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ON SOME RELAXATIONS COMMONLY USED IN THE STUDY OF LINEAR SYSTEMS

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This note proposes a quite general mathematical proposition which can be a starting point to prove many well-known results encountered while studying the theory of linear systems through matrix inequalities, including the S-procedure, the projection lemma and few others. Moreover, the problem of robustness with respect to several parameter uncertainties is revisited owing to this new theorem, leading to LMI (Linear Matrix Inequality)-based conditions for robust stability or performance analysis with respect to ILFR (Implicit Linear Fractional Representation)-based parametric uncertainty. These conditions, though conservative, are computationally very tractable and make a good compromise between conservatism and engineering applicability.

Keywords: LMI relaxations, robust analysis, parametric uncertainty

Classification: 93C05, 93C35, 93D09

1. INTRODUCTION

In modern linear control theory, many results are stated through matrix inequalities, especially Linear Matrix Inequalities (LMI: [6]). The reason is that existing solvers enable the designer to handle such inequalities as computationally tractable conditions. Therefore, they are often encountered in many fields of automatic control, especially robust analysis and design. A typical approach is to formulate an analysis or control problem as a set of possibly infinitely many inequalities and to use strong theorems to transform this set into a finite set of exploitable inequalities such as LMIs. As examples of such strong theorems, the S-procedure and the projection lemma can be cited. The S-procedure [41], initially proposed by Yakubovich, has a very long history in automatic control, as highlighted in [22]. Some modern versions of the S-procedure such as the ones proposed in [25, 38] are quite general and particularly well adapted to the study of linear systems. They can be efficiently used to prove the Kalman–Yakubovich–Popov (KYP) lemma [35] and the Generalised KYP lemma [25]. Indeed, the S-procedure has always been connected to the KYP lemma [22]. It must be mentioned that there exist more sophisticated relaxation tools and the S-procedure is only a first level of Lasserre's hierarchy of relaxations [27]. However, due to its simplicity, it remains very popular in the control community. The projection lemma, also called matrix elimination lemma

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[1, 6, 39] is particularly useful to synthesize some controllers [1, 2, 10] or to analyse the robustness of uncertain linear models with respect to polytopic uncertainties, through the notion of parameter-dependent Lyapunov functions [5, 10, 17, 32]. It is also very popular in the control community.

In this paper, a quite general theorem is proposed as an attempt to unify several propositions in the literature. It particularly encompasses the full block S-procedure [25, 38] and the projection lemma [1, 6, 39]. It can also be used to prove a more recent result proposed by Graham et al. [20]. This attempt to unify several contributions is to be connected to the paper by Feng et al. [15] as well as to the recent book [12] which particularly focuses on the use of projection lemma to LMI-based robust control. The second section states the theorem and a proof is provided. The third section establishes the connection to the literature. The fourth section gives new insights to analyse the robustness of linear systems with respect to uncertain parameters involved through Implicit Linear Fractional Representation (ILFR). This attempt to bring in an implicit formalism in the relaxation schemes is also to be connected to [15] or to [33]. The obtained LMI condition is very tractable from a computational point of view. It does not really improves the existing results in terms of conservatism but it provides a rather simple and systematic way of reducing analysis problems where parameter-dependent ILFR uncertainties are involved to classic polytopic-like conditions. Therefore, although the uncertainty can be sophisticated, the simplicity and the conservatism are to be compared to the well-known condition in [32]. (see the discussion in the fifth section). The sixth section is devoted to numeric illustration. The paper is concluded in the seventh section.

Notation: M' is the conjugate transpose of M and M^H denotes the epxression $M + M'$. I_q is the identity matrix of order q. I and 0 are identity and zero matrices of appropriate dimensions. $M > 0$ ($M < 0$) means that the matrix M is symmetric and positive (negative) definite. $\mathcal{H}^n$ is the set of Hermitian matrices of dimension n. $\mathrm{Ker}(M)$ can denote either the matrix the columns of which span the right orthogonal nullspace of M or this nullspace itself. $M_\perp$ is the matrix which span the left orthogonal nullspace of M. $\mathrm{Span}(M)$ is the set spanned by the columns of M. Symbol $\otimes$ denotes Kronecker's product. M^+ is some full rank pseudo inverse of M such that $MM^+M = M$ and $M^+MM^+ = M^+$. The set $\mathcal{B}^\star$ denotes $\mathcal{B}\backslash\{0\}$. The notation $\oplus_{i=1}^q M_i = M_1 \oplus \cdots \oplus M_q$ is used to denote $\mathrm{blocdiag}_{i=1,\dots,q}\, M_i$. At last, $\mathbf{i}$ is the imaginary unit.

2. THE PROPOSED THEOREM

The main result is straight away presented. This is a quite hard and general theorem which encompasses many results of the literature, as it will be seen in the next section.

Theorem 1. Let the next mathematical objects be defined:

- $\mathcal{E}$, a subspace of $\mathbb{C}^n$;
- $\mathbf{\Delta}$, a compact set of complex matrices, the elements of which are matrices denoted by Δ;

- $\Theta(\Delta) \in \mathcal{H}_n$, which continuously depends on Δ over $\boldsymbol{\Delta}$;
- $V_L(\Delta) \in \mathbb{C}^{l\times n}$ and $V_R(\Delta) \in \mathbb{C}^{r\times n}$, full rank matrices which continuously depend on Δ over $\boldsymbol{\Delta}$;
- $\bar{V}(\Delta) = \begin{bmatrix} V_L(\Delta) \\ V_R(\Delta) \end{bmatrix}$;
- $\mathcal{S}_L(\Delta) \subset \mathbb{C}^{l}$, a family of subspaces defined over $\boldsymbol{\Delta}$ through Δ-continuously dependent matrix $K_L(\Delta)$ as $\mathcal{S}_L(\Delta) = \mathrm{Ker}(K_L(\Delta)) \cap \mathcal{E}$;
- $\mathcal{S}_R(\Delta) \subset \mathbb{C}^{r}$, a family of subspaces defined over $\boldsymbol{\Delta}$ through Δ-continuously dependent matrix $K_R(\Delta)$ as $\mathcal{S}_R(\Delta) = \mathrm{Ker}(K_R(\Delta)) \cap \mathcal{E}$;
- $\mathcal{B}_L(\Delta) = \{x \in \mathcal{E} : V_L(\Delta)x \in \mathcal{S}_L(\Delta)\}$, $\Delta \in \boldsymbol{\Delta}$;
- $\mathcal{B}_R(\Delta) = \{x \in \mathcal{E} : V_R(\Delta)x \in \mathcal{S}_R(\Delta)\}$, $\Delta \in \boldsymbol{\Delta}$;
- $\mathcal{B}(\Delta) = \mathcal{B}_L(\Delta)^\star \cup \mathcal{B}_R(\Delta)^\star$, $\Delta \in \boldsymbol{\Delta}$;
- $\bar{\mathcal{X}}(\Delta) = \left\{ \xi = \begin{bmatrix} z = z(\Delta) \in \mathcal{S}_L(\Delta) \\ t = t(\Delta) \in \mathcal{S}_R(\Delta) \end{bmatrix} : \exists x \in \mathcal{E} : \bar{V}(\Delta)x = \xi) \right\}, \Delta \in \boldsymbol{\Delta}.$

The two following statements are equivalent :

i)

$$x'\Theta(\Delta)x < 0 \quad \forall x \in \mathcal{B}(\Delta), \quad \forall \Delta \in \boldsymbol{\Delta}. \tag{1}$$

ii)

$$\exists H(\Delta) \in \mathbb{C}^{l\times r} \ : \left\{ \begin{array}{ll} x'((V_L'(\Delta)H(\Delta)V_R(\Delta))^H + \Theta(\Delta))x < 0, & \forall x \in \mathcal{E}^\star, \forall \Delta \in \boldsymbol{\Delta}, \\ \xi' \begin{bmatrix} 0 & H(\Delta) \\ H'(\Delta) & 0 \end{bmatrix} \xi \geq 0 \quad \forall \xi \in \bar{\mathcal{X}}(\Delta), & \forall \Delta \in \boldsymbol{\Delta}. \end{array} \right. \tag{2}$$

a) Moreover, if Θ, V_R and V_L are Δ-independant, then H can be assumed to be Δ-independant.

b) Furthermore, if $V_L(\Delta)$ equals $V_R(\Delta)$ and if $\mathcal{S}_L(\Delta) = \mathcal{S}_R(\Delta) = \mathcal{S}(\Delta) = \{0\}$, then there exists a sufficiently large β such that matrix H can be written $H = -\beta I$.

Remark 1. All the choices for $\Theta(\Delta)$, $V_L(\Delta)$, $V_R(\Delta)$, $K_L(\Delta)$ and $K_R(\Delta)$ are not appropriate to make (1) meaningful. Indeed, let $M(\Delta)$ be a matrix satisfying $x'\Theta(\Delta)x \geq 0$ for any x such that $M(\Delta)x = 0$. Then condition (1) implies that $\mathrm{Ker}(M(\Delta)) \cap \mathcal{B}_L(\Delta) = \{0\}$ and $\mathrm{Ker}(M(\Delta)) \cap \mathcal{B}_R(\Delta) = \{0\}$. Otherwise, there would exist $x^\bullet \in \mathcal{B}(\Delta)$ such that $M(\Delta)x^\bullet = 0$ and thus $x^{\bullet\prime}\Theta(\Delta)x^\bullet \geq 0$, which contradicts (1). As a simple example, assume that $\Theta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and that $M(\Delta) = V_L(\Delta) = V_R(\Delta) = K_L(\Delta) = K_R(\Delta) =$

$\begin{bmatrix} 0 & \Delta \end{bmatrix}$. Then any x which writes $x = \begin{bmatrix} I \\ 0 \end{bmatrix} q$ belongs to $\mathrm{Ker}(M(\Delta))$ but also to $\mathcal{B}(\Delta)$. However, it is clear that $x'\Theta(\Delta)x \geq 0$ so (1) cannot hold.

The meaning of this remark is that for inappropriate choices, it is needless to test (1) through (2) since it can clearly never hold.

Before to prove Theorem 1, some issue is addressed. The idea behind Theorem 1 is that **i)** usually corresponds to a property to be tested. Moreover, it is hardly tractable since checking (1) amounts to check an infinite number of inequalities over a restricted subset. Condition (2) can be more tractable, especially when H is Δ-independent, even if this last assumption might introduce a possible level of conservatism. This theorem might not be exploited in its whole generality but offers nice applications in some special cases (see the fifth section). However, the reader must keep in mind that it is mainly introduced as a general framework to encompass several strong results of the literature.

Proof. In this proof, arguments and notations are sometimes borrowed from [38] and [1].

ii) $\Rightarrow$ **i)**

Assume that the first inequality in (2) holds. If it is satisfied over $\mathcal{E}^\star$, it is *a fortiori* satisfied over $\mathcal{B}(\Delta)$ i.e.

$$x'(V_L'(\Delta)H(\Delta)V_R(\Delta))^H x + x'\Theta(\Delta)x < 0 \quad \forall x \in \mathcal{B}(\Delta), \forall \Delta \in \mathbf{\Delta}. \tag{3}$$

Besides, if $x \in \mathcal{B}(\Delta)$, then $\xi = \begin{bmatrix} z = V_L(\Delta)x \\ t = V_R(\Delta)x \end{bmatrix} \in \bar{\mathcal{X}}(\Delta)$. From the second inequality in (2), it comes

$$x'(V_L'(\Delta)H(\Delta)V_R(\Delta))^H x = (z'H(\Delta)t)^H \geq 0 \quad \forall x \in \mathcal{B}(\Delta), \forall \Delta \in \mathbf{\Delta}. \tag{4}$$

Hence, (3) together with (4) lead to (1).

i) $\Rightarrow$ **ii)**

Assume that (1) holds. Then there exists $\alpha > 0$ small enough such that

$$f(x) = x'T(\Delta)x < 0 \quad \forall x \in \mathcal{B}(\Delta), \quad \forall \Delta \in \mathbf{\Delta}, \tag{5}$$

where $T = \frac{1}{2}(\Theta(\Delta) + \alpha I)$. Indeed, suppose that α cannot be found and consider the sequence which, to any $k \in \mathbb{N}$, associates a matrix $\Delta_k \in \mathbf{\Delta}$, a vector $x_k \in \mathcal{B}(\Delta_k)$ and a scalar $\alpha_k = \frac{1}{k}$ such that

$$x_k'T(\Delta_k)x_k \geq 0. \tag{6}$$

Since $\mathbf{\Delta}$ is compact, one can extract a subsequence $(\Delta_{k_i}, x_{k_i}, \alpha_{k_i})$ whose limit $(\Delta^\bullet, x^\bullet, \alpha^\bullet)$, defined by $\lim_{i\to\infty}(\Delta_{k_i}, x_{k_i}, \alpha_{k_i})$, by continuity, is such that, on the one hand, $\Delta^\bullet \in \mathbf{\Delta}$ and, on the other hand, $x^\bullet \in \mathcal{B}(\Delta^\bullet)$. The former is straightforward. To prove the latter, notice that the definition of $\mathcal{S}_L(\Delta)$ and $\mathcal{S}_R(\Delta)$ leads to

$$\mathcal{B}(\Delta) = \mathrm{Ker}(K_L(\Delta)V_L(\Delta))^\star \cup \mathrm{Ker}(K_R(\Delta)V_R(\Delta))^\star$$

and thus $x_k \in \mathcal{B}(\Delta_k) \Leftrightarrow K_L(\Delta_k)V_L(\Delta_k)x_k = 0$ or $K_R(\Delta_k)V_R(\Delta_k)x_k = 0$. So, owing to the continuity of $K_L(\Delta)V_L(\Delta)$ and $K_R(\Delta)V_R(\Delta)$ with respect to Δ, the limit of the subsequence $x^\bullet$ is such that $K_L(\Delta^\bullet)V_L(\Delta^\bullet)x^\bullet = 0$ or $K_R(\Delta^\bullet)V_R(\Delta^\bullet)x^\bullet = 0$, which means that $x^\bullet \in \mathcal{B}(\Delta^\bullet)$, $\Delta^\bullet \in \boldsymbol{\Delta}$. For this instance, and noting that $\alpha^\bullet$ obvioulsy equals zero, inequality (6) becomes

$$x^{\bullet\prime}\Theta(\Delta)x^\bullet \geq 0, \quad x^\bullet \in \mathcal{B}(\Delta^\bullet),\ \Delta^\bullet \in \boldsymbol{\Delta}. \tag{7}$$

This is in contradiction with (1) so (6) cannot hold i. e. (5) holds for some $\alpha > 0$.

To continue the proof, the space $\mathcal{E}$ is decomposed over a basis whose columns are given by matrix $B(\Delta)$, expressed as follows,

$$B(\Delta) = \begin{bmatrix} B_{L\cup R}(\Delta) & B_c(\Delta) \end{bmatrix}, \tag{8}$$

where:

- $B_{L\cup R}(\Delta)$ is a basis of $(\mathrm{Ker}(V_L(\Delta)) \cup \mathrm{Ker}(V_R(\Delta))) \cap \mathcal{E}$;
- $B_c(\Delta)$ is such that $B(\Delta)$ is a basis of $\mathcal{E}$.

Moreover, going further with that decomposition, let $B_{L\cup R}(\Delta)$ be itself divided into three matrices as follows,

$$B_{L\cup R}(\Delta) = \begin{bmatrix} B_{L\cap R}(\Delta) & B_{L\backslash R}(\Delta) & B_{R\backslash L}(\Delta) \end{bmatrix}, \tag{9}$$

where

- $B_{L\cap R}(\Delta)$ is a basis of $\mathrm{Ker}(V_L(\Delta)) \cap \mathrm{Ker}(V_R(\Delta)) \cap \mathcal{E}$;
- $B_{L\backslash R}(\Delta)$ is such that $B_L(\Delta) = \begin{bmatrix} B_{L\cap R}(\Delta) & B_{L\backslash R}(\Delta) \end{bmatrix}$ is a basis of $\mathrm{Ker}(V_L(\Delta)) \cap \mathcal{E}$;
- $B_{R\backslash L}(\Delta)$ is such that $B_R(\Delta) = \begin{bmatrix} B_{L\cap R}(\Delta) & B_{R\backslash L}(\Delta) \end{bmatrix}$ is a basis of $\mathrm{Ker}(V_R(\Delta)) \cap \mathcal{E}$.

Therefore, one can write

$$\forall x \in \mathcal{E},\ \exists \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \hline \mu_4 \end{bmatrix} = \begin{bmatrix} \mu_{123} \\ \hline \mu_4 \end{bmatrix} :$$

$$x = B_{L\cap R}(\Delta)\mu_1 + B_{L\backslash R}(\Delta)\mu_2 + B_{R\backslash L}(\Delta)\mu_3 + B_c(\Delta)\mu_4 \tag{10}$$

$$\Leftrightarrow x = B_{L\cup R}(\Delta)\mu_{123} + B_c(\Delta)\mu_4. \tag{11}$$

Another possible induced decomposition is given by

$$\Leftrightarrow x = B_L(\Delta)\mu_{12} + B_{L_c}(\Delta)\mu_{34} = B_R(\Delta)\mu_{13} + B_{R_c}(\Delta)\mu_{24} \tag{12}$$

where

$$B_{L_c}(\Delta) = \begin{bmatrix} B_{R\backslash L}(\Delta) & B_c(\Delta) \end{bmatrix}, \quad B_{R_c}(\Delta) = \begin{bmatrix} B_{L\backslash R}(\Delta) & B_c(\Delta) \end{bmatrix}, \tag{13}$$

$$\mu_{12} = \begin{bmatrix} \mu_1' & \mu_2' \end{bmatrix}', \quad \mu_{34} = \begin{bmatrix} \mu_3' & \mu_4' \end{bmatrix}',$$

$$\mu_{13} = \begin{bmatrix} \mu_1' & \mu_3' \end{bmatrix}' \quad \text{and} \quad \mu_{24} = \begin{bmatrix} \mu_2' & \mu_4' \end{bmatrix}'. \tag{14}$$

Then, from (11), the function $f(x)$ defined in (5) can be written

$$\begin{aligned} f(x) = \mu_{123}' B_{L\cup R}'(\Delta)T(\Delta)B_{L\cup R}(\Delta)\mu_{123} + \mu_{123}' B_{L\cup R}'(\Delta)T(\Delta)B_c(\Delta)\mu_4 \\ \mu_4' B_c'(\Delta)T(\Delta)B_{L\cup R}'(\Delta)\mu_{123} + \mu_4' B_c'(\Delta)T(\Delta)B_c(\Delta)\mu_4, \end{aligned} \tag{15}$$

from which the next partial derivative with respect to μ_{123} is deduced:

$$\frac{\partial f}{\partial \mu_{123}} = (2B_{L\cup R}'(\Delta)T(\Delta)B_{L\cup R}(\Delta)\mu_{123} + 2B_{L\cup R}'(\Delta)T(\Delta)B_c(\Delta)\mu_4). \tag{16}$$

A critical point is attained for

$$\frac{\partial f}{\partial \mu_{123}} = 0, \tag{17}$$

i. e. for the special value of μ_{123}:

$$\hat{\mu}_{123} = -(B_{L\cup R}'(\Delta)T(\Delta)B_{L\cup R}(\Delta))^+ B_{L\cup R}'(\Delta)T(\Delta)B_c(\Delta)\mu_4. \tag{18}$$

From (16), one can also infer that

$$\left.\frac{\partial^2 f}{\partial \mu_{123}^2}\right|_{\mu_{123}=\hat{\mu}_{123}} = 2B_{L\cup R}(\Delta)T(\Delta)B_{L\cup R}(\Delta). \tag{19}$$

Each vector of the form $x = B_{L\cup R}(\Delta)q$ is an element of $(\mathrm{Ker}(V_L(\Delta)) \cup \mathrm{Ker}(V_R(\Delta))) \cap \mathcal{E}$ so either $V_L(\Delta)x = 0$ or $V_R(\Delta)x = 0$, which implies that either $V_L(\Delta)x \in \mathcal{S}_L(\Delta)$ or $V_R(\Delta)x \in \mathcal{S}_R(\Delta)$, and therefore either $x \in \mathcal{B}_L(\Delta)$ or $x \in \mathcal{B}_R(\Delta)$. Thus $x \in \mathcal{B}(\Delta)$. So taking (5) into account, the right handside member of (19) is a negative definite matrix over $\mathcal{E}$. Consequently, the above-mentioned critical point is a maximum of $f(x)$ with respect to μ_{123}, which, using (11) applied at this critical point, leads to

$$\max_{\mu_{123}} f(x) = (\hat{\mu}_{123}' B_{L\cup R}'(\Delta) + \mu'(4)B_c'(\Delta))T(\Delta)(B_{L\cup R}\hat{\mu}_{123} + B_c(\Delta)\mu_4).$$

Taking (18) into account yields

$$\max_{\mu_{123}} f(x) = \mu_4' M(\Delta)\mu_4. \tag{20}$$

with

$$M(\Delta) = B_c'(\Delta)T(\Delta)B_c(\Delta) - B_c'(\Delta)T(\Delta)B_{L\cup R}(\Delta)(B_{L\cup R}'(\Delta)T(\Delta)B_{L\cup R}(\Delta))^+ B_{L\cup R}'(\Delta)T(\Delta)B_c(\Delta)$$

$$\Leftrightarrow M(\Delta) = B_c'(\Delta)\underbrace{\left(T(\Delta) - T(\Delta)B_{L\cup R}(\Delta)(B_{L\cup R}'(\Delta)T(\Delta)B_{L\cup R}(\Delta))^+ B_{L\cup R}'(\Delta)T(\Delta)\right)}_{\hat{M}(\Delta)} B_c(\Delta)$$

$$\Leftrightarrow M(\Delta) = B_c'(\Delta)\hat{M}(\Delta)B_c(\Delta).$$

Notice that from (14),

$$\mu_4 = \begin{bmatrix} 0 & I \end{bmatrix} \mu_{34} = \begin{bmatrix} 0 & I \end{bmatrix} \mu_{24}.$$

So if the maximum of $f(x)$ is expressed with respect to μ_{34} and μ_{24} instead of simply μ_4, then it can be written

$$\begin{bmatrix} 0 \\ B_c'(\Delta) \end{bmatrix} \hat{M}(\Delta) \begin{bmatrix} 0 & B_c(\Delta) \end{bmatrix} \mu_{24}.$$

Since x complies with the two expressions in (12) and since $V_L(\Delta)B_L(\Delta) = 0$ and $V_R(\Delta)B_R(\Delta) = 0$, it comes

$$\begin{array}{rcl} V_L(\Delta)x & = & V_L(\Delta)B_{L_c}\mu_{34}, \\ V_R(\Delta)x & = & V_R(\Delta)B_{R_c}\mu_{24}. \end{array}$$

Therefore, once again, the maximum can be rewritten as

$$\begin{array}{c} \max\limits_{\mu_{123}} f(x) \\ = \big((V_L(\Delta)B_{L_c})^+ V_L(\Delta)x\big)' \begin{bmatrix} 0 \\ B_c'(\Delta) \end{bmatrix} \hat{M}(\Delta) \begin{bmatrix} 0 & B_c(\Delta) \end{bmatrix} \big((V_R(\Delta)B_{R_c})^+ V_R(\Delta)x\big) \\ \Leftrightarrow \max\limits_{\mu_{123}} f(x) = x'V_L'(\Delta)H(\Delta)V_R(\Delta)x, \end{array} \tag{21}$$

with

$$H(\Delta) = -(V_L(\Delta)B_{L_c}(\Delta))'^{+} \begin{bmatrix} 0 \\ B_c'(\Delta) \end{bmatrix} \hat{M}(\Delta) \begin{bmatrix} 0 & B_c(\Delta) \end{bmatrix} (V_R(\Delta)B_{R_c}(\Delta))^+. \tag{22}$$

Then, it comes

$$\Leftrightarrow x'(V_L'(\Delta)H(\Delta)V_R(\Delta))^H x = -(\max_{\mu_{123}}(f(x)))^H \leq -\left(f(x)\right)^H \ \forall x \in \mathcal{E}^\star \tag{23}$$

$$\Leftrightarrow x'(V_L'(\Delta)H(\Delta)V_R(\Delta))^H x \leq -(x'T(\Delta)x)^H, \ \forall x \in \mathcal{E}^\star. \tag{24}$$

Keeping in mind that $T(\Delta) = \frac{1}{2}(\Theta(\Delta) + \alpha I)$ for some $\alpha > 0$, it comes

$$\Leftrightarrow x'\left((V_L'(\Delta)H(\Delta)V_R(\Delta))^H + \Theta(\Delta)\right) x \leq -\alpha I < 0, \ \forall x \in \mathcal{E}^\star \tag{25}$$

which proves the first inequality in (2).

Consider any pair of vectors $\{z; t\}$ such that $\xi = [z' \ \ t']' \in \bar{\mathcal{X}}(\Delta)$. Both subvectors can be written $z = V_L(\Delta)x$ and $t = V_R(\Delta)x$ respectively, with the same x that can be expressed $x = B(\Delta)\mu$. Keeping in mind that, since $z \in \mathcal{S}_L(\Delta)$ and $t \in \mathcal{S}_R(\Delta)$, then $x \in \mathcal{B}(\Delta) \cup \{0\}$. Besides, still because $z \in \mathcal{S}_L(\Delta)$ and $t \in \mathcal{S}_R(\Delta)$ and since $V_L(\Delta)B_L(\Delta) = 0$ and $V_R(\Delta)B_R(\Delta) = 0$, from (12) we get $z = V_L(\Delta)x = V_L(\Delta)B_{L_c}(\Delta)\mu_{34}$ and $t = V_R(\Delta)x = V_R(\Delta)B_{R_c}(\Delta)\mu_{24}$ which yields

$$\xi' \begin{bmatrix} 0 & H(\Delta) \\ H'(\Delta) & 0 \end{bmatrix} \xi = (z'H(\Delta)t)^H = -\left(\mu_4' B_c'(\Delta)\hat{M}(\Delta)B_c(\Delta)\mu_4\right)^H = -\max_{\mu_{123}}(f(x))^H \geq 0 \tag{26}$$

which proves the second inequality in (2).

Case `a)`:
Moreover, it is clear that in the event of $\Theta(\Delta)$, $V_L(\Delta_L)$ and $V_R(\Delta_R)$ equaling Δ-independent matrices Θ, V_L and V_R respectively, then, it is no longer required that matrix B defined in (8) depends on Δ and thus matrices $\hat{M}(\Delta)$, $M(\Delta)$ and $H(\Delta)$ also equal Δ-independent matrices $\hat{M}$, M and H.

Case `b)`:
If $V_L(\Delta) = V_R(\Delta)$, one gets $B_{L\cup R}(\Delta) = B_{L\cap R}(\Delta) = B_L(\Delta) = B_R(\Delta)$ and it is possible to choose $B_{L_c}(\Delta) = B_{R_c}(\Delta) = B_c(\Delta)$ so as to get a Δ-independent product $V_L(\Delta)B_c(\Delta) = V_R(\Delta)B_c(\Delta)$. Besides, since $\mathbf{\Delta}$ is compact, and since $\Theta(\Delta)$ (and thus also $T(\Delta)$, $B_c(\Delta)$ and $\hat{M}(\Delta)$) as well as $V_L(\Delta)$ are continuous with respect to Δ over $\mathbf{\Delta}$, then one can find a scalar β large enough so that

$$\beta \geq \bar{\lambda}((V_L(\Delta)B_c(\Delta))'^{+}B_c'(\Delta)'\hat{M}(\Delta)B_c'(\Delta)((V_L(\Delta)B_c(\Delta))^{+}), \tag{27}$$

where $\bar{\lambda}(.)$ denotes the maximum eigenvalue. Therefore, if H is chosen as follows,

$$H(\Delta) = H = -\beta I \quad \forall \Delta \in \mathbf{\Delta}, \tag{28}$$

then it comes

$$x'(V_L'(\Delta)H(\Delta)V_L(\Delta))^H x \leq x'((V_L(\Delta)B_c(\Delta))'^{+}B_c'(\Delta)'\hat{M}(\Delta)B_c'(\Delta)(V_L(\Delta)B_c(\Delta))^{+})^H x,$$

which, from the reasoning of the proof of the part **i)**$\Rightarrow$**ii)** (equations (21) to (25)), implies

$$\begin{aligned} & x'(V_L'(\Delta)H(\Delta)V_L(\Delta))^H x \leq (x'T(\Delta)x)^H,\ \forall x \in \mathcal{E}^\star \\ \Leftrightarrow\ & x'V_L'(\Delta)H(\Delta)V_L(\Delta)x + x'\Theta(\Delta)x \leq -\alpha I < 0,\ \forall x \in \mathcal{E}^\star \end{aligned} \tag{29}$$

which proves the first inequality in (2).

Since $\mathcal{S}_L(\Delta) = \mathcal{S}_R(\Delta) = \{0\}$, the set $\bar{\mathcal{X}}(\Delta)$ reduces to $\{0\}$ and the second inequality in (2) necessarily holds and is of no interest in that case. □

Remark 2. The compactness assumption allows the achievement of the proof in the "difficult path" i.e. **i)**$\Rightarrow$ **ii)** but, without this assumption, the implication **ii)**$\Rightarrow$ **i)** remains true. For a non compact set $\mathbf{\Delta}$, the proof cannot be followed but the question is to know if the equivalence can still be true. It probably depends on the set but this is an open question.

Remark 3. Matrix H is usually called a *multiplier*.

3. CONNECTION TO THE LITERATURE

Theorem 1 is surely general and it is difficult to catch in such a statement what can really be exploited in the context of automatic control. However, for the reader who is familiar with the literature related to robustness or to KYP lemma, this formulation might look like other strong and well-known propositions. In this section, several known results are stated as corollaries of Theorem 1. The justifications are provided. Most of these connections correspond to cases where matrix H is Δ-independent.

3.1. Projection lemma and Finsler's lemma

The first result to be connected to the present work is the so-called *projection lemma* or *matrix elimination lemma*.

Corollary 1. (Apkarian and Gahinet [1], Boyd et al. [6] or Skelton et al. [39, Theorem 2.3.12]) Let Θ belong to $\mathcal{H}_n$ and V_L and V_R be full rank matrices. The following statements are equivalent:

a)

$$\left\{ \begin{array}{ll} \mathrm{Ker}(V_L)'\Theta\mathrm{Ker}(V_L) < 0 & (\text{or } V_L'V_L > 0), \\ \mathrm{Ker}(V_R)'\Theta\mathrm{Ker}(V_R) < 0 & (\text{or } V_R'V_R > 0). \end{array} \right. \tag{30}$$

b)

$$\exists H : \, (V_L' H V_R)^H + \Theta < 0. \tag{31}$$

Proof. In this case, one considers $\mathcal{E} = \mathbb{C}^n$ and the matrices are Δ-independent. Condition (30) can be rewritten

$$\left\{ \begin{array}{lll} x'\Theta x & < & 0, \quad \forall x \in \mathrm{Ker}^*(V_L) \\ y'\Theta y & < & 0, \quad \forall y \in \mathrm{Ker}^*(V_R) \end{array} \right. \tag{32}$$

which is equivalent to

$$x'\Theta x \; < \; 0, \quad \forall x \in \mathrm{Ker}^*(V_L) \cup \mathrm{Ker}^*(V_R).$$

The condition above is exactly condition **i)** of Theorem 1 with

$$\mathcal{B} = \mathrm{Ker}^*(V_L) \cup \mathrm{Ker}^*(V_R).$$

As a consequence we have

$$\mathcal{B}_L = \mathrm{Ker}(V_L), \; \mathcal{B}_R = \mathrm{Ker}(V_R), \; \mathcal{S}_L(\Delta) = \{0\} = \mathcal{S}_R(\Delta) = \{0\}, \; \bar{\mathcal{X}}(\Delta) = \{0\}. \tag{33}$$

Applying Theorem 1 leads to conclude that condition **i)** (i. e. (32) is equivalent to condition **ii)** which must here be interpreted. Indeed, with the special case defined by (33), it can be seen that the second inequality in **ii)** is always satisfied since ξ reduces to zero. Moreover, the first condition in **ii)** clearly reduces to (31) by refering to the case `b)` of Theorem 1 where Θ, V_L, V_R and then H are Δ-independent. □

Corollary 1 is particularly useful to prove interesting results in the realm of robust analysis and state feedback robust control, especially against polytopic uncertainty [10, 17, 32]. It is also exploited by some techniques of static or dynamic output feedback controller design [1, 9]. See [34] for a quite recent survey on the possible applications of this lemma to the study of linear systems, as well as [15] for applications to descriptor linear systems. See also the very recent book [12] for a complete approach of robust control based upon this lemma.

Note that if one particularises the previous theorem to the case where $V_L = V_R = V$, then (30) becomes

$$\mathrm{Ker}(V)'\Theta\mathrm{Ker}(V) < 0 \tag{34}$$

and (31) becomes

$$\exists X :\ V'XV + \Theta < 0, \tag{35}$$

with $X = X' = H + H'$. The equivalence between (34) and (35) corresponds to [39, corollaire 2.3.5]. If a little more attention is paid to Theorem 1 (statement `b)`), it can be seen that a possible structure for X is $X = -\sigma I$ with $\sigma = 2\beta > 0$ large enough. This particular instance corresponds to Finsler's lemma [16, 39].

3.2. S-procedure for linear systems

Corollary 2. (Scherer [38]) Let the next mathematical objects be defined:

- $\mathcal{E}$, a subspace of $\mathbb{C}^n$;
- $\boldsymbol{\Delta}$, a compact set of complex matrices Δ;
- $\Theta(\Delta) \in \mathcal{H}_n$, a matrix which continuously depends on Δ over $\boldsymbol{\Delta}$;
- $V \in \mathbb{C}^{l\times n}$;
- $\mathcal{S}(\Delta) \subset \mathbb{C}^l$, a family of subspaces defined over $\boldsymbol{\Delta}$ through Δ-continuously dependent matrix $K(\Delta)$ as $\mathcal{S}(\Delta) = \mathrm{Ker}(K(\Delta)) \cap \mathcal{E}$;
- $\mathcal{B}(\Delta) = \{x \in \mathcal{E} : Vx \in \mathcal{S}(\Delta)\}$, $\Delta \in \boldsymbol{\Delta}$;

The two following statements are equivalent:

a)

$$x'\Theta(\Delta)x < 0 \quad \forall x \in \mathcal{B}(\Delta)^\star, \quad \forall\Delta \in \boldsymbol{\Delta}. \tag{36}$$

b)

$$\exists X\ :\ \begin{cases} x'(V'XV + \Theta(\Delta))x < 0, \quad \forall x \in \mathcal{E}^\star, \forall\Delta \in \boldsymbol{\Delta}, \\ z'Xz \geq 0 \quad \forall z \in \mathcal{S}(\Delta), \qquad\qquad \forall\Delta \in \boldsymbol{\Delta}. \end{cases} \tag{37}$$

Proof. This is a special instance of Theorem 1. Indeed, in this case, one can consider $l = r$, $V_L = V_R = V$, $\mathcal{B}(\Delta) = \mathcal{B}_L(\Delta) = \mathcal{B}_R(\Delta)$ and $\mathcal{S}(\Delta) = \mathcal{S}_L(\Delta) = \mathcal{S}_R(\Delta)$. Then the set $\mathcal{B}(\Delta)$ in the present corollary matches that in Theorem 1 and inequality (36) matches (1). Moreover, in this case, the set $\bar{\mathcal{X}}(\Delta)$ is defined as a set of vectors $\xi = \begin{bmatrix} z' & z' \end{bmatrix}'$ where $z \in \mathcal{S}(\Delta)$. Matrix H is constant (case `a)` of Theorem 1). So matrix X can comply with $X = H^H$. Then inequality (2) can be rewritten as (37). Therefore, by virtue of Theorem 1, inequality (36) is then equivalent to (37). □

Such an equivalence is called *abstract full block S-procedure* [38]. If $\mathcal{S}(\Delta)$ reduces to $\{0\}$ then [39, corollaire 2.3.5] is recovered again.

In practice, in automatic control, $\mathbf{\Delta}$ is directly defined by the second inequality in (37). Indeed, in many problems of robust analysis, a special choice is made: The set $\mathcal{S}(\Delta)$ is defined through

$$K(\Delta) = \begin{bmatrix} I & -\Delta \end{bmatrix}. \tag{38}$$

The second inequality in (37) becomes

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}' X \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \quad \forall \Delta \in \mathbf{\Delta}, \tag{39}$$

and, in numerous cases, the previous inequality is even the definition of $\mathbf{\Delta}$ itself. Therefore, this second inequality in (37) is satisfied by definition. Moreover, V is often chosen as

$$V = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}, \tag{40}$$

so that this special instance of the S-procedure is called the *concrete full block S-procedure*. It is particularly useful to analyse the robustness of some properties of linear systems with respect to LFR (Linear Fractional Representation)-based uncertainties (such as stability, $\mathcal{D}$-stability, $\mathcal{H}_\infty$ performance, and so on) [37].

3.3. Generalized KYP lemma

Considering the discussion about the concrete S-procedure in the previous subsection, an even more special instance deserves attention, namely the finite frequency KYP lemma:

Corollary 3. (Iwasaki et al. [26]) Let the pair (A, B) be controllable and Θ be a symmetric matrix. The two following statements are equivalent:

a)

$$\begin{bmatrix} (\mathbf{i}\omega I - A)^{-1}B \\ I \end{bmatrix}' \Theta \begin{bmatrix} (\mathbf{i}\omega I - A)^{-1}B \\ I \end{bmatrix} < 0 \qquad \forall \underline{\omega} \leq \omega \leq \overline{\omega}. \tag{41}$$

b)

$$\exists P = P', Q = Q' > 0 :$$
$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}' \begin{bmatrix} -Q & P + \mathbf{i}\left(\dfrac{\underline{\omega} + \overline{\omega}}{2}\right) Q \\ P - \mathbf{i}\left(\dfrac{\underline{\omega} + \overline{\omega}}{2}\right) Q & -\underline{\omega}\overline{\omega} Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta < 0. \tag{42}$$

Proof. It is an application of the concrete full block S-procedure but with

$$V = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}. \tag{43}$$

Indeed, in this case Δ matches sI, with $s = \mathbf{i}\omega I$ and since ω is considered only in the range $(-\underline{\omega}; \overline{\omega})$ in property (41), it means that $\mathbf{\Delta}$ indirectly defines a closed segment of the imaginary axis $\mathfrak{I}$ in which $\mathbf{i}\omega$ lies and which is the intersection between $\mathfrak{I}$ and a disc of centre $\left(0, \gamma = \dfrac{\underline{\omega} + \overline{\omega}}{2}\right)$ and of radius $r = \left(\dfrac{\overline{\omega} - \underline{\omega}}{2}\right)$. Consider the definition of $K(\Delta)$ given in (38), i.e. in this case, $K(\Delta) = K(s) = [I \;\; -sI]$ with $s = \mathbf{i}\omega I$. The considered segment can then be defined through:

$$\begin{bmatrix} sI \\ I \end{bmatrix}' X(P,Q) \begin{bmatrix} sI \\ I \end{bmatrix} \geq 0, \quad \forall P = P', \forall Q = Q' > 0, \tag{44}$$

where

$$X = X(P,Q) = \begin{bmatrix} -Q & P + \mathbf{i}\gamma Q \\ P - \mathbf{i}\gamma Q & -\underline{\omega}\overline{\omega}Q \end{bmatrix}. \tag{45}$$

Indeed, (44) and (45) amount to

$$(-s's - \mathbf{i}\gamma s + \mathbf{i}\gamma s' - \underline{\omega}\overline{\omega})Q + (s + s')P \leq 0, \; \forall P = P', \forall Q = Q' > 0. \tag{46}$$

Since this inequality is meant to hold for any $P = P'$ and for any $Q = Q' > 0$, it particularly holds for $P = 0$ meaning that it necessarily comes

$$-s's - \mathbf{i}\gamma s + \mathbf{i}\gamma s' - \underline{\omega}\overline{\omega} \leq 0.$$

This is the inequality which defines the disc of centre $(0, \gamma)$ and of radius r to which s thus belongs. Besides, for $Q = \alpha I$, with α as low as desired (making the first term of the left handside member of (46) negligible), then the two possible instances $P = I$ and $P = -I$ lead to conclude that

$$s + s' = 0,$$

constraining s to also belong to $\mathfrak{I}$. Reciprocally, if s belongs to both the disc and $\mathfrak{I}$ then (46) obviously holds. The special structure of multiplier (45) is then appropriate to define the considered segment. Therefore, it can be used as a *lossless* definition of $\mathbf{\Delta}$. As in the previous paragraph, it can be seen that $z \in \mathcal{S}(\Delta)$ is parametrized by a vector q as follows: $z = [sI \;\; I]'q$. So the characterization of the frequency range proposed in (44−45) involves an expression which matches the second inequality in (37), provided that X complies with (45). In other words, with such a characterization of $\mathbf{\Delta}$, X can only comply with (45). The second inequality in (37) becomes useless since it is implied by the definition of $\mathbf{\Delta}$ itself. The application of the S-procedure (Corollary 2) reduces to the equivalence between (36 and the first inequality (37), while ensuring that X matches (45) to work on the suitable set $\mathbf{\Delta}$. Now focus on this equivalence. With the choice (43), one gets

$$\mathcal{B}(\Delta) = \mathrm{Ker}(K^*(\Delta)V) = \mathrm{Span}^*\left(\begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}\right) = \mathrm{Span}^*\left(\begin{bmatrix} (\mathbf{i}\omega I - A)^{-1}B \\ I \end{bmatrix}\right),$$

keeping in mind that, due to the defintion of $\mathbf{\Delta}$, ω satisfies $\underline{\omega} \leq \omega \leq \overline{\omega}$. So in this case, (36) writes as (41). Besides, (37) is nothing but (42), so the previously mentioned equivalence between (36) and (37) under constraint (45) is the equivalence between **a)** and **b)**. □

Following the same kind of reasoning, it is possible to address a more general case (by changing the defintion of $\mathbf{\Delta}$ so as to characterize other regions than imaginary ranges) and to prove the famous Generalized KYP lemma [25].

Moreover, when $\underline{\omega} \to -\infty$ and $\overline{\omega} \to +\infty$, then $\mathbf{\Delta}$ is such that $\mathbf{i}\omega$ describes the imaginary axis $\mathfrak{I}$. Such a set is not compact but it can be extended to a compact set in the sense of Alexandrov by adding $\{\infty\}$, and in that case, Corollary 3 reduces to the classic continous KYP lemma [35] which encompasses many propositions such as the Bounded real lemma [1] and so on. Of course, the discrete counterpart of the KYP lemma [35] can also be proved in the same fashion.

3.4. Lyapunov and Stein's inequalities

To recover an even more classic result, as an amusement, the next corollary can be proved.

Corollary 4. (Ostrowski and Schneider [30], Hill [24]) A complex matrix A has no eigenvalue on the extended imaginary axis $\mathfrak{I} \cup \{\infty\}$ if and only if

$$\exists P = P' : \quad A'P + PA < 0. \tag{47}$$

Proof. A has no eigenvalue on $\mathfrak{I} \cup \{\infty\}$ if and only if

$$\det(\lambda I - A) \neq 0 \quad \forall \lambda \in \mathfrak{I} \cup \{\infty\} \tag{48}$$

$$\Leftrightarrow (\lambda' I - A')(-I)(\lambda I - A) < 0 \quad \forall \lambda \in \mathfrak{I} \cup \{\infty\}$$

$$\Leftrightarrow \begin{bmatrix} \lambda' I & I \end{bmatrix} \underbrace{\begin{bmatrix} I \\ -A' \end{bmatrix} (-I) \begin{bmatrix} I & -A \end{bmatrix}}_{\Theta} \begin{bmatrix} \lambda I \\ I \end{bmatrix} < 0 \quad \forall \lambda \in \mathfrak{I} \cup \{\infty\}. \tag{49}$$

Following the same lines as in the proof of Corollary 3, the extended imaginary axis $\mathfrak{I} \cup \{\infty\}$ is losslessly defined by

$$\mathfrak{I} \cup \{\infty\} = \left\{ \lambda \in \mathbb{C} \cup \{\infty\} \,:\, \begin{bmatrix} \lambda' I & I \end{bmatrix} \underbrace{\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}}_{X} \begin{bmatrix} \lambda I \\ I \end{bmatrix} \geq 0, \forall P = P' \right\}. \tag{50}$$

By applying Theorem 1 with the choice $V = I$, $\mathbf{\Delta} = \{\lambda I : \lambda \in \mathfrak{I} \cup \{\infty\}\}$, one deduces that A has no eigenvalue on $\mathfrak{I} \cup \{\infty\}$ if and only if

$$X + \Theta < 0. \tag{51}$$

By virtue of Corollary 1, and noting that

$$\begin{bmatrix} A \\ I \end{bmatrix} = \operatorname{Ker}\left(\begin{bmatrix} I & -A \end{bmatrix} \right),$$

inequality (51) is equivalent to (47). □

This result is to be connected to the notion of $\delta\mathcal{D}$-regularity or $\mathcal{S}$-regularity of a matrix introduced in [3]. If the extended closed right half complex plane $\mathbb{C}^+ \cup \{\infty\}$

is considered instead of $\mathfrak{I} \cup \{\infty\}$, then the above reasoning can be followed but with imposing $P > 0$ in (50). Then Corollary 4 is modified so as to state that A is Hurwitz-stable if and only if there exists $P = P' > 0$ such that (47) holds, which is nothing but Lyapunov's inequality [28]. For the discrete counterpart, an analogous reasoning enables the recovery of Stein's inequality [24, 40].

3.5. A new theorem

Now, another theorem is proposed, which can also be deduced (but a little less directly) from Theorem 1. This theorem is a basis for an approach to robust analysis. It does not (at least obvioulsy) lead to less conservative results than other results of the literature, but, as it will be seen in the next section, it offers the possibility to derive tools which, on the one hand, take sophisticated uncertainties into account and, on the other hand, are as simple as the ones proposed in [32] for polytopic uncertainties.

Theorem 2. Let the next mathematical objects be defined:

- $\mathcal{X}$, a subset of the set $\mathcal{Y}$, defined by

$$\mathcal{X} \subset \left(\mathcal{Y} = \left\{ X = X' \in \begin{bmatrix} X_{11} & X_{12} \\ X'_{12} & X_{22} \end{bmatrix} : X_{11} < 0 \right\} \right) \tag{52}$$

  such that the set $\boldsymbol{\Delta}$ of complex matrices Δ defined by

$$\boldsymbol{\Delta} = \left\{ \Delta : \begin{bmatrix} \Delta \\ I \end{bmatrix}' X \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \forall X \in \mathcal{X} \right\}, \tag{53}$$

  is compact and nonempty;

- $\Theta \in \mathcal{H}_n$;

- A, B, E and F matrices such that the square ILFR-based matrix $(E-\Delta A)^{-1}(\Delta B-F)$ is well-posed over $\boldsymbol{\Delta}$ (meaning that $(E-\Delta A)^{-1}$ exists over $\boldsymbol{\Delta}$).

The two following statements are equivalent :

a)

$$\begin{bmatrix} (E-\Delta A)^{-1}(\Delta B - F) \\ I \end{bmatrix} \Theta \begin{bmatrix} (E-\Delta A)^{-1}(\Delta B - F) \\ I \end{bmatrix} < 0, \quad \forall \Delta \in \boldsymbol{\Delta}. \tag{54}$$

b)

$$\exists H : \left\{ H \begin{bmatrix} (E-\Delta A) & (F-\Delta B) \end{bmatrix} \right\}^H + \Theta < 0 \quad \forall \Delta \in \boldsymbol{\Delta}. \tag{55}$$

Proof. The idea is of course to draw an analogy with Theorem 1 where $\mathcal{E} = \mathbb{C}^n$.
a)⇒b):

Consider that $V_L = V_R = V$ with

$$V = \begin{bmatrix} E & F \\ A & B \end{bmatrix}. \tag{56}$$

Moreover, consider the choice (38). Then the set

$$\mathcal{B}(\Delta) = \mathrm{Span}\left(\begin{bmatrix} (E-\Delta A)^{-1}(\Delta B - F) \\ I \end{bmatrix}\right), \quad \Delta \in \mathbf{\Delta}. \tag{57}$$

matches the definition in Corollary 2 with $\mathcal{S}_L(\Delta) = \mathcal{S}_R(\Delta) = [I \quad -\Delta]$, and inequality (54) corresponds to (36).

Besides, by definition, there exists some *multiplier* X which satisfies the second inequality in (37) due to the definition of $\mathbf{\Delta}$, i. e. (53).

Therefore, the application of Corollary 2 here consists in the equivalence between (54) and the existence of a Δ-independent multiplier $X \in \mathcal{X}$ such that the first inequality in (37) holds. This inequality here becomes

$$\Theta + \begin{bmatrix} E & F \\ A & B \end{bmatrix}' X \begin{bmatrix} E & F \\ A & B \end{bmatrix} < 0. \tag{58}$$

Since Δ belongs to $\mathbf{\Delta}$ and since X belongs to $\mathcal{X} \subset \mathcal{Y}$, then it comes

$$\Delta' X_{11} \Delta + \Delta' X_{12} + X_{12}' \Delta + X_{22} \geq 0, \quad \forall \Delta \in \mathbf{\Delta} \tag{59}$$

which, by using Schur's complement argument, is equivalent to

$$\Phi(\Delta) = \left[\begin{array}{c|c} X_{11} & -X_{11}\Delta \\ \hline -\Delta' X_{11} & -X_{22} - \Delta' X_{12} - X_{12}'\Delta \end{array}\right] \leq 0, \quad \forall \Delta \in \mathbf{\Delta}. \tag{60}$$

It implies

$$V' \Phi(\Delta) V \leq 0, \quad \forall \Delta \in \mathbf{\Delta}. \tag{61}$$

Adding inequalities (58) and (61) leads to

$$\Theta + V' \underbrace{(X + \Phi(\Delta))}_{Z(\Delta)} V < 0 \quad \forall \Delta \in \mathbf{\Delta}. \tag{62}$$

Note that $Z(\Delta)$ satisfies

$$Z(\Delta) = \left[\begin{array}{c|c} 2X_{11} & X_{12} - X_{11}\Delta \\ \hline X_{12}' - \Delta' X_{11} & -\Delta' X_{12} - X_{12}'\Delta \end{array}\right] = \left(\begin{bmatrix} X_{11} \\ X_{12}' \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix}\right)^H \quad \forall \Delta \in \mathbf{\Delta}. \tag{63}$$

With this expression at hand, inequality (62) can be written

$$\Theta + \left(\begin{bmatrix} E & F \\ A & B \end{bmatrix}' \begin{bmatrix} X_{11} \\ X_{12}' \end{bmatrix} \begin{bmatrix} (E-\Delta A) & (F - \Delta B) \end{bmatrix}\right)^H < 0, \quad \forall \Delta \in \mathbf{\Delta}, \tag{64}$$

which is nothing but (55) with the choice

$$H = \begin{bmatrix} E & F \\ A & B \end{bmatrix}' \begin{bmatrix} X_{11} \\ X_{12}' \end{bmatrix}. \tag{65}$$

b)⇒a):

Now assume that $V_L = I$ and that

$$V_R(\Delta) = \begin{bmatrix} (E - \Delta A) & (F - \Delta B) \end{bmatrix}, \quad \Delta \in \mathbf{\Delta}. \tag{66}$$

Also make the choice that $\mathcal{S}_L(\Delta) = \{0\}$ and $\mathcal{S}_R(\Delta) = \{0\}$. Then the set $\mathcal{B}(\Delta)$ as defined by (57) matches the set $\mathcal{B}_R(\Delta)$ in Theorem 1. Since $\mathcal{B}_L(\Delta)$ reduces to $\{0\}$, then $\mathcal{B}(\Delta)$ in (57) is also the same as $\mathcal{B}(\Delta)$ in Theorem 1.

The second inequality in condition **ii)** of Theorem 1 necessarily holds since $\mathcal{S}_L(\Delta) = \{0\}$ and $\mathcal{S}_R(\Delta) = \{0\}$ imply $\bar{\mathcal{X}}(\Delta) = \{0\}$. At last, inequality (55) can be written as the first inequality in condition **ii)** of Theorem 1. Then the application of Theorem 1 leads to inequality (1), which, in the present case, is nothing but (54). □

In Theorem 2, the set $\mathcal{X}$ is implicitly defined through $\mathbf{\Delta}$. In other words, from (53), any X in $\mathcal{X}$ is such that

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}' X \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0, \quad \forall \Delta \in \mathbf{\Delta}. \tag{67}$$

Reciprocally, still from (53), any $\Delta \in \mathbf{\Delta}$ is such that (67) is satisfied. This is a particular and ideal case where the structure of X perfectly characterizes $\mathbf{\Delta}$. The set $\mathcal{X}$ is then *lossless*. This is a generalization of what was used in the proof of Corollary 3.

Of course, the definition of a lossless set $\mathcal{X}$ (i. e. a lossless structure of X) is not always easy to obtain and is a well-known problem which induces conservatism in various conditions for robust analysis proposed in the literature. However, in the following corollary, the use of a lossless $\mathcal{X}$ is possible. This corollary turns to be the alternative KYP lemma which was originally proposed in [20] (Note that it also led to interesting and significant extensions in [21, 19]), This alternative KYP lemma can be seen as an alternative to Corollary 3.

Corollary 5. (Graham et al. [20]) Let the pair (A, B) be controllable and let $\Theta \in \mathcal{H}_n$ affinely depend on ω over the range $(\underline{\omega}; \overline{\omega})$. The statements **a**) and **b**) as expressed in Corollary 3 are equivalent to the following one:

c)

$$\exists H : \Theta + \left(H \begin{bmatrix} I \\ \mathbf{i}\omega I \end{bmatrix}' \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right)^H < 0 \qquad \forall \omega \in \{\underline{\omega}; \overline{\omega}\}. \tag{68}$$

P r o o f. The equivalence between **a**) and **b**) is of course proved by virtue of Corollary 3. The equivalence between **a**) and **c**) results from the application of Theorem 2 with the next change of variables :

$$\Delta \leftarrow \mathbf{i}\omega I, \quad A \leftarrow I, \quad B \leftarrow 0 \quad E \leftarrow A, \quad F \leftarrow B, \tag{69}$$

and X as defined in (45) which is known to be lossless (it is also shown in the proof of Corollary 3). With such an instance, Theorem 2 proves the equivalence between **a**) and **c**) but for any ω belonging to the range $(\underline{\omega};\overline{\omega})$. However, if the inequality in **c**) is satisfied for the whole range $(\underline{\omega};\overline{\omega})$, it is *a fortiori* satisfied for the extreme values $\underline{\omega}$ and $\overline{\omega}$ and, the other way around, if it is satisfied for the extreme values, it is also satisfied anywhere inside the range by simple convex combination. □

4. NEW INSIGHTS FOR ROBUSTNESS ANALYSIS AGAINST PARAMETRIC UNCERTAINTIES

In this section, Theorem 2 is exploited to analyse the robustness of some properties of uncertain linear systems with respect to time invariant ILFR-based parametric uncertainties. It is shown that, by exploiting Theorem 2, such an uncertainty can be handled with the same simplicity as if the parameters were involved in a linear fashion in the model.

Corollary 6. Let the matrix M be Hermitian and $\boldsymbol{\Delta}$ be the set of real matrices Δ which are defined by

$$\Delta = \mathop{\oplus}_{j=1}^{q} \delta_j I_{n_j} \tag{70}$$

where the *real* parameters δ_j satisfy

$$\underline{\delta}_j \leq \delta_j \leq \overline{\delta}_j, \quad \forall j \in \{1, \dots, q\}. \tag{71}$$

Also define the $N = 2^q$ matrices Δ_i, $i = 1, \dots, N$, as the vertices of the set $\boldsymbol{\Delta}$, which is actually the polytope of matrices obtained when the parameters δ_j describe their respective ranges. At last, define the uncertain matrix $\mathbf{A}$ as the ILFR

$$\mathbf{A}(\Delta) = D + C(E - \Delta A)^{-1}(\Delta B - F), \tag{72}$$

which is assumed to be well-posed over $\boldsymbol{\Delta}$. Consider the two next statements:

i)
$$\begin{bmatrix} \mathbf{A}(\Delta) \\ I \end{bmatrix}' M \begin{bmatrix} \mathbf{A}(\Delta) \\ I \end{bmatrix} < 0, \quad \forall \Delta \in \boldsymbol{\Delta}. \tag{73}$$

ii) There exists a Δ-independent matrix H such that

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}' M \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \left\{ H \begin{bmatrix} (E - \Delta_i A) & (F - \Delta_i B) \end{bmatrix} \right\}^H < 0 \qquad \forall i \in \{1, \dots, N\}. \tag{74}$$

Then statement **ii)** is sufficient for statement **i)** to hold and if $q = 1$, then the converse is also true.

Proof. The first issue to address is to make the connection between $\boldsymbol{\Delta}$ as defined in the present corollary and the definition of $\boldsymbol{\Delta}$ proposed in Theorem 2. Actually, the present set $\boldsymbol{\Delta}$ is a subset (and only a subset!) of $\boldsymbol{\Delta}$ defined in Theorem 2 with the special choice

$$X = \left[\begin{array}{c|c} X_{11} = -\overset{q}{\underset{j=1}{\oplus}} Q_j & X_{12} = \overset{q}{\underset{j=1}{\oplus}} (c_j Q_j + G_j) \\ \hline X'_{12} = \overset{q}{\underset{j=1}{\oplus}} (c_j Q_j + G'_j) & X_{22} = \overset{q}{\underset{j=1}{\oplus}} g_j Q_j \end{array}\right], \tag{75}$$

and with

$$Q_j \in \mathbb{R}^{n_j \times n_j},\ Q_j = Q'_j > 0,\ G_j \in \mathbb{R}^{n_j \times n_j},\ G_j = -G'_j\, \forall j \in \{1, \dots, q\}, \tag{76}$$

and

$$c_j = \frac{\underline{\delta}_j + \overline{\delta}_j}{2},, \quad g_j = \left(\frac{\overline{\delta}_j - \underline{\delta}_j}{2}\right)^2 - c_j^2, \tag{77}$$

meaning that each δ_j is in a disc of centre c_j with radius $r_j = \left(\frac{\overline{\delta}_j - \underline{\delta}_j}{2}\right)$. This is now a quite classic characterization which was proposed for instance in [11] or in a slightly different fashion in [25].

Besides, also in a very classic fashion, each matrix Δ belonging to the present set $\boldsymbol{\Delta}$ can be written as a convex combination of the vertices Δ_i:

$$\Delta = \sum_{i=1}^{N} \alpha_i \Delta_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i = 1. \tag{78}$$

Therefore, since the barycentric coordinates α_i are positive, if the convex combination of the inequalities in (84) is built with these coordinates, then condition (55) is satisfied over $\boldsymbol{\Delta}$ with

$$\Theta = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}' M \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{79}$$

If inequality (74) holds, then inequality (55) holds for an overset of $\boldsymbol{\Delta}$ and therefore also for $\boldsymbol{\Delta}$ itself. Thus, by virtue of Theorem 2, inequality (54) is satisifed, which can also be written as (73).

To complete the proof, the case $q = 1$ must be considered. In this case, the structure (75) defines a lossless set $\mathcal{X}$ which losslessly characterizes the set $\boldsymbol{\Delta}$ (see [25] or follow a similar reasoning as in the proof of Corollary 3). Hence, the application of Theorem 2 leads to the equivalence between both statements. □

To clearly understand why necessity is not verified for $q > 1$, consider the case where Δ is defined by (70) and (71) with $q = 2$, $n_1 = n_1$ and $\overline{\delta}_j = -\underline{\delta}_j = 1 \Rightarrow c_j = 0$, $g_j = 1$. A possible instance of X could be

$$X = \left[\begin{array}{c|c} X_{11} = -I_2 \otimes Q & X_{12} = 0 \\ \hline X'_{12} = 0 & X_{22} = I_2 \otimes Q \end{array}\right], \tag{80}$$

meaning that $Q_1 = Q_2 = Q > 0$ and $G_1 = G_2 = 0$. Indeed, it is a particular element of the set $\mathcal{X}$ defined by (52) and (75). However, for such multiplier, matrix

$$\Delta = \begin{bmatrix} \alpha I & \alpha I \\ \alpha I & \alpha I \end{bmatrix}, \tag{81}$$

with $\alpha \in \mathbb{R}$ checking $|\alpha| \leq \frac{1}{2}$ satisfies (67) whereas it does not belong to $\mathbf{\Delta}$. If the obtained H is such that the underlying X complies with (80), it means that the property to be tested is proved to be satisfied for uncertainties which are "beyond" the considered set $\mathbf{\Delta}$ (e. g. Δ in (81)), which is not required. Therefore, some degree of conservatism is introduced.

Corollary 6 can be used to assess the robustness of many properties, at least when matrix M is Δ-independent. Note that the case $q = 1$ corresponds to the problem addressed in [18].

For instance, the next corollary can be stated.

Corollary 7. Let the set $\mathbf{\Delta}$ and the matrix $\mathbf{A}(\Delta)$ be defined as in Corollary 6. Then matrix $\mathbf{A}(\Delta)$ is *quadratically* Hurwitz stable against $\mathbf{\Delta}$ if (and only if when $q = 1$) there exist matrices $P = P' > 0$ and H such that

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}' \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \left\{ H \left[\ (E - \Delta_i A) \ (F - \Delta_i B) \ \right] \right\}^H < 0 \qquad \forall i \in \{1, \ldots, N\}. \tag{82}$$

Proof. This is a special application of Corollary 6 with

$$M = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}. \tag{83}$$

In this case, inequality (73) is nothing but Lyapunov's inequality and P is a Δ-independent Lyapunov matrix valid for the whole set $\mathbf{\Delta}$. □

But of course, it would be very interesting to obtain the same result with parameter-dependent Lyapunov functions, i. e. $P(\Delta)$ instead of P. It is unfortunately not so straightforward since M is assumed to be Δ-independent in Theorem 6. This comes from the fact that Θ is assumed to be Δ-independent in Theorem 2. The question is to know if it is possible that Θ depends on Δ in Theorem 2. In [19, 21], Corollary 5 (i. e. alternative KYP lemma) is extended to the case where Θ is affine with respect to the frequency, while preserving necessity of the condition. But in the general case where Θ would depend on several parameters in a more sophisticated way, the necessity can anyway not hold (see the counterexample further mentioned).

Nevertheless, the sufficiency in itself provides a quite interesting result:

Corollary 8. Let the set $\mathbf{\Delta}$ and the matrix $\mathbf{A}(\Delta)$ be defined as in Corollary 6. Then matrix $\mathbf{A}(\Delta)$ is robustly Hurwitz stable against $\mathbf{\Delta}$ if N symmetric positive matrices P_i, $i = 1, \ldots, N$ and a matrix H exist such that

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}' \begin{bmatrix} 0 & P_i \\ P_i & 0 \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \left\{ H \left[\ (E - \Delta_i A) \ (F - \Delta_i B) \ \right] \right\}^H < 0 \qquad \forall i \in \{1, \ldots, N\}. \tag{84}$$

P r o o f . It is similar to the proof of the sufficiency part of Corollary 6. By using convex combination, inequality (55) is recovered with

$$\Theta(\Delta) = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}' \begin{bmatrix} 0 & P(\Delta) \\ P(\Delta) & 0 \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}, \tag{85}$$

and with

$$P(\Delta) = \sum_{i=1}^{N} \alpha_i P_i. \tag{86}$$

By virtue of Theorem 2, inequality (54) holds and it can also be written

$$\mathbf{A}'(\Delta)P(\Delta) + P(\Delta)\mathbf{A}(\Delta) < 0 \quad \forall \Delta \in \mathbf{\Delta}, \tag{87}$$

which is nothing but Lyapunov's inequality applied to $\mathbf{A}(\Delta)$. Thus, $\mathbf{A}(\Delta)$ is robustly Hurwitz-stable against $\mathbf{\Delta}$. □

5. DISCUSSION AND COMMENTS

The first comment is that this robust Hurwitz stability test can be easily extended to many other performances tests such as $\mathcal{H}_\infty$ level [1], pole clustering constraints in convex regions i. e. $\mathcal{D}$-stability [8, 17, 32] or even pole clustering constraints in non convex regions [3, 4, 5]. Indeed, all those performances have been expressed in terms of inequalities such as (1) or (54). Thus, Theorem 2 can be exploited following the same lines as for Corollary 8.

The second comment is that the result presented by Corollary 8 is conservative and thus cannot be positively compared in that sense to other results of the literature such as for instance [7, 36]. But these references exhibit quite sophisticated conditions which, though interesting, are often very cumbersome and which might not find their way to the engineering world, unlike conditions proposed in [17, 32] which are very tractable from a numeric point of view and thus very easy to test with usual solvers.

Going on with dilated LMI conditions which can reasonably be considered by an engineer, the connection to [13] has to be mentioned. Corollary 8 might be seen as an extension of [13, Theorem 3] to the case of ILFR. Also, Corollary 8 is very close to [13, Theorem 4]. In [13, Theorem 4], the possibility to consider a multiaffine Lyapunov matrix with respect to the parameters δ_j is offered and the multiplier H, which is also multiaffine, is considered on the vertices of $\mathbf{\Delta}$. However, an additional condition is imposed on the multiplier which, if easily satisfied for some special cases [13], might be somewhat constraining in the general case i. e. for any LFR matrices. On the contrary, in Corollary 8 the multiplier H is Δ-independent but the more general case of an *implicit* LFR is considered. Whatever the present comments are, the connection with [13] is strong and much attention should be paid to this paper.

Moreover, even if a sophisticated ILFR uncertainty might increase the size of the matrices A, B, C, D, E, F and Δ, the number of inequalities to be tested remains the same, i. e. $N = 2^q$ where q is the number of involved parameters. It means that the number of LMIs (which contributes more to computational heavyness than the size of the matrices) does not depend on the complexity of the ILFR. Only the number of parameters is really significant. In that sense, once again, it makes the present condition similar to the very frequently used condition for $\mathcal{D}$-stability proposed in [32]. It is even equivalent in the case where $\mathbf{A}(\Delta)$ is an affine function of the parameters δ_j. But, clearly, it can be used for a far larger class of uncertainties. For this reason, the authors think that it is a very good compromise between conservatism and computational heavyness.

Of course, the way the matrices A, B, C, D, E, F can be found is a possibly hard challenge when the parameters appear in a very sophisticated fashion in $\mathbf{A}(\Delta)$ but this is an unfortunately classic problem encountered in the whole of the literature devoted to robustness analysis. The derivation of LFR-based expressions is hidden behind many contributions to robust analysis and design. The reader must be aware that Implicit LFR are very useful to decrease the dimension of Δ when building expression (72) [29, 31] and a great attention can be paid to [23].

6. ILLUSTRATIVE EXAMPLES

To illustrate the efficiency of Corollary 8, the following academic example is proposed. Consider the set $\mathbf{\Delta}$ as defined by (70), $q = 2$ and

$$\begin{cases} \underline{\delta}_1 = -0.7289 \leq \delta_1 \leq \overline{\delta}_1 = 0.7289, \\ \underline{\delta}_2 = 0.3600 \leq \delta_2 \leq \overline{\delta}_2 = 1.6400, \end{cases} \tag{88}$$

as well as the following uncertain matrix:

$$\mathbf{A}(\Delta) = \left[\begin{array}{c|c|c} -10 + \dfrac{\delta_1}{1-\delta_1} & \dfrac{(3\delta_1 - 2)(\delta_2 + 2)}{\delta_2(1-\delta_1)} & 1 \\ \hline -3 & -3 & 0 \\ \hline -4 & \dfrac{\delta_2 + 2}{\delta_2} & -1 \end{array}\right]. \tag{89}$$

This matrix is well-posed over $\mathbf{\Delta}$ and a possible (minimal) ILFR is given by

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} = \left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 1 & 0 & -10 & 0 & 1 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 1 & -4 & 0 & -1 \\ \hline 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 \end{array}\right], \tag{90}$$

meaning that $n_1 = n_2 = 1$. Using condition (84) (involving ($N = 4$) LMIs) which is found feasible, matrix $\mathbf{A}(\Delta)$ is proved to be robustly stable against $\mathbf{\Delta}$. To appreciate the weak conservatism of the condition, we plot the eigenvalues of $\mathbf{A}(\Delta)$ for 1600 random instances of $\Delta = \delta_1 \oplus \delta_2$ inside $\mathbf{\Delta}$. The result is depicted on Figure 1. It can be seen that some clusters of eigenvalues tend to touch the imaginary axis, highlighting the tightness of the condition.

Note that if the size of the polytope is very slightly increased by enlarging either the range $(\underline{\delta}_1; \overline{\delta}_1)$ or the range $(\underline{\delta}_2; \overline{\delta}_2)$ on both sides, not only the LMI system is found infeasible but plotting the clusters of eigenvalues shows that instability is reached.

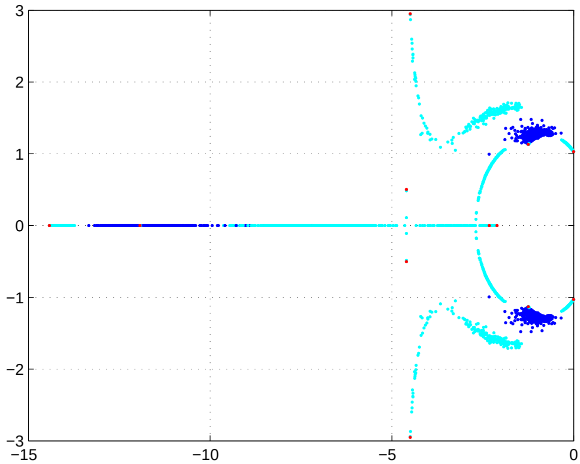

Fig. 1. Migration of the eigenvalues of $\mathbf{A}(\Delta)$ over $\boldsymbol{\Delta}$ (first example).

Another academic example is now considered. The set of uncertain parameters is defined by

$$\begin{cases} \underline{\delta}_1 = -0.5145 \leq \delta_1 \leq \overline{\delta}_1 = 0.5145, \\ \underline{\delta}_2 = 0.4340 \leq \delta_2 \leq \overline{\delta}_2 = 1.5660, \end{cases} \tag{91}$$

and the uncertain matrix to be analysed is given by

$$\mathbf{A}(\Delta) = \left[\begin{array}{c|c|c} -5 + \dfrac{\delta_1}{1-\delta_1} & 0 & 4 \\ \hline -3 & -3 + \dfrac{2}{1+\delta_2} & 0 \\ \hline -4 + \dfrac{\delta_1}{1-\delta_1} + \dfrac{1}{2\delta_2} & 2 - \dfrac{1}{2\delta_2} & -1 + \dfrac{1}{\delta_2} \end{array}\right]. \tag{92}$$

This matrix is well-posed over the set of uncertain parameters and a possible ILFR is given by

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -5 & 0 & 4 \\ 0 & 1 & 0 & -3 & -3 & 0 \\ 0 & 0 & 1 & -4 & 2 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 1 & -2 \end{array}\right], \tag{93}$$

meaning that, in this case, $n_1 = 1$ but $n_2 = 2$, i. e.

$$\Delta = \delta_1 \oplus \delta_2 I_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix}. \tag{94}$$

This ILFR might not be minimal. Using condition (84) (involving ($N = 4$) LMIs) which is found feasible, matrix $\mathbf{A}(\Delta)$ is proved to be robustly stable against $\boldsymbol{\Delta}$. To appreciate the weak conservatism of the condition, we plot the same clusters of eigenvalues as in the previous example, highlighting once again that the condition is weakly conservative (see Figure 2).

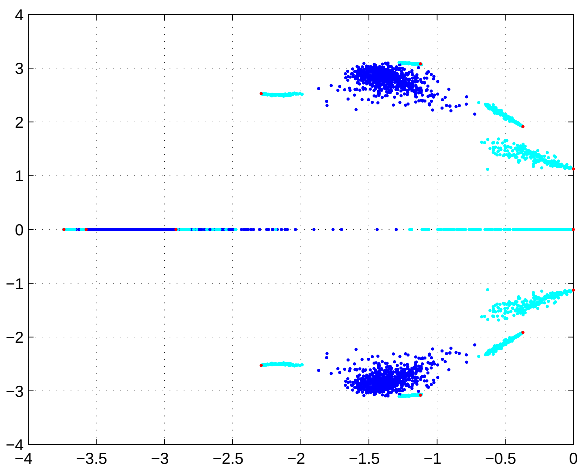

Fig. 2. Migration of the eigenvalues of $\mathbf{A}(\Delta)$ over $\boldsymbol{\Delta}$ (second example).

Note that, once again if the size of the polytope is very slightly increased by enlarging either the range $(\underline{\delta}_1; \overline{\delta}_1)$ or the range $(\underline{\delta}_2; \overline{\delta}_2)$ on both sides, not only the LMI system is found infeasible but plotting the clusters of eigenvalues shows that instability is reached.

These rather sophisticated examples could tend to show that condition (84) is not conservative. This is not true. Indeed, if a little attention is paid to the proof of Corollary 8, then it can be seen that the structure of $P(\Delta)$ is $\sum_{i=1}^{N} \alpha_i P_i$. Such a class of Lyapunov matrices is not general enough to assess robust stability, even in the case of one single real uncertain parameter, as pointed out in [14] through a counter-example. Nevertheless, condition (84) is very tractable and induces a conservatism which is often more than reasonable.

7. CONCLUSION

In this paper, a quite general proposition was established. It can be a starting point to prove many useful theorems encountered in the theory of linear systems, especially when robustness aspects are considered. Moreover, a simple and strong LMI condition for a matrix to be stable against ILFR-based parametric uncertainy was proposed.

This work is mostly achieved in order to give coherence to some existing results but the perspectives have to be regarded with attention. An investigation could be followed to know if this work could be exploited to propose extensions of the presented conditions from robust analysis to robust synthesis.

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