = \int_0^n \frac{D_{x+t}}{D_x} \left(\mu^I_{x+t} S^I_t + \mu^{II}_{x+t} S^{II}_t\right) dt + \frac{D_{x+n}}{D_x} S^{III}_n$$

berechnet wird. In dieser bedeuten μ^I_{x+t} und μ^{II}_{x+t} die Intensitäten des Eintreffens der Ereignisse, auf welche die Summen S^I_t und S^{II}_t versichert werden, doch kann auch $S^{II}_t = o$ sein. Die Summe der Intensitäten $\mu^I_{x+t} + \mu^{II}_{x+t} = \mu_{x+t}$ soll die gesamte Ausscheideintensität aus der Dekremententafel für das Alter $x + t$ Jahre sein, so daß die bekannte Beziehung