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Partial Fuzzy Metric Space and Some Fixed Point Results

Shaban Sedghi, Nabi Shobkolaei, Ishak Altun

Abstract. In this paper, we introduce the concept of partial fuzzy metric on a nonempty set $X$ and give the topological structure and some properties of partial fuzzy metric space. Then some fixed point results are provided.

1 Introduction and preliminaries

We recall some basic definitions and results from the theory of fuzzy metric spaces, used in the sequel.

Definition 1. [5] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0, 1],$
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1].$

Two typical examples of continuous t-norms are $a * b = ab$ and $a * b = \min\{a, b\}.$

Definition 2. [1] A triple $(X, M, *)$ is called a fuzzy metric space (in the sense of George and Veeramani) if $X$ is a nonempty set, $*$ is a continuous t-norm and $M: X^2 \times (0, \infty) \to [0, 1]$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0,$

1. $M(x, y, t) > 0,$
2. $M(x, y, t) = 1 \iff x = y,$

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3. \( M(x, y, t) = M(y, x, t) \),

4. \( M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s) \),

5. \( M(x, y, \cdot) : (0, \infty) \to [0, 1] \) is a continuous mapping

If the fourth condition is replaced by

4’. \( M(x, z, \max\{t, s\}) \geq M(x, y, t) \star M(y, z, s) \),

then the space \((X, M, \ast)\) is said to be a non-Archimedean fuzzy metric space. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.

The following properties of \( M \) noted in the theorem below are easy consequences of the definition.

**Theorem 1.** Let \((X, M, \ast)\) be a fuzzy metric space.

1. \( M(x, y, t) \) is nondecreasing with respect to \( t \) for each \( x, y \in X \),

2. If \( M \) is non-Archimedean, then \( M(x, y, t) \geq M(x, z, t) \ast M(z, y, t) \) for all \( x, y, z \in X \) and \( t > 0 \).

**Example 1.** Let \((X, d)\) be an ordinary metric space and \( a \ast b = ab \) for all \( a, b \in [0, 1] \). Then the fuzzy set \( M \) on \( X^2 \times (0, \infty) \) defined by

\[
M(x, y, t) = \exp\left( -\frac{d(x, y)}{t} \right),
\]

is a fuzzy metric on \( X \).

**Example 2.** Let \( a \ast b = ab \) for all \( a, b \in [0, 1] \) and \( M \) be the fuzzy set on \( \mathbb{R}^+ \times \mathbb{R}^+ \times (0, \infty) \) (where \( \mathbb{R}^+ = (0, \infty) \)) defined by

\[
M(x, y, t) = \min\{x, y\} \max\{x, y\},
\]

for all \( x, y \in \mathbb{R}^+ \). Then \((\mathbb{R}^+, M, \ast)\) is a fuzzy metric space.

Let \((X, M, \ast)\) be a fuzzy metric space. For \( t > 0 \), the open ball \( B(x, r, t) \) with centre \( x \in X \) and radius \( 0 < r < 1 \) is defined by

\[
B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.
\]

Let \( \tau \) be the set of all \( A \subset X \) with \( x \in A \) if and only if there exist \( t > 0 \) and \( 0 < r < 1 \) such that \( B(x, r, t) \subset A \). Then \( \tau \) is a topology on \( X \) (induced by the fuzzy metric \( M \)). A sequence \( \{x_n\} \) in \( X \) converges to \( x \) if and only if \( M(x_n, x, t) \to 1 \) as \( n \to \infty \), for each \( t > 0 \). It is called a Cauchy sequence if for each \( 0 < \varepsilon < 1 \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \) for each \( n, m \geq n_0 \). This definition of Cauchy sequence is identical with that given by George and Veeramani.
The fuzzy metric space \((X, M, *)\) is said to be complete if every Cauchy sequence is convergent.

The fixed point theory in fuzzy metric spaces started with the paper of Grabieć [2]. Later on, the concept of fuzzy contractive mappings, initiated by Gregori and Sapena in [3], have become of interest for many authors, see, e.g., the papers [3], [7], [8], [9], [10], [11].

In our paper we present the concept of partial fuzzy metric space and some properties of it. Then we give some fundamental fixed point theorem on complete partial fuzzy metric space.

2 Partial fuzzy metric space

In this section we introduce the concept of partial fuzzy metric space and give its properties.

**Definition 3.** A partial fuzzy metric on a nonempty set \(X\) is a function

\[
P_M : X \times X \times (0, \infty) \rightarrow [0, 1]
\]

such that for all \(x, y, z \in X\) and \(t, s > 0\)

(PM1) \(x = y \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t)\),

(PM2) \(P_M(x, x, t) \geq P_M(x, y, t)\),

(PM3) \(P_M(x, y, t) = P_M(y, x, t)\),

(PM4) \(P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, t) * P_M(z, y, s)\).

(PM5) \(P_M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]\) is continuous.

A partial fuzzy metric space is a 3-tuple \((X, P_M, *)\) such that \(X\) is a nonempty set and \(P_M\) is a partial fuzzy metric on \(X\). It is clear that, if \(P_M(x, y, t) = 1\), then from (PM1) and (PM2) \(x = y\). But if \(x = y\), \(P_M(x, y, t)\) may not be 1. A basic example of a partial fuzzy metric space is the 3-tuple \((\mathbb{R}^+, P_M, *)\), where

\[
P_M(x, y, t) = \frac{t}{t + \max\{x, y\}}
\]

for all \(t > 0, x, y \in \mathbb{R}^+\) and \(a * b = ab\).

From (PM4) for all \(x, y, z \in X\) and \(t > 0\), we have:

\[
P_M(x, y, t) * P_M(z, z, t) \geq P_M(x, z, t) * P_M(z, y, t).
\]

Let \((X, M, *)\) and \((X, P_M, *)\) be a fuzzy metric space and partial fuzzy metric space, respectively. Then mappings \(P_M_i : X \times X \times (0, \infty) \rightarrow [0, 1]\) \((i \in \{1, 2\})\) defined by

\[
P_M_1(x, y, t) = M(x, y, t) * P_M(x, y, t)
\]

and

\[
P_M_2(x, y, t) = M(x, y, t) * a
\]

are partial fuzzy metrics on \(X\), where \(0 < a < 1\).
Theorem 2. The partial fuzzy metric $P_M(x, y, t)$ is nondecreasing with respect to $t$ for each $x, y \in X$ and $t > 0$, if the continuous $t$-norm $*$ satisfies the following condition for all $a, b, c \in [0, 1]$

$$a * b \geq a * c \Rightarrow b \geq c.$$ 

Proof. From (PM4) for all $x, y, z \in X$ and $t, s > 0$, we have:

$$P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, s) * P_M(z, y, t).$$

Let $t > s$, then taking $z = y$ in above inequality we have

$$P_M(x, y, t) * P_M(y, y, t) \geq P_M(x, y, s) * P_M(y, y, t),$$

hence by assume we get $P_M(x, y, t) \geq P_M(x, y, s)$. □

It is easy to see that every fuzzy metric is a partial fuzzy metric, but the converse may not be true. In the following examples, the partial fuzzy metrics fails to satisfy properties of fuzzy metric.

Example 3. Let $(X, p)$ is a partial metric space in the sense of Matthews [6] and $P_M : X \times X \times (0, \infty) \to [0, 1]$ be a mapping defined as

$$P_M(x, y, t) = \frac{t}{t + p(x, y)},$$

or

$$P_M(x, y, t) = \exp\left(-\frac{p(x, y)}{t}\right).$$

If $a * b = ab$ for all $a, b \in [0, 1]$, then clearly $P_M$ is a partial fuzzy metric, but it may not be a fuzzy metric.

Lemma 1. Let $(X, P_M, *)$ be a partial fuzzy metric space with $a * b = ab$ for all $a, b \in [0, 1]$. If we define $p : X^2 \to [0, \infty)$ by

$$p(x, y) = \sup_{\alpha \in (0, 1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t)) \, dt,$$

then $p$ is a partial metric on $X$ for fixed $0 < a < 1$.

Proof. It is clear from the definition that $p(x, y)$ is well defined for each $x, y \in X$ and $p(x, y) \geq 0$ for all $x, y \in X$.

1. For all $t > 0$

$$p(x, x) = p(x, y) = p(y, y) \iff P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t) \iff x = y.$$

2. 

$$p(x, x) = \sup_{\alpha \in (0, 1)} \int_{\alpha}^{1} \log_a(P_M(x, x, t)) \, dt$$

$$\leq \sup_{\alpha \in (0, 1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t)) \, dt$$

$$= p(x, y).$$
we have

\[ \lim \]

therefore

\[
\{ \log \}
\]

hence

Let

Example 4.

not unique.

Cauchy sequence. In particular, it shows that the limit of a convergent sequence is convergent sequence.

\[
\{ \text{we define } L \}
\]

This proves that \( p \) is a partial metric on \( X \).

\[ \square \]

Definition 4. Let \((X, P_M, \ast)\) be a partial fuzzy metric space.

1. A sequence \( \{x_n\} \) in a partial fuzzy metric space \((X, P_M, \ast)\) converges to \( x \) if and only if \( P_M(x, x, t) = \lim_{n \to \infty} P_M(x_n, x, t) \) for every \( t > 0 \).

2. A sequence \( \{x_n\} \) in a partial fuzzy metric space \((X, P_M, \ast)\) is called a Cauchy sequence if \( \lim_{n, m \to \infty} P_M(x_n, x_m, t) \) exists.

3. A partial fuzzy metric space \((X, P_M, \ast)\) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \( X \) converges to a point \( x \in X \).

Suppose that \( \{x_n\} \) is a sequence in partial fuzzy metric space \((X, P_M, \ast)\), then we define \( L(x_n) = \{x \in X : x_n \to x\} \). In the following example shows that every convergent sequence \( \{x_n\} \) in a partial fuzzy metric space \((X, P_M, \ast)\) fails to satisfy Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not unique.

Example 4. Let \( X = [0, \infty) \) and \( P_M(x, y, t) = \frac{t}{1 + \max(x, y)} \), then it is clear that \((X, P_M, \ast)\) is a partial fuzzy metric space where \( \ast \) \( b = ab \) for all \( a, b \in [0, 1] \). Let \( \{x_n\} = \{1, 2, 1, 2, \ldots\} \). Then clearly it is convergent sequence and for every \( x \geq 2 \) we have

\[
\lim_{n \to \infty} P_M(x_n, x, t) = P_M(x, x, t),
\]

therefore

\[
L(x_n) = \{x \in X : x_n \to x\} = [2, \infty).
\]

but \( \lim_{n, m \to \infty} P_M(x_n, x_m, t) \) is not exist, that is, \( \{x_n\} \) is not Cauchy sequence.
The following Lemma shows that under certain conditions the limit of a convergent sequence is unique.

**Lemma 2.** Let \( \{ x_n \} \) be a convergent sequence in partial fuzzy metric space \((X, P_M, \ast)\) such that \( a \ast b \geq a \ast c \Rightarrow b \geq c \) for all \( a, b, c \in [0, 1], \) \( x_n \to x \) and \( x_n \to y. \) If
\[
\lim_{n \to \infty} P_M(x_n, x_n, t) = P_M(x, x, t) = P_M(y, y, t),
\]
then \( x = y. \)

**Proof.** As
\[
P_M(x, y, t) \ast P_M(x_n, x_n, t) \geq P_M(x, x, t) \ast P_M(y, x_n, t),
\]
taking limit as \( n \to \infty, \) we have
\[
P_M(x, y, t) \ast P_M(x, x, t) \geq P_M(x, x, t) \ast P_M(y, y, t).
\]
By given assumptions and from (PM2), we have
\[
P_M(y, y, t) \geq P_M(x, y, t) \geq P_M(y, y, t),
\]
which shows that
\[
P_M(x, y, t) = P_M(y, y, t) = P_M(x, x, t),
\]
therefore \( x = y. \) \( \square \)

**Lemma 3.** Let \( \{ x_n \} \) and \( \{ y_n \} \) be two sequences in partial fuzzy metric space \((X, P_M, \ast)\) such that \( a \ast b \geq a \ast c \Rightarrow b \geq c \) for all \( a, b, c \in [0, 1], \)
\[
\lim_{n \to \infty} P_M(x_n, x, t) = \lim_{n \to \infty} P_M(x_n, x_n, t) = P_M(x, x, t),
\]
and
\[
\lim_{n \to \infty} P_M(y_n, y, t) = \lim_{n \to \infty} P_M(y_n, y_n, t) = P_M(y, y, t),
\]
then \( \lim_{n \to \infty} P_M(x_n, y_n, t) = P_M(x, y, t). \) In particular, for every \( z \in X \)
\[
\lim_{n \to \infty} P_M(x_n, z, t) = \lim_{n \to \infty} P_M(x, z, t).
\]

**Proof.** As
\[
P_M(x_n, y_n, t) \ast P_M(x, x, t) \geq P_M(x_n, x, t) \ast P_M(x, y_n, t),
\]
therefore
\[
P_M(x_n, y_n, t) \ast P_M(x, x, t) \ast P_M(y, y, t) \geq P_M(x_n, x, t) \ast P_M(x, y_n, t) \ast P_M(y, y, t)
\]
\[
\geq P_M(x_n, x, t) \ast P_M(x, y, t) \ast P_M(y, y, t).
\]
Thus
\[
\limsup_{n \to \infty} P_M(x_n, y_n, t) \ast P_M(x, x, t) \ast P_M(y, y, t)
\]
\[
\geq \limsup_{n \to \infty} P_M(x_n, x, t) \ast P_M(x, y, t) \ast \limsup_{n \to \infty} P_M(y, y_n, t)
\]
\[
= P_M(x, x, t) \ast P_M(x, y, t) \ast P_M(y, y, t),
\]
hence
\[ \limsup_{n \to \infty} P_M(x_n, y_n, t) \geq P_M(x, y, t). \]
Also, as
\[ P_M(x, y, t) \ast P_M(x_n, x_n, t) \geq P_M(x, x_n, t) \ast P_M(x_n, y, t), \]
therefore
\[ P_M(x, y, t) \ast P_M(x_n, x_n, t) \ast P_M(y_n, y_n, t) \]
\[ \geq P_M(x, x_n, t) \ast P_M(x_n, y, t) \ast P_M(y_n, y_n, t) \]
\[ \geq P_M(x, x_n, t) \ast P_M(x_n, y_n, t) \ast P_M(y, y, t). \]
Thus
\[ P_M(x, y, t) \ast P_M(x, x, t) \ast P_M(y, y, t) \]
\[ = P_M(x, y, t) \ast \limsup_{n \to \infty} P_M(x_n, x_n, t) \ast \limsup_{n \to \infty} P_M(y_n, y_n, t) \]
\[ \geq \limsup_{n \to \infty} P_M(x, x_n, t) \ast \limsup_{n \to \infty} P_M(x_n, y_n, t) \ast \limsup_{n \to \infty} P_M(y_n, y_n, t) \]
\[ = P_M(x, x, t) \ast \limsup_{n \to \infty} P_M(x_n, y_n, t) \ast P_M(y, y, t). \]
Therefore
\[ P_M(x, y, t) \geq \limsup_{n \to \infty} P_M(x_n, y_n, t). \]
That is,
\[ \limsup_{n \to \infty} P_M(x_n, y_n, t) = P_M(x, y, t). \]
Similarly, we have
\[ \limsup_{n \to \infty} P_M(x_n, y_n, t) = P_M(x, y, t). \]
Hence the result follows. \(\square\)

**Definition 5.** Let \((X, P_M, \ast)\) be a partial fuzzy metric space. \(P_M\) is said to be upper semicontinuous on \(X\) if for every \(x \in X\),
\[ P_M(p, x, t) \geq \limsup_{n \to \infty} P_M(x_n, x_n, t), \]
whenever \(\{x_n\}\) is a sequence in \(X\) which converges to a point \(p \in X\).

### 3 Fixed point results

Let \((X, P_M, \ast)\) be a partial fuzzy metric space and \(\emptyset \neq S \subseteq X\). Define
\[ \delta_{P_M}(S, t) = \inf \{ P_M(x, y, t) : x, y \in S \} \]
for all \(t > 0\). For an \(A_n = \{ x_n, x_{n+1}, \ldots \} \) in partial fuzzy metric space \((X, P_M, \ast)\),
let \(r_n(t) = \delta_{P_M}(A_n, t)\). Then \(r_n(t)\) is finite for all \(n \in \mathbb{N}\), \(\{r_n(t)\}\) is nonincreasing,
\(r_n(t) \to r(t)\) for some \(0 \leq r(t) \leq 1\) and also \(r_n(t) \leq P_M(x_l, x_k, t)\) for all \(l, k \geq n\).
Let $F$ be the set of all continuous functions $F: [0, 1]^3 \times [0, 1] \to [-1, 1]$ such that $F$ is nondecreasing on $[0, 1]^3$ satisfying the following condition:

- $F((u, u, u), v) \leq 0$ implies that $v \geq \gamma(u)$ where $\gamma : [0, 1] \to [0, 1]$ is a nondecreasing continuous function with $\gamma(s) > s$ for $s \in [0, 1]$.

**Example 5.** Let $\gamma(s) = s^h$ for $0 < h < 1$, then the functions $F$ defined by

$$F((t_1, t_2, t_3), t_4) = \gamma(\min\{t_1, t_2, t_3\}) - t_4$$

and

$$F((t_1, t_2, t_3), t_4) = \gamma\left(\sum_{i=1}^{n} a_i t_i \right) - t_4,$$

where $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$, belong to $F$.

Now we give our main theorem.

**Theorem 3.** Let $(X, P_M, \ast)$ be a complete bounded partial fuzzy metric space, $P_M$ is upper semicontinuous function on $X$ and $T$ be a self map of $X$ satisfying

$$F(P_M(x, y, t), P_M(Tx, x, t), P_M(Tx, y, t), P_M(Tx, Ty, t)) \leq 0$$

for all $x, y \in X$, where $F \in F$. Then $T$ has a unique fixed point $p$ in $X$ and $T$ is continuous at $p$.

**Proof.** Let $x_0 \in X$ and $Tx_n = x_{n+1}$. Let $r_n(t) = \delta_{P_M}(A_n, t)$, where $A_n = \{x_n, x_{n+1}, \ldots\}$. Then we know $\lim_{n \to \infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $T$ has a fixed point. Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Taking $x = x_{n-1}$, $y = x_{n+m-1}$ in (1) where $n \geq k$ and $m \in \mathbb{N}$, we have

$$F\left(\begin{array}{c}
P_M(x_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, x_{n-1}, t), \\
P_M(Tx_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, Tx_{n+m-1}, t)
\end{array}\right)$$

$$= F\left(\begin{array}{c}
P_M(x_{n-1}, x_{n+m-1}, t), P_M(x_n, x_{n-1}, t), \\
P_M(x_n, x_{n+m-1}, t), P_M(x_n, x_{n+m}, t)
\end{array}\right) \leq 0$$

Thus we have

$$F(r_{n-1}(t), t, r_n(t), P_M(x_n, x_{n+m}, t)) \leq 0,$$

since $F$ is nondecreasing on $[0, 1]^3$. Also, since $r_n(t)$ is nonincreasing, we have

$$F(r_{k-1}(t), t, r_{k-1}(t), P_M(x_n, x_{n+m}, t)) \leq 0,$$

which implies that

$$P_M(x_n, x_{n+m}, t) \geq \gamma(r_{k-1}(t)).$$
Thus for all $n \geq k$, we have
\[
\inf_{n \geq k} \{ P_M(x_n, x_{n+m}, t) \} = r_k(t) \geq \gamma(r_{k-1}(t)).
\]
Letting $k \to \infty$, we get $r(t) \geq \gamma(r(t))$. If $r(t) \neq 1$, then $r(t) \geq \gamma(r(t)) > r(t)$, which is a contradiction. Thus $r(t) = 1$ and hence $\lim_{n \to \infty} \gamma_n(t) = 1$. Thus given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $r_n(t) > 1 - \varepsilon$. Then we have for $n \geq n_0$ and $m \in \mathbb{N}$, $P_M(x_n, x_{n+m}, t) > 1 - \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $X$, there exists a $p \in X$ such that
\[
\lim_{n \to \infty} P_M(x_n, p, t) = P_M(p, p, t).
\]
Taking $x = x_n, y = p$ in (1), we have
\[
F(P_M(x_n, p, t), P_M(Tx_n, p, t), P_M(Tx_n, x_n, t), P_M(Tx_n, Tp, t))
= F(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)) \leq 0.
\]
Hence, we have
\[
\limsup_{n \to \infty} F(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t))
= F(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n \to \infty} P_M(x_{n+1}, Tp, t)) \leq 0.
\]
Since
\[
F(P_M(p, p, t), P_M(p, p, t), P_M(p, p, t), \limsup_{n \to \infty} P_M(x_{n+1}, Tp, t))
\leq F(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n \to \infty} P_M(x_{n+1}, Tp, t)) \leq 0,
\]
which implies
\[
P_M(p, Tp, t) \geq \limsup_{n \to \infty} P_M(x_{n+1}, Tp, t) \geq \gamma(P_M(p, p, t)).
\]
On the other hand, we have
\[
P_M(p, p, t) \geq P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)).
\]
Hence $P_M(p, p, t) = 1$. Also, since
\[
P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)) = \gamma(1) = 1,
\]
this implies that $P_M(p, Tp, t) = 1$, therefore, we get $Tp = p$.

For the uniqueness, let $p$ and $w$ be fixed points of $T$. Taking $x = p, y = w$ in (1), we have
\[
F(P_M(p, w, t), P_M(Tp, p, t), P_M(Tp, w, t), P_M(Tp, Tw, t))
= F(P_M(p, w, t), P_M(p, p, t), P_M(p, w, t), P_M(p, w, t)) \leq 0.
\]
Since $F$ is nondecreasing on $[0, 1]^3$, we have

$$F(P_M(p, w, t), P_M(p, w, t), P_M(p, w, t), P_M(p, w, t)) \leq 0,$$

which implies

$$P_M(p, w, t) \geq \gamma(P_M(p, w, t)) > P_M(p, w, t)$$

which is a contradiction. Thus we have $P_M(p, w, t) = 1$, therefore, $p = w$. Now, we show that $T$ is continuous at $p$. Let $\{y_n\}$ be a sequence in $X$ and $\lim_{n \to \infty} y_n = p$. Taking $x = p, y = y_n$ in (1), we have

$$F(P_M(p, y_n, t), P_M(Tp, p, t), P_M(Tp, y_n, t), P_M(Tp, Ty_n, t)) = F(P_M(p, y_n, t), P_M(p, p, t), P_M(p, y_n, t), P_M(p, Ty_n, t)) \leq 0,$$

hence

$$F(P_M(p, p, t), P_M(p, p, t), P_M(p, p, t), \lim_{n \to \infty} P_M(p, Ty_n, t)) = F\left(\lim_{n \to \infty} P_M(p, y_n, t), \lim_{n \to \infty} P_M(p, p, t), \lim_{n \to \infty} P_M(p, y_n, t), \lim_{n \to \infty} P_M(p, Ty_n, t)\right) \leq 0,$$

which implies

$$\lim_{n \to \infty} P_M(p, Ty_n, t) \geq \gamma(P_M(p, p, t)) = \gamma(1) = 1.$$

Thus,

$$\lim_{n \to \infty} P_M(p, Ty_n, t) = 1.$$

Similarly, taking limit inf, we have

$$\lim_{n \to \infty} P_M(p, Ty_n, t) = 1.$$

Therefore, $\lim_{n \to \infty} P_M(Ty_n, p, t) = 1$, this implies that

$$\lim_{n \to \infty} P_M(Ty_n, Tp, t) = 1 = P_M(p, p, t) = P_M(Tp, Tp, t).$$

Thus $\lim_{n \to \infty} Ty_n = p = Tp$. Hence $T$ is continuous at $p$. \hfill \Box

**Corollary 1.** Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and $T$ be a self map of $X$ satisfying for all $x, y \in X$,

$$F(P_M(x, y, t), P_M(T^m x, x, t), P_M(T^m x, y, t), P_M(T^m x, T^m y, t)) \leq 0$$

where $F \in \mathcal{F}$. Then $T$ has a unique fixed point $p$ in $X$ and $T^m$ is continuous at $p$.

**Proof.** From Theorem 3, $T^m$ has a unique fixed point $p$ in $X$ and $T^m$ is continuous at $p$. Since $Tp = TTTp = T^3p$, $T$ is also a fixed point of $T^m$. By the uniqueness it follows $Tp = p$. \hfill \Box
In Theorem 3, if we take \( F((t_1, t_2, t_3), t_4) = \gamma(\min\{t_1, t_2, t_3\}) - t_4 \) then we have the next result.

**Corollary 2.** Let \((X, P_M, \ast)\) be a complete bounded partial fuzzy metric space and \(T\) be a self map of \(X\) satisfying for all \(x, y \in X\),

\[
P_M(Tx, Ty, t) \geq \gamma(\min\{P_M(x, y, t), P_M(Tx, x, t), P_M(Tx, y, t)\}).
\]

Then \(T\) has a unique fixed point \(p\) in \(X\) and \(T\) is continuous at \(p\).

**Example 6.** Let \(X = \mathbb{R}^+\). Define \(P_M : X^2 \times [0, \infty) \to [0, 1]\) by

\[
P_M(x, y, t) = \exp\left(-\frac{\max\{x, y\}}{t}\right)
\]

for all \(x, y \in X\) and \(t > 0\). Then \((X, P_M, \ast)\) is a complete partial fuzzy metric space where \(a \ast b = ab\). Define map \(T : X \to X\) by \(Tx = \frac{x}{2}\) for \(x \in X\) and let \(\gamma : [0, 1] \to [0, 1]\) defined by \(\gamma(s) = s^{\frac{1}{2}}\). It is easy to see that

\[
P_M(Tx, Ty, t) = \sqrt{\exp\left(-\frac{\max\{\frac{x}{2}, \frac{y}{2}\}}{t}\right)}
\]

\[
= \sqrt{\sqrt{P_M(x, y, t)}}
\]

\[
\geq \sqrt{\min\{P_M(x, y, t), P_M(Tx, x, t), P_M(Tx, y, t)\}}.
\]

Thus \(T\) satisfy all the hypotheses of Corollary 2 and hence \(T\) has a unique fixed point.

**Corollary 3.** Let \((X, P_M, \ast)\) be a complete bounded partial fuzzy metric space, \(m \in \mathbb{N}\) and \(T\) be a self map of \(X\) satisfying for all \(x, y \in X\),

\[
P_M(T^m x, T^m y, t) \geq \gamma(\min\{P_M(x, y, t), P_M(T^m x, x, t), P_M(T^m x, y, t)\}).
\]

Then \(T\) has a unique fixed point \(p\) in \(X\) and and \(T^m\) is continuous at \(p\).

**Corollary 4.** Let \((X, P_M, \ast)\) be a complete bounded partial fuzzy metric space and \(T\) be a self map of \(X\) satisfying for all \(x, y \in X\),

\[
P_M(Tx, Ty, t) \geq \sqrt{a_1 P_M(x, y, t) + a_2 P_M(Tx, x, t) + a_3 P_M(Tx, y, t)},
\]

such that for every \(a_i \geq 0\), \(\sum_{i=1}^{3} a_i = 1\). Then \(T\) has a unique fixed point \(p\) in \(X\) and \(T\) is continuous at \(p\).

**Corollary 5.** Let \((X, M, \ast)\) be a complete bounded fuzzy metric space and \(T\) be a self map of \(X\) satisfying for all \(x, y \in X\) the

\[
F(M(x, y, t), M(Tx, x, t), M(Tx, y, t), M(Tx, Ty, t)) \leq 0
\]

where \(F \in \mathcal{F}\). Then \(T\) has a unique fixed point \(p\) in \(X\) and \(T\) is continuous at \(p\).
References


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