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SINGER-THORPE BASES FOR SPECIAL EINSTEIN CURVATURE TENSORS IN DIMENSION 4

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Abstract. Let (M,g) be a 4-dimensional Einstein Riemannian manifold. At each point p of M, the tangent space admits a so-called Singer-Thorpe basis (ST basis) with respect to the curvature tensor R at p. In this basis, up to standard symmetries and antisymmetries, just 5 components of the curvature tensor R are nonzero. For the space of constant curvature, the group O(4) acts as a transformation group between ST bases at T_pM and for the so-called 2-stein curvature tensors, the group $\mathrm{Sp}(1) \subset \mathrm{SO}(4)$ acts as a transformation group between ST bases. In the present work, the complete list of Lie subgroups of SO(4) which act as transformation groups between ST bases for certain classes of Einstein curvature tensors is presented. Special representations of groups SO(2), T^2, Sp(1) or U(2) are obtained and the classes of curvature tensors whose transformation group into new ST bases is one of the mentioned groups are determined.

Keywords: Einstein manifold; 2-stein manifold; Singer-Thorpe basis

MSC 2010: 53C25

1. Introduction

Singer and Thorpe, see [9], have proved the following:

Theorem 1.1. *If (M,g) is a 4-dimensional Einstein Riemannian manifold and R its curvature tensor at some fixed point p, then there is an orthonormal basis $\mathcal{B} = \{e_1,e_2,e_3,e_4\}$ in T_pM such that the complementary sectional curvatures are equal, i.e. $K_{12}=K_{34}$, $K_{13}=K_{24}$, $K_{14}=K_{23}$, and all the corresponding components R_{ijkl} with exactly three distinct indices are equal to zero.*

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Such a basis is referred to as a Singer-Thorpe basis or as an ST basis. In the following, we shall study ST bases in a purely algebraic way. Following Gilkey (see [5], page 17), we introduce the following:

Definition 1.2. An *algebraic curvature tensor* on a vector space $\mathbb{V}$ with a positive scalar product $\langle , \rangle$ is a tensor R of the type $(0, 4)$ on $\mathbb{V}$ which satisfies the same symmetries and antisymmetries as the Riemannian curvature tensor of a Riemannian manifold, i.e.

$$R(U, V, W, Z) = -R(V, U, W, Z) = R(W, Z, U, V),$$
$$R(U, V, W, Z) + R(V, W, U, Z) + R(W, U, V, Z) = 0$$

for all $U, V, W, Z \in \mathbb{V}$. Further, a triplet $(\mathbb{V}, \langle , \rangle, R)$ as above (or, an algebraic curvature tensor R on $\mathbb{V}$) is said to be Einstein if the corresponding Ricci tensor ϱ on $\mathbb{V}$ satisfies the identity $\varrho = \lambda \langle , \rangle$ for some $\lambda \in \mathbb{R}$.

Now, analogously to [9], one can prove the following algebraic version of Theorem 1:

Theorem 1.3. *Let $\mathbb{V}$ be a 4-dimensional vector space provided with a positive scalar product $\langle , \rangle$. Let R be an Einstein algebraic curvature tensor on $\mathbb{V}$. Then there is an orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ such that the nontrivial components of R with respect to $\mathcal{B}$ are, up to standard symmetries and antisymmetries, the following:*

$$(1.1) \qquad R_{1212} = R_{3434} = A, \quad R_{1313} = R_{2424} = B, \quad R_{1414} = R_{2323} = C,$$
$$R_{1234} = F, \quad R_{1423} = G, \quad R_{1342} = H,$$

where A, B, C, F, G, H are some constants which satisfy $F + G + H = 0$. On the other hand, all components R_{ijkl} with exactly three distinct indices are equal to zero.

Definition 1.4. An orthonormal basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ of $\mathbb{V}$ with the properties given above is called an ST basis on $\mathbb{V}$ corresponding to the curvature tensor R.

Definition 1.5. Let $(\mathbb{V}, \langle , \rangle, R)$ be an Einstein triplet. Then $\mathbb{V}$ is called 2-stein if it satisfies the following additional condition:

$$\mathcal{F}(X) = \sum_{i,j=1}^{n} (R(X, e_i, X, e_j))^2,$$

where $\mathcal{B} = \{e_1, \dots, e_n\}$ is any orthonormal basis, is independent of the choice of the unit vector $X \in \mathbb{V}$. (Cf. [1].)

Then, we have the following (cf. Lemma 7 in [7]):

Proposition 1.6. *An Einstein triplet* $(\mathbb{V}, \langle , \rangle, R)$ *of dimension* 4 *is* 2*-stein if and only if*

$$\pm F = A - \frac{\tau}{12}, \quad \pm H = B - \frac{\tau}{12}, \quad \pm G = C - \frac{\tau}{12} \tag{1.2}$$

holds with respect to any ST basis of $\mathbb{V}$*. Here* $\tau = \sum_{i=1}^{n} \varrho(e_i, e_i)$.

Now, let $\{J_1, J_2, J_3\}$ be a quaternionic structure on $(\mathbb{V}, \langle , \rangle)$ compatible with a fixed orientation defined by

$$\begin{aligned} J_1 X &= -x_2 e_1 + x_1 e_2 - x_4 e_3 + x_3 e_4, \\ J_2 X &= -x_3 e_1 + x_4 e_2 + x_1 e_3 - x_2 e_4, \\ J_3 X &= -x_4 e_1 - x_3 e_2 + x_2 e_3 + x_1 e_4 \end{aligned} \tag{1.3}$$

for any $X = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \in \mathbb{V}$, where $\{e_1, \dots, e_4\}$ is an ST basis compatible with the given orientation. Then, the following fact is also well known ([6], [8]):

Proposition 1.7. *Let* $(\mathbb{V}, \langle , \rangle, R)$ *be an Einstein triplet. Then the following two assertions are equivalent:*

(i) *For any quaternionic structure on* $\mathbb{V}$ *given by* (1.3) *and any unit vector* $X \in \mathbb{V}$*, the quadruplet* $\{X, J_1 X, J_2 X, J_3 X\}$ *is an ST basis for R;*
(ii) $(\mathbb{V}, \langle , \rangle, R)$ *is* 2*-stein.*

Motivated by this result and also by the research in the so-called weakly Einstein spaces (see [3], [4]), Sekigawa put the following, more general question: Let (M, g) be a 4-dimensional Einstein manifold, not necessarily 2-stein, and let $\{e_1, \dots, e_4\}$ be an arbitrary fixed ST basis at any point $p \in M$. Determine the relation between all ST bases $\{\bar{e}_1, \dots, \bar{e}_4\}$ at p and the fixed ST basis $\{e_1, \dots, e_4\}$.

In [2], the present author and Kowalski studied the transformation group between ST bases for the *family of all* Einstein curvature tensors in a given ST basis. The method was just the technical analysis of the additional conditions which must be satisfied by the matrix from O(4) to satisfy the properties mentioned. The result was a discrete matrix group and we will recall this result in the next section after the preliminaries.

In the present work, we study ST bases using another approach. We first determine the candidates of 1-parameter Lie subgroups of SO(4) for transformations between

ST bases for some classes of algebraic Einstein curvature tensors. After finding the 6 candidates for such 1-parameter groups, we prove that these groups and the groups generated by them are the unique Lie groups which transform some classes of algebraic Einstein curvature tensors into new ST bases. The conditions for these special algebraic Einstein curvature tensors naturally appear during the process.

In the last section, we return to the question by Sekigawa. It appears that the desired transformations essentially depend on the properties of the chosen Einstein curvature tensor R. Hence, the original question cannot be studied directly in general, but separately for various possible special forms of the tensor R. A surprising fact is that these special forms may, for a given tensor R, be different in different ST bases. To illustrate this phenomenon, we combine the new results with the result from [2]. To answer completely the question of Sekigawa, it remains to determine all possible discrete groups of transformations, possibly for special Einstein curvature tensors.

2. Algebraic preliminaries and the basic finite group

We first notice that the relation between two ST bases is characterized by an orthogonal transformation (i.e., by an orthogonal matrix). Let $P = (p^i_j) \in \mathrm{SO}(4)$ be the matrix of an orthogonal transformation acting on the set of orthonormal bases of $(\mathbb{V}, \langle , \rangle)$ in the natural way. Hence, if $\mathcal{B} = \{e_i\}_{i=1}^4$ is an orthonormal basis, the new orthonormal basis $\mathcal{B}P = \mathcal{B}' = \{e'_j\}_{j=1}^4$ is given as $e'_j = \sum\limits_{i=1}^4 e_i p^i_j$. If $\mathcal{G}$ is a group or a set of matrices, we will also denote by $\mathcal{BG}$ the set of all bases $\mathcal{B}P$ for $P \in \mathcal{G}$. Let us denote by P^{ij}_{kl} the 2×2 submatrix of the matrix P formed by the elements in the rows i, j and in the columns k, l. Let us denote by d^{ij}_{kl} its determinant. We now recall algebraic results derived in [2].

Lemma 2.1 ([2]). *Let $\mathcal{B}$ be an ST basis for an Einstein algebraic curvature tensor R in which the components of R are given by formula* (1.1). *Then the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P$ are given by the formula*

$$
\begin{aligned}
R'_{ijkl} &= (d^{12}_{ij} \cdot d^{12}_{kl} + d^{34}_{ij} \cdot d^{34}_{kl})A + (d^{13}_{ij} \cdot d^{13}_{kl} + d^{24}_{ij} \cdot d^{24}_{kl})B \\
&\quad + (d^{14}_{ij} \cdot d^{14}_{kl} + d^{23}_{ij} \cdot d^{23}_{kl})C + (d^{12}_{ij} \cdot d^{34}_{kl} + d^{34}_{ij} \cdot d^{12}_{kl})F \\
&\quad + (d^{14}_{ij} \cdot d^{23}_{kl} + d^{23}_{ij} \cdot d^{14}_{kl})G + (d^{13}_{ij} \cdot d^{42}_{kl} + d^{42}_{ij} \cdot d^{13}_{kl})H.
\end{aligned} \tag{2.1}
$$

P r o o f. It follows by the straightforward check using formulas

$$R'_{ijkl} = R(e'_i, e'_j, e'_k, e'_l),$$

where the components of the vector e'_i are p^u_i (the i-th column of the given matrix P). $\square$

Definition 2.2. Given a matrix $P \in \mathrm{SO}(4)$, we define, for each admissible set of indices $\{i,j,k,l\}$, the quantities $\widetilde{A}_{ijkl}$, $\widetilde{B}_{ijkl}$, $\widetilde{C}_{ijkl}$, $\widetilde{F}_{ijkl}$, $\widetilde{G}_{ijkl}$, $\widetilde{H}_{ijkl}$ as the coefficients in formula (2.1). This formula becomes

$$R'_{ijkl} = \widetilde{A}_{ijkl}A + \widetilde{B}_{ijkl}B + \widetilde{C}_{ijkl}C + \widetilde{F}_{ijkl}F + \widetilde{G}_{ijkl}G + \widetilde{H}_{ijkl}H. \tag{2.2}$$

Lemma 2.3 ([2]). *Let $\mathcal{B}$ be an ST basis for an Einstein algebraic curvature tensor R. For any matrix $P \in \mathrm{SO}(4)$, the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P$ satisfy*

$$R'_{1212} = R'_{3434}, \quad R'_{1313} = R'_{2424}, \quad R'_{1414} = R'_{2323}.$$

Let $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$ be an ST basis for an Einstein algebraic curvature tensor R on $(V, \langle,\rangle)$. According to Lemma 2.3, we are interested in transformations $P \in \mathrm{SO}(4)$ such that the tensor R in the new bases $\mathcal{B}' = \mathcal{B}P$ have all components with just three different indices equal to zero, namely $R'_{ijkl} = 0$ for the following 12 choices of indices $ijkl$:

$$\begin{array}{cccccc} 1213, & 1214, & 1223, & 1224, & 1314, & 1323, \\ 1334, & 1424, & 1434, & 2324, & 2334, & 2434. \end{array} \tag{2.3}$$

Equivalently, all the bases $\mathcal{B}' = \mathcal{B}P$ should be new ST bases for the tensor R.

We now recall the basic finite group of transformations, determined in [2], which can be applied to an ST basis of *any* Einstein algebraic curvature tensor R, and we obtain a new ST basis for this tensor. Let us denote by $\mathcal{H}_1 \subset \mathrm{SO}(4)$ the group of all permutation matrices (i.e., the matrices corresponding to permutations of the vectors e_1, e_2, e_3, e_4) and by $\mathcal{H}_2 \subset \mathrm{SO}(4)$ the group of all diagonal matrices with ± 1 on the diagonal. Obviously, $|\mathcal{H}_1| = 24$ and $|\mathcal{H}_2| = 16$. We further denote $\mathcal{H}_3 = \mathcal{H}_1 \cdot \mathcal{H}_2 = \mathcal{H}_2 \cdot \mathcal{H}_1$. We easily see that $\mathcal{H}_3$ is a group and $|\mathcal{H}_3| = 16 \cdot 24 = 384$. It is not hard to verify that, for all $P \in \mathcal{H}_3$, $\mathcal{B}P$ are ST bases for R. Let us further consider two special transformations given by the matrices

$$P_4 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad P_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Direct calculation using formula (2.1) shows that the components of the tensor R in the basis $\mathcal{B}' = \mathcal{B}P_4$ are

$$\begin{aligned} A' &= R'_{1212} = R'_{3434} = \frac{1}{2}(B + C - G + H), \\ B' &= R'_{1313} = R'_{2424} = \frac{1}{2}(A + C - F + G), \end{aligned} \tag{2.4}$$

$$C' = R'_{1414} = R'_{2323} = \frac{1}{2}(A+B+F-H), \quad F' = R'_{1234} = \frac{1}{2}(C-B+F),$$
$$G' = R'_{1423} = \frac{1}{2}(B-A+G), \quad H' = R'_{1342} = \frac{1}{2}(A-C+H)$$

and $R'_{ijkl} = 0$ for all choices of $ijkl$ from formula (2.3). In the basis $\mathcal{B}' = \mathcal{B}P_5$, the components of the tensor R are

$$(2.5) \quad A' = R'_{1212} = R'_{3434} = A, \quad B' = R'_{1313} = R'_{2424} = \frac{1}{2}(B+C+G-H),$$
$$C' = R'_{1414} = R'_{2323} = \frac{1}{2}(B+C+H-G), \quad F' = R'_{1234} = F,$$
$$G' = R'_{1423} = \frac{1}{2}(B-C-F), \quad H' = R'_{1342} = \frac{1}{2}(C-B-F)$$

and $R'_{ijkl} = 0$ for all choices of $ijkl$ from formula (2.3). We see that both $\mathcal{B}P_4$ and $\mathcal{B}P_5$ are ST bases for R. Let us denote by P'_4 the transformation whose first three columns are the same as those of the transformation P_4 and the last column has the opposite sign. Obviously, $P'_4 \in P_4\mathcal{H}_2$ and $\mathcal{B}P'_4$ is also an ST basis for R. Now we recall the main results from [2].

Lemma 2.4 ([2]). *The group $\mathcal{H}_4$ generated by $\mathcal{H}_3$ and P_4 is the union of cosets $\mathcal{H}_3 \cup \mathcal{H}_3P_4 \cup \mathcal{H}_3P'_4$. The group $\mathcal{H}_5$ generated by $\mathcal{H}_4$ and P_5 is the union of cosets $\mathcal{H}_4 \cup \mathcal{H}_4P_5$.*

Theorem 2.5 ([2]). *Let $\mathcal{B}$ be a fixed basis. Let us consider the 5-parameter family of all Einstein algebraic curvature tensors (given by an arbitrary choice of the parameters $A, \dots, H$) for which this basis is an ST basis. The group which transforms all the tensors R from this family into new ST bases is just the group $\mathcal{H}_5$.*

This statement can be obviously reformulated in the following way, which will be useful later: Let $\mathcal{B}$ be an ST basis for an arbitrary given Einstein algebraic curvature tensor R. For any transformation $P \in \mathcal{H}_5$, the new basis $\mathcal{B}P$ is also an ST basis for R.

3. Special Lie groups and corresponding invariant tensors

We consider the matrix

$$(3.1) \qquad X = \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ -s_1 & 0 & s_4 & s_5 \\ -s_2 & -s_4 & 0 & s_6 \\ -s_3 & -s_5 & -s_6 & 0 \end{pmatrix} \in \mathfrak{so}(4)$$

for some numbers $s_1, \dots, s_6 \in \mathbb{R}$, and the corresponding 1-parameter group $P(t) = \exp(tX)$. We consider further a fixed ST basis $\mathcal{B}$ for a given Einstein algebraic curvature tensor R and we want to determine necessary conditions for the new bases $\mathcal{B}P(t)$ to be ST bases. We use the approximation of the 1-parameter group $P(t) = \exp(tX)$ by the Taylor polynomial of the first order, hence $P(t) = E + tX + o(t^2)$. We calculate the components of the tensor R in the new bases $\mathcal{B}P(t)$. Using formula (2.1) and the matrix $P(t)$, we obtain in particular

$$
\begin{aligned}
R_{1214} &= (s_2(G - F) + s_5(A - C))t + o(t^2),\\
R_{1223} &= (s_2(C - A) + s_5(F - G))t + o(t^2),\\
R_{1224} &= (s_3(B - A) + s_4(H - F))t + o(t^2),\\
R_{1312} &= (s_3(F - H) + s_4(A - B))t + o(t^2),\\
R_{1314} &= (s_1(H - G) + s_6(B - C))t + o(t^2),\\
R_{1323} &= (s_1(B - C) + s_6(H - G))t + o(t^2).
\end{aligned}
$$

We see that necessary conditions under which a 1-parameter group transforms the ST basis $\mathcal{B}$ into new ST bases can be written as

$$
\begin{aligned}
&(s_5 - s_2)(A - C + F - G) = 0, \quad (s_5 + s_2)(A - C - F + G) = 0,\\
&(s_4 - s_3)(A - B + H - F) = 0, \quad (s_4 + s_3)(A - B - H + F) = 0,\\
&(s_6 + s_1)(B - C + H - G) = 0, \quad (s_6 - s_1)(B - C - H + G) = 0.
\end{aligned} \tag{3.2}
$$

Definition 3.1. Let us denote by G_1, G_2, G_3, H_1, H_2, H_3 the 1-parameter subgroups of the matrix group SO(4), each of them defined as $\exp(tX)$ for $X \in \mathfrak{so}(4)$ from formula (3.1) with just two nonzero parameters s_i satisfying the corresponding condition in the second column of the table (3.3) below and with other four parameters s_i equal to zero.

Proposition 3.2. *Let $\mathcal{B}$ be an ST basis for an Einstein algebraic curvature tensor R and let $\mathcal{G}$ be some of the 1-parameter matrix groups $G_1, \dots, H_3$. If all the new bases $\mathcal{B}\mathcal{G}$ are ST bases, the tensor R must satisfy the corresponding condition in the following table,*

$$
\begin{aligned}
G_1\colon &\quad s_1 + s_6 = 0, \quad B - C - H + G = 0,\\
G_2\colon &\quad s_3 + s_4 = 0, \quad A - B + H - F = 0,\\
G_3\colon &\quad s_2 - s_5 = 0, \quad A - C - F + G = 0,\\
H_1\colon &\quad s_1 - s_6 = 0, \quad B - C + H - G = 0,\\
H_2\colon &\quad s_3 - s_4 = 0, \quad A - B - H + F = 0,\\
H_3\colon &\quad s_2 + s_5 = 0, \quad A - C + F - G = 0.
\end{aligned} \tag{3.3}
$$

P r o o f. For each choice of the group $\mathcal{G}$, there are four parameters s_i equal to zero and hence four of the conditions (3.2) are satisfied identically. One condition in (3.2) is satisfied due to the choice of the two nonzero parameters s_i in the definition of $\mathcal{G}$ and the last condition must be satisfied by the components of the tensor R. □

It can be easily seen that for other 1-parameter groups, the curvature tensor R must satisfy at least two of the above conditions and we will see that these groups are included in a bigger group of transformations generated by these special 1-parameter groups. We now formulate basic algebraic facts about these special groups.

Proposition 3.3. *Each of the matrix groups G_i, H_j is a faithful representation of* $\mathrm{SO}(2)$ *in* $\mathrm{SO}(4)$. *Each of the groups G_i commutes with each of the groups H_j. For fixed i and j, the groups G_i and H_j generate a faithful representation of the torus T^2 in* $\mathrm{SO}(4)$. *All the groups G_i generate a faithful representation of the group* $\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4)$*; for simplicity, we will denote it by* $\overline{\mathrm{Sp}}(1)$. *All the groups H_i generate another faithful representation of the group* $\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4)$*; for simplicity, we will denote it by* $\widetilde{\mathrm{Sp}}(1)$. *The matrix groups* $\overline{\mathrm{Sp}}(1)$ *and* $\widetilde{\mathrm{Sp}}(1)$ *commute.*

P r o o f. The first statement is obvious, the second statement follows either from a direct calculation or it can be verified on the Lie algebra level. It follows immediately that each fixed G_i with a fixed H_j generate a torus. On the Lie algebra level, we can also verify that the generators of the groups G_i form a 3-dimensional Lie subalgebra of $\mathfrak{so}(4)$, and it is isomorphic to $\mathfrak{so}(3) \simeq \mathfrak{su}(2) \simeq \mathfrak{sp}(1)$. The second algebra isomophic to $\mathfrak{sp}(1)$ is generated by the generators of the groups H_i. □

For the reader's convenience, we write down the elements $g \in \overline{\mathrm{Sp}}(1)$, $h \in \widetilde{\mathrm{Sp}}(1)$ and $h' \in H_1$ explicitly:

$$g = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}, \quad h = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}, \tag{3.4}$$

$$h' = \begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix},$$

for any $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 + c^2 + d^2 = 1$ and for any $t \in \mathbb{R}$.

Proposition 3.4. *Each of the matrix groups G_i is conjugate to any other group G_j via an even permutational matrix from $\mathcal{H}_1$. Each of the groups G_i is conjugate to any of the matrix groups H_i via an odd permutational matrix from $\mathcal{H}_1$. The matrix groups $\overline{\mathrm{Sp}}(1)$ and $\widetilde{\mathrm{Sp}}(1)$ are conjugate via an odd permutational matrix.*

Proof. The statement for the groups G_i or H_i, respectively, is easy and the statement for the groups $\overline{\mathrm{Sp}}(1)$ and $\widetilde{\mathrm{Sp}}(1)$ follows by applying the transposition $t = (34)$ to the rows and to the columns of matrices in formula (3.4). □

In the rest of this section, we are going to derive the geometrical consequences of formulas (3.3). We start with a technical, but crucial calculation.

Proposition 3.5. *Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$ and let $\mathcal{G} = H_1$. Each of the new bases $\mathcal{B}\mathcal{G}$ is an ST basis for the tensor R if and only if it satisfies*

$$B - C + H - G = 0. \tag{3.5}$$

If the condition (3.5) is satisfied, then in any new bases $\mathcal{B}\mathcal{G}$ the new components A', B', C', F', H', G' of the tensor R are the same as the original components A, B, C, F, H, G.

Proof. We have seen in equations (3.2) that the condition (3.5) is necessary, now we will prove the sufficiency. Let the tensor R satisfy (3.5) and let $P = h' \in H_1$. By straightforward calculations with the matrix P we obtain

$$d_{12}^{12} = d_{34}^{34} = 1 \tag{3.6}$$

and for any pair ij different from 12 or 34, we have

$$d_{34}^{12} = d_{12}^{34} = d_{ij}^{12} = d_{ij}^{34} = d_{12}^{ij} = d_{34}^{ij} = 0. \tag{3.7}$$

Further, we obtain

$$\begin{aligned} d_{13}^{13} = d_{14}^{14} = d_{23}^{23} = d_{24}^{24} &= \cos^2(t), \\ d_{24}^{13} = d_{13}^{24} = -d_{23}^{14} = -d_{14}^{23} &= \sin^2(t), \\ d_{14}^{13} = d_{23}^{13} = -d_{14}^{24} = -d_{23}^{24} &= \sin(t)\cos(t), \\ -d_{13}^{14} = d_{24}^{14} = -d_{13}^{23} = d_{24}^{23} &= \sin(t)\cos(t). \end{aligned} \tag{3.8}$$

Let now just three indices from $ijkl$ be distinct. We are going to determine the coefficients in formula (2.2). We obtain easily from formulas (3.6) and (3.7), using (2.1) and (2.2), that

$$\widetilde{A}_{ijkl} = \widetilde{F}_{ijkl} = 0.$$

Further, if $ij = 12$ or $kl = 34$ (in 8 of the 12 choices in formula (2.3)), we obtain from formulas (3.6)–(3.8), that

$$\widetilde{B}_{ijkl} = \widetilde{C}_{ijkl} = \widetilde{H}_{ijkl} = \widetilde{G}_{ijkl} = 0.$$

And finally, in the last four cases for $ijkl$, namely $ijkl = 1314$, 1323, 1424, 2324, we obtain

$$\begin{aligned}
\widetilde{B}_{1314} = -\widetilde{C}_{1314} = \widetilde{H}_{1314} = -\widetilde{G}_{1314} &= (\cos^2(t) - \sin^2(t))\sin(t)\cos(t),\\
\widetilde{B}_{1323} = -\widetilde{C}_{1323} = \widetilde{H}_{1323} = -\widetilde{G}_{1323} &= (\cos^2(t) - \sin^2(t))\sin(t)\cos(t),\\
\widetilde{B}_{1424} = -\widetilde{C}_{1424} = \widetilde{H}_{1424} = -\widetilde{G}_{1424} &= (\sin^2(t) - \cos^2(t))\sin(t)\cos(t),\\
\widetilde{B}_{2324} = -\widetilde{C}_{2324} = \widetilde{H}_{2324} = -\widetilde{G}_{2324} &= (\sin^2(t) - \cos^2(t))\sin(t)\cos(t).
\end{aligned}$$

We see that in each of the 12 choices from (2.3), formula (2.2) simplifies to

$$R'_{ijkl} = \widetilde{B}_{ijkl}(B - C + H - G) = 0,$$

which implies that the new basis is an ST basis.

And finally, we determine the nonzero components of the tensor R in the new basis. First, let $ijkl = 1212$ or $ijkl = 1234$. Again from the formulas (3.6) and (3.7), we obtain

$$\begin{gathered}
\widetilde{A}_{1212} = \widetilde{F}_{1234} = 1, \quad \widetilde{B}_{1212} = \widetilde{C}_{1212} = \widetilde{F}_{1212} = \widetilde{H}_{1212} = \widetilde{G}_{1212} = 0,\\
\widetilde{A}_{1234} = \widetilde{B}_{1234} = \widetilde{C}_{1234} = \widetilde{H}_{1234} = \widetilde{G}_{1234} = 0.
\end{gathered}$$

Hence we have

$$A' = R'_{1212} = A, \quad F' = R'_{1234} = F.$$

Let now $ijkl$ be one of the choices 1313, 1414, 1342 or 1423. We calculate using formulas (3.6)–(3.8):

$$\begin{aligned}
&\widetilde{A}_{1313} = \widetilde{A}_{1414} = \widetilde{A}_{1342} = \widetilde{A}_{1423} = 0,\\
&\widetilde{F}_{1313} = \widetilde{F}_{1414} = \widetilde{F}_{1324} = \widetilde{F}_{1423} = 0,\\
&\widetilde{B}_{1313} = \widetilde{C}_{1414} = \widetilde{H}_{1342} = \widetilde{G}_{1423} = \cos^4(t) + \sin^4(t)\\
&\qquad = (\sin^2(t) + \cos^2(t))^2 - 2\cos^2(t)\sin^2(t) = 1 - 2\cos^2(t)\sin^2(t),\\
&\widetilde{C}_{1313} = \widetilde{B}_{1414} = \widetilde{G}_{1342} = \widetilde{H}_{1423} = 2\cos^2(t)\sin^2(t),\\
&\widetilde{H}_{1313} = \widetilde{G}_{1414} = \widetilde{B}_{1342} = \widetilde{C}_{1423} = -2\cos^2(t)\sin^2(t),\\
&\widetilde{G}_{1313} = \widetilde{H}_{1414} = \widetilde{C}_{1342} = \widetilde{B}_{1423} = 2\cos^2(t)\sin^2(t).
\end{aligned}$$

We see, using (2.2), that the remaining new components are

$$\begin{aligned} B' &= R'_{1313} = B - 2\cos^2(t)\sin^2(t)(B - C + H - G) = B,\\ C' &= R'_{1414} = C + 2\cos^2(t)\sin^2(t)(B - C + H - G) = C,\\ H' &= R'_{1342} = H - 2\cos^2(t)\sin^2(t)(B - C + H - G) = H,\\ G' &= R'_{1423} = G + 2\cos^2(t)\sin^2(t)(B - C + H - G) = G. \end{aligned}$$

□

Proposition 3.6. *Let $\mathcal{G} = H_1$ be the special representation of the group* SO(2) *described above and let R be any tensor from the family satisfying the corresponding homogeneous linear condition* (3.5) *in a given ST basis $\mathcal{B}$. Let $p \in \mathcal{H}_5$ and $\mathcal{G}' = p^{-1}\mathcal{G}p$. The matrix group $\mathcal{G}'$ is also one of the six special representations $G_1, \ldots, H_3$ of the group* SO(2) *and it transforms ST basis $\mathcal{B}p$ for the tensor R into new ST bases. In all these bases $\mathcal{B}p\mathcal{G}'$, the tensor R satisfies the corresponding condition* (3.3) *for the group $\mathcal{G}'$.*

Proof. The group $\mathcal{G}$ transforms the corresponding family of special tensors R satisfying the given condition from an ST basis $\mathcal{B}$ into new ST bases $\mathcal{B}\mathcal{G}$. Further, all the bases $\mathcal{B}\mathcal{G}p$ are also ST bases for all considered tensors R and these bases can be written as $\mathcal{B}\mathcal{G}p = \mathcal{B}pp^{-1}\mathcal{G}p = \mathcal{B}p\mathcal{G}'$. One can view the latter as transformations of the ST basis $\mathcal{B}p$ for the same family of tensors R by the group $\mathcal{G}'$, which is also a faithful representation of SO(2). Conditions (3.3) describe the unique candidates for representations of SO(2) which transform a given ST basis into new ST bases for the whole family of algebraic Einstein curvature tensors R satisfying one homogeneous linear condition (and depending on 4 parameters). We see that $\mathcal{G}'$ must be also one of these representations. The last statement is a direct corollary. □

As an example, we show that both statements can be checked also directly for any of the conditions (3.3) and any generator of the group $\mathcal{H}_5$. Let $\mathcal{G} = H_1$ and let the tensor R in an ST basis $\mathcal{B}$ satisfy

$$B - C + H - G = 0.$$

For example, we use for p the transposition $p = (23)$ on the basis $\mathcal{B}$. The tensor R changes by the equations

$$A' = B, \quad B' = A, \quad C' = C, \quad F' = -H, \quad H' = -F, \quad G' = -G.$$

We see that the tensor R in the new basis $\mathcal{B}p$ satisfies the equation

$$A' - C' - F' + G' = 0,$$

which corresponds to the group $\mathcal{G}' = G_3$. For the generators $p = P_4$ or $p = P_5$, the procedure is similar, using formulas (2.4) and (2.5).

Theorem 3.7. *Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$. The group* SO(2) *acts as a transformation group between ST bases if and only if the tensor R satisfies at least one of the following equations, each equation corresponding to a particular representation of the group* SO(2)*:*

$$
\begin{aligned}
(3.9)\qquad H_1\colon\quad & B - C + H - G = 0, & G_1\colon\quad & B - C - H + G = 0,\\
H_2\colon\quad & A - B - H + F = 0, & G_2\colon\quad & A - B + H - F = 0,\\
H_3\colon\quad & A - C + F - G = 0, & G_3\colon\quad & A - C - F + G = 0.
\end{aligned}
$$

P r o o f. The results for the group H_1 were obtained in Proposition 3.5. The analogous results for the other mentioned groups follow easily by using Propositions 3.4 and 3.6 and conjugation of the mentioned groups. □

Definition 3.8. We will call the three equations on the left in formulas (3.9) the conditions of the type $\widetilde{\mathrm{Sp}}(1)$ and the three equations on the right the conditions of the type $\overline{\mathrm{Sp}}(1)$.

Theorem 3.9. *Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$. The torus group $T^2 = \mathrm{SO}(2) \times \mathrm{SO}(2)$ acts as a transformation group between ST bases if and only if the tensor R satisfies at least one of the equations of the type $\overline{\mathrm{Sp}}(1)$ and at least one of the equations of the type $\widetilde{\mathrm{Sp}}(1)$. The group T^2 can be represented with respect to the basis $\mathcal{B}$ in one of the nine possible ways.*

P r o o f. Obviously, we obtain a representation $\mathcal{G}$ of the torus T^2 as $G_i \times H_j$ for each fixed choice of i and j. If all the new bases $\mathcal{B}T^2$ are ST bases, necessary conditions for the tensor R are in formulas (3.3). On the other hand, if the two mentioned equations are satisfied, all the new bases $\mathcal{B}\mathcal{G}$ are ST bases, as a corollary of Proposition 3.5 and Theorem 3.7. Any considered representation $\mathcal{G}$ of the group T^2 also preserves the values of all components of the tensor R. In particular, it preserves both the corresponding equations for the components of the tensor R. □

We notice that each of the triplets of equations in formulas (3.9) are dependent, because any two of the groups G_i, or H_i, generate a representation of the group Sp(1) in SO(4). We also remark that the Einstein curvature tensor satisfying all conditions of the type G_i, or H_i, is a 2-stein curvature tensor (see formulas (1.2)). ST bases of 2-stein curvature tensors were studied in [8], see Lemma 5 there. The following theorem is a reformulation and we put it here for the completeness of the exposition. Because it is based on Proposition 3.5, it is useful also as an alternative proof of Lemma 5 in [8].

Theorem 3.10. *Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$. The group* $\mathrm{Sp}(1)$ *acts as a transformation group between ST bases if and only if the tensor R satisfies the equations of type* $\overline{\mathrm{Sp}}(1)$ *or the equations of type* $\widetilde{\mathrm{Sp}}(1)$. *The group* $\mathrm{Sp}(1)$ *can be represented with respect to the basis $\mathcal{B}$ in one of two possible ways.*

P r o o f. Obviously, we have two choices for a representation $\mathcal{G}$ of the group $\mathrm{Sp}(1)$ in $\mathrm{SO}(4)$, either the one generated by G_i or the one generated by H_i. For each choice, if all the new bases $\mathcal{BG}$ are ST bases, necessary conditions for the tensor R are in formulas (3.3) and are just the conditions of the corresponding type. The rest of the proof is the same as for Theorem 3.9. □

Theorem 3.11. *Let R be an Einstein curvature tensor in an ST basis $\mathcal{B}$. The group* $\mathrm{U}(2) \simeq \mathrm{Sp}(1) \times \mathrm{SO}(2)$ *acts as a transformation group between ST bases if and only if the tensor R satisfies either the equations of type* $\overline{\mathrm{Sp}}(1)$ *and at least one of the equations of type* $\widetilde{\mathrm{Sp}}(1)$ *or the equations of type* $\widetilde{\mathrm{Sp}}(1)$ *and at least one of the equations of type* $\overline{\mathrm{Sp}}(1)$. *The group* $\mathrm{U}(2)$ *can be represented with respect to the basis $\mathcal{B}$ in one of six possible ways.*

P r o o f. Obviously, with each of the two choices for the representation of the group $\mathrm{Sp}(1)$, we have three choices for a representation of the group $\mathrm{SO}(2)$. In each case, we obtain a representation $\mathcal{G}$ of the group $\mathrm{U}(2)$. For each choice, if all the new bases $\mathcal{BG}$ are ST bases, necessary conditions for the tensor R are in formulas (3.3). The rest of the proof is the same as for Theorem 3.9. □

For completeness, we end this paragraph by the observation that the conditions of type $\mathcal{G}$ together with the conditions of type $\mathcal{H}$ imply conditions

$$A = B = C, \quad F = G = H = 0$$

which hold in a space of constant curvature. The group of transformations between ST bases is the full group $\mathrm{O}(4)$, which is a well known fact and in the context of the present section it follows easily from the previous theorems.

4. The set of all ST bases for a fixed tensor R

Now we try to give at least a partial answer to the original question by Sekigawa and approach the description of the set of all ST bases for a given Einstein algebraic curvature tensor R.

In the previous section we have seen that there are special tensors, which admit the particular representation $\mathcal{G}$ of one of the groups SO(2), T^2, Sp(1) or U(2) as a group of transformations from a given ST basis into another ST basis. At the same time, in each case, these transformations preserve the components of the tensor R. On the other hand, to each new ST basis we can apply any transformation $h \in \mathcal{H}_5$ and obtain another ST basis in which the tensor R may have components different from those in the original ST basis.

With respect to this new basis, the matrix Lie group $\mathcal{G}'$ of transformations is $\mathcal{G}' = h^{-1}\mathcal{G}h$ and it is isomorphic to $\mathcal{G}$ (see Proposition 3.6).

We can conclude that for each tensor R and a fixed ST basis $\mathcal{B}$, transformations of the type $p = gh$, for $g \in \mathcal{G},\ h \in \mathcal{H}_5$, transform the given ST basis $\mathcal{B}$ into a new ST basis. One can notice that for each of the considered representations $\mathcal{G}$, there is an element $g_5 \in \mathcal{G}$ which is the matrix P_5 with possibly rearranged rows and columns. Further, any such element g_5 satisfies $\mathcal{H}_5 = \mathcal{H}_4 \cup g_5\mathcal{H}_4$. We see that the above transformations $p = gh$ can be simplified to

$$p = gh, \quad g \in \mathcal{G}, \quad h \in \mathcal{H}_4.$$

It can be also observed that these transformations can be composed in the following way: Let $p = gh$ ($g \in \mathcal{G}$, $h \in \mathcal{H}_4$) be a transformation from an ST basis $\mathcal{B}$ into a new ST basis $\mathcal{B}'$ (written with respect to the basis $\mathcal{B}$) and let $p' = g'h'$ ($g' \in \mathcal{G}'$, $h' \in \mathcal{H}_4$) be a transformation from an ST basis $\mathcal{B}'$ into a new ST basis $\mathcal{B}''$ (written with respect to the basis $\mathcal{B}'$). Then $g' = h^{-1}\overline{g}h$ for some $\overline{g} \in \mathcal{G}$ and the transformation $p \circ p'$ from the ST basis $\mathcal{B}$ into the ST basis $\mathcal{B}''$ (written with respect to the basis $\mathcal{B}$) is

$$p \circ p' = (gh) \circ (g'h') = ghh^{-1}\overline{g}hh' = g\overline{g}hh'.$$

An open question remains whether the ST bases obtained in this way from one given ST basis $\mathcal{B}$ are all possible ST bases for the given tensor R. The discrete group $\mathcal{H}_5$ was found as a transformation which can be applied to an ST basis of *arbitrary* tensor R and obtain a new ST basis. It is not disproved that for special tensors R there may exist more *discrete* transformations which can be taken for h above.

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