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Zeros of Solutions and Their Derivatives of Higher Order Non-homogeneous Linear Differential Equations

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Abstract. This paper is devoted to studying the growth and oscillation of solutions and their derivatives of higher order non-homogeneous linear differential equations with finite order meromorphic coefficients. Illustrative examples are also treated.

1 Introduction and main results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [9], [11], [17]. Let f be a meromorphic function, we define

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\varphi})|\,d\varphi\,,$$
$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log r\,,$$

and

$$T(r,f)=m(r,f)+N(r,f)\quad (r>0)$$

is the Nevanlinna characteristic function of f, where $\log^+ x=\max(0,\log x)$ for $x\geq 0$, and $n(t,f)$ is the number of poles of $f(z)$ lying in $|z|\leq t$, counted according to their multiplicity. Also, we define

$$N\Big(r,\frac{1}{f}\Big)=\int_0^r\frac{n\big(t,\frac{1}{f}\big)-n\big(0,\frac{1}{f}\big)}{t}\,dt+n\Big(0,\frac{1}{f}\Big)\log r,$$
$$\overline{N}\Big(r,\frac{1}{f}\Big)=\int_0^r\frac{\overline{n}\big(t,\frac{1}{f}\big)-\overline{n}\big(0,\frac{1}{f}\big)}{t}\,dt+\overline{n}\Big(0,\frac{1}{f}\Big)\log r,$$

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where $n\left(t,\frac{1}{f}\right)$ $\left(\overline{n}\left(t,\frac{1}{f}\right)\right)$ is the number of zeros (distinct zeros) of $f(z)$ lying in $|z|\leq t$, counted according to their multiplicity. In addition, we will use

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N\left(r,\frac{1}{f}\right)}{\log r}$$

and

$$\overline{\lambda}(f)=\limsup_{r\to+\infty}\frac{\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}$$

to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of $f(z)$. In the following, we give the necessary notations and basic definitions.

Definition 1. [9], [17] Let f be a meromorphic function. Then the order $\rho(f)$ and the lower order $\mu(f)$ of $f(z)$ are defined respectively by

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

Definition 2. [7], [17] Let f be a meromorphic function. Then the hyper-order of $f(z)$ is defined by

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}.$$

Definition 3. [9], [13] The type of a meromorphic function f of order ρ $(0<\rho<\infty)$ is defined by

$$\tau(f)=\limsup_{r\to+\infty}\frac{T(r,f)}{r^{\rho}}.$$

Definition 4. [7] Let f be a meromorphic function. Then the hyper-exponent of convergence of zero-sequence of $f(z)$ is defined by

$$\lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r}.$$

Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

The study of oscillation of solutions of linear differential equations has attracted many interests since the work of Bank and Laine [1], [2], for more details see [11]. One of the main subject of this research is the zeros distribution of solutions and their derivatives of linear differential equations. In this paper, we first discuss the growth of solutions of higher-order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \tag{1}$$

where $A_j(z)$ $(j = 1, \ldots, k-1)$, $A_0(z) \not\equiv 0$ and $F(z)(\not\equiv 0)$ are meromorphic functions of finite order. Some results on the growth of entire and meromorphic solutions of (1) have been obtained by several researchers (see [5], [6], [10], [11], [16]). In the case that the coefficients $A_j(z)$ $(j = 0, \ldots, k-1)$ are polynomials and $F(z) \equiv 0$, the growth of solutions of (1) has been extensively studied (see [8]). In 1992, Hellerstein et al. (see [10]) proved that every transcendental solution of (1) is of infinite order, if there exists some $d \in \{0, 1, \ldots, k-1\}$ such that

$$\max_{j \neq d}\{\rho(A_j), \rho(F)\} < \rho(A_d) \leq \frac{1}{2}.$$

Recently, Wang and Liu proved the following.

Theorem 1. [16, Theorem 1.6] *Suppose that $A_0, A_1, \ldots, A_{k-1}, F(z)$ are meromorphic functions of finite order. If there exists some A_s $(0 \leq s \leq k-1)$ such that*

$$b = \max_{j \neq s}\left\{\rho(F), \rho(A_j), \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) < \frac{1}{2}.$$

Then

1. *Every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of (1) satisfies $\mu(A_s) \leq \rho_2(f) \leq \rho(A_s)$. Furthermore, if $F \not\equiv 0$, then we have $\mu(A_s) \leq \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A_s)$.*

2. *If $s \geq 2$, then every non-transcendental meromorphic solution f of (1) is a polynomial with degree $\deg f \leq s-1$. If $s = 0$ or 1 then every nonconstant solution of (1) is transcendental.*

For more details there are many interesting papers, please see [11] and references contained in it. Recently, in [12], the authors studied equations of type

$$f'' + A(z)f' + B(z)f = F(z)\,, \tag{2}$$

where $A(z), B(z)\,(\not\equiv 0)$ and $F(z)\,(\not\equiv 0)$ are meromorphic functions of finite order. They proved under different conditions that every nontrivial meromorphic solution f of (2) satisfies

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = +\infty \quad (j \in \mathbb{N})$$

with at most one exception. It's interesting now to study the stability of the exponent of convergence of the sequence of zeros (resp. distinct zeros) of solutions for higher order differential equation (1) with their derivatives. The main purpose of this paper is to deal with this problem. Before we state our results we need to define the following notations.

$$A_i^j(z) = A_i^{j-1}(z) + \left(A_{i+1}^{j-1}(z)\right)' - A_{i+1}^{j-1}(z)\frac{\left(A_0^{j-1}(z)\right)'}{A_0^{j-1}(z)}, \text{ for } j = 1, 2, 3, \dots, \tag{3}$$

where $i = 0, 1, \dots, k-1$ and

$$F^j(z) = \left(F^{j-1}(z)\right)' - F^{j-1}(z)\frac{\left(A_0^{j-1}(z)\right)'}{A_0^{j-1}(z)}, \text{ for } j = 1, 2, 3, \dots, \tag{4}$$

where $A_i^0(z) = A_i(z)$ $(i = 0, 1, \dots, k-1)$, $F^0(z) = F(z)$ and $A_k^j(z) = 1$. We obtain the following results.

Theorem 2. (Main Theorem) *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be meromorphic functions of finite order such that $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$, where $j \in \mathbb{N}$. If f is a meromorphic solution of* (1) *with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \quad (j = 0, 1, 2, \dots).$$

Furthermore, if f is of finite order with

$$\rho(f) > \max_{i=0,\dots,k-1}\left\{\rho(A_i), \rho(F)\right\},$$

then

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = \rho(f) \quad (j = 0, 1, 2, \dots).$$

Remark 1. The condition "$A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$ where $j \in \mathbb{N}$" in Theorem 2 is necessary. For example, the entire function $f(z) = \mathrm{e}^{\mathrm{e}^z} - 1$ satisfies

$$f^{(3)} - \mathrm{e}^z f'' - f' - \mathrm{e}^{2z} f = \mathrm{e}^{2z},$$

where $A_2(z) = -\mathrm{e}^z$, $A_1(z) = -1$, $A_0(z) = -\mathrm{e}^{2z}$ and $F(z) = \mathrm{e}^{2z}$. So

$$A_0^1(z) = -\mathrm{e}^{2z} + 2 \quad F^1(z) \equiv 0.$$

On the other hand, we have $\lambda(f') = 0 < \lambda(f) = \infty$.

Here, we will give some sufficient conditions on the coefficients which guarantee $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0, (j = 1, 2, 3, \dots)$.

Theorem 3. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that $\rho(A_0) > \max\limits_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\}$. Then all nontrivial solutions of* (1) *satisfy*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one possible exceptional solution f_0 such that

$$\rho(f_0) = \max\{\overline{\lambda}(f_0), \rho(A_0)\}.$$

Furthermore, if $\rho(A_0) \le \frac{1}{2}$, then every transcendental solution of (1) *satisfies*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots).$$

Remark 2. The condition $\rho(A_0) > \max\limits_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\}$ does not ensure that all solutions of (1) are of infinite order. For example, we can see that $f_0(z) = \mathrm{e}^{-z^2}$ satisfies the differential equation

$$f^{(3)} + 2zf'' + 3f' + \big(\mathrm{e}^{z^2} - 2z\big)f = 1,$$

where

$$\overline{\lambda}(f_0) = 0 < \rho(f_0) = \rho(A_0) = 2.$$

Combining Theorem 1 and Theorem 3, we obtain the following result.

Corollary 1. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that*

$$\max_{i=1,\dots,k-1}\{\rho(F), \rho(A_i)\} < \mu(A_0) < \frac{1}{2}.$$

Then, every transcendental solution f of (1) *satisfies*

$$\mu(A_0) \le \overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) \le \rho(A_0) \quad (j \in \mathbb{N}).$$

Furthermore, if

$$\max_{i=1,\dots,k-1}\{\rho(F), \rho(A_i)\} < \mu(A_0) = \rho(A_0) < \frac{1}{2},$$

then every transcendental solution f of (1) *satisfies*

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) = \rho(A_0)\,.$$

Theorem 4. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that $A_1, \dots, A_{k-1}$ and F are polynomials and A_0 is transcendental. Then all nontrivial solutions of* (1) *satisfy*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one possible solution f_0 of finite order.

Corollary 2. *Let P be a nonconstant entire function, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} + e^{P(z)} f = Q(z) \quad (k \in \mathbb{N}).$$

1. *If P is polynomial, then*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = \infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, 2, \dots),$$

where ρ is positive integer not exceeding the degree of P.

2. *If P is transcendental with $\rho(P) < \frac{1}{2}$, then*

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \infty \quad (j = 0, 1, 2, \dots).$$

Theorem 5. *Let $j \geq 1$ be an integer, let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that $\rho(F) < \rho(A_i) \leq \rho(A_0)$ $(i = 1, \dots, k-1)$ and*

$$\tau(A_0) > \begin{cases} \sum\limits_{l \in I_j} \beta_l \tau(A_{l+1}) & \text{if } j < k, \\ \sum\limits_{l \in I_k} \beta_l \tau(A_{l+1}) & \text{if } j \geq k, \end{cases}$$

where $\beta_l = \sum_{p=l}^{j-1} C_p^l$ with $C_p^l = \frac{p!}{(p-l)!\,l!}$,

$$I_k = \big\{0 \leq l \leq k-2 : \rho(A_{l+1}) = \rho(A_0)\big\}$$

and

$$I_j = \big\{0 \leq l \leq j-1 : \rho(A_{l+1}) = \rho(A_0)\big\}.$$

If f is a nontrivial solution of (1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies

$$\overline{\lambda}\big(f^{(m)}\big) = \lambda\big(f^{(m)}\big) = +\infty \quad (m = 0, 1, 2, \dots, j)$$

and

$$\overline{\lambda}_2\big(f^{(m)}\big) = \lambda_2\big(f^{(m)}\big) = \rho \quad (m = 0, 1, 2, \dots, j).$$

From Theorem 5, we obtain the following result of paper [12].

Corollary 3. [12] *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be entire functions with finite order such that $\rho(B) = \rho(A) > \rho(F)$ and $\tau(B) > k\tau(A)$, $k \geq 1$ is an integer. If f is a nontrivial solution of (2) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, \dots, k)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, \dots, k).$$

In the next theorem, we denote by $\sigma(f)$ the following quantity

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log m(r,f)}{\log r}.$$

Theorem 6. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be meromorphic functions of finite order such that $\sigma(A_0) > \max_{i=1,\dots,k-1}\{\sigma(A_i), \sigma(F)\}$. If f is a meromorphic solution of (1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, 2, \dots).$$

In the following, we mean by two meromorphic functions f and g share a finite value a CM (counting multiplicities) when $f-a$ and $g-a$ have the same zeros with the same multiplicities. It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. Rubel and Yang [14], [17] proved that if f is an entire function and shares two finite values CM with its derivative, then $f = f'$. We give here a different result.

Theorem 7. *Let k be a positive integer and let f be entire function. If f and $f^{(k)}$ share the value $a \neq 0$ CM, then*

(a) *$\rho(f) = 1$ or*

(b) *with at most one exception*

$$\overline{\lambda}(f-a) = \overline{\lambda}\big(f^{(j)}\big) = \infty \quad (j = 1, 2, \dots).$$

2 Preliminary lemmas

Lemma 1. [9] *Let f be a meromorphic function and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\big(\log T(r,f) + \log r\big)$, possibly outside of an exceptional set $E \subset (0,+\infty)$ with finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2. [3], [5] *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions.*

1. *If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{5}$$

with $\rho(f) = +\infty$, then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

2. *If f is a meromorphic solution of (5) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty \quad \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Lemma 3. [15] *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation (5) with*

$$\max_{j=0,1,\ldots,k-1} \big\{\rho(A_j), \rho(F)\big\} < \rho(f) < +\infty,$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

Lemma 4. [6] *Let $A, B_1, \ldots, B_{k-1}$, $F \not\equiv 0$ be entire functions of finite order, where $k \geq 2$. Suppose that either* (a) *or* (b) *below holds:*

(a) $\rho(B_j) < \rho(A)$ $(j = 1, \ldots, k-1)$;

(b) $B_1, \ldots, B_{k-1}$ *are polynomials and A is transcendental.*

Then we have

1. *All solutions of the differential equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_1 f' + A f = F$$

satisfy

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

with at most one possible solution f_0 of finite order.

2. *If there exists an exceptional solution f_0 in case 1, then f_0 satisfies*

$$\rho(f_0) \leq \max\big\{\rho(A), \rho(F), \overline{\lambda}(f_0)\big\} < \infty. \tag{6}$$

Furthermore, if $\rho(A) \neq \rho(F)$ and $\overline{\lambda}(f_0) < \rho(f_0)$, then

$$\rho(f_0) = \max\big\{\rho(A), \rho(F)\big\}.$$

Lemma 5. *Let $A_0, A_1, \ldots, A_{k-1}$ be the coefficients of (1). For any integer j, the following inequalities hold*

$$m\big(r, A_1^j\big) \leq \begin{cases} \sum\limits_{i=0}^{j} C_j^i m(r, A_{i+1}) + O(\log r) & \text{if } j < k, \\ \sum\limits_{i=0}^{k-2} C_j^i m(r, A_{i+1}) + O(\log r) & \text{if } j \geq k, \end{cases} \tag{7}$$

where A_1^j is defined in (3).

Proof. First, we prove the case $j < k$. We have from (3)

$$A_i^j = A_i^{j-1} + A_{i+1}^{j-1}\left(\frac{\left(A_{i+1}^{j-1}\right)'}{A_{i+1}^{j-1}} - \frac{\left(A_0^{j-1}\right)'}{A_0^{j-1}}\right) \quad (i \in \mathbb{N}).$$

By using Lemma 1, we have for all $j \in \mathbb{N}$

$$m\left(r, A_i^j\right) \le m\left(r, A_i^{j-1}\right) + m\left(r, A_{i+1}^{j-1}\right) + O(\log r). \tag{8}$$

In order to prove Lemma 5, we apply mathematical induction. For $j = 2$, we have from (8)

$$\begin{aligned} m\left(r, A_1^2\right) &\le m\left(r, A_1^1\right) + m\left(r, A_2^1\right) + O(\log r) \\ &\le m(r, A_1) + 2m(r, A_2) + m(r, A_3) + O(\log r) \\ &= C_2^0 m(r, A_1) + C_2^1 m(r, A_2) + C_2^2 m(r, A_3) + O(\log r) \\ &= \sum_{i=0}^{2} C_2^i m(r, A_{i+1}) + O(\log r). \end{aligned}$$

Suppose that (7) is true and we show that for $j + 1 < k$

$$m\left(r, A_1^{j+1}\right) \le \sum_{i=0}^{j+1} C_{j+1}^i m(r, A_{i+1}) + O(\log r).$$

By using (3) and (7) we have

$$\begin{aligned} m\left(r, A_1^{j+1}\right) &\le m\left(r, A_1^j\right) + m\left(r, A_2^j\right) + O(\log r) \\ &\le \sum_{i=0}^{j} C_j^i m(r, A_{i+1}) + \sum_{i=0}^{j} C_j^i m(r, A_{i+2}) + O(\log r) \\ &= C_j^0 m(r, A_1) + \sum_{i=1}^{j} C_j^i m(r, A_{i+1}) \\ &\quad + \sum_{i=0}^{j-1} C_j^i m(r, A_{i+2}) + C_j^j m(r, A_{j+2}) + O(\log r) \\ &= C_j^0 m(r, A_1) + \sum_{i=1}^{j} C_j^i m(r, A_{i+1}) \\ &\quad + \sum_{i=1}^{j} C_j^{i-1} m(r, A_{i+1}) + C_j^j m(r, A_{j+2}) + O(\log r). \end{aligned}$$

Of course if $j + 1 \ge k$, then $m(r, A_{j+1}) = m(r, A_{j+2}) = 0$. Since $C_j^0 = C_{j+1}^0$ and $C_j^j = C_{j+1}^{j+1}$, then we have

$$\begin{aligned} m\left(r, A_1^{j+1}\right) \le C_{j+1}^0 m(r, A_1) &+ \sum_{i=1}^{j} \left(C_j^i + C_j^{i-1}\right) m(r, A_{i+1}) \\ &+ C_{j+1}^{j+1} m(r, A_{j+2}) + O(\log r). \end{aligned}$$

Using the identity $C_j^i + C_j^{i-1} = C_{j+1}^i$, we get

$$\begin{aligned} m\big(r, A_1^{j+1}\big) &\leq C_{j+1}^0 m(r, A_1) + \sum_{i=1}^{j} C_{j+1}^i m(r, A_{i+1}) + C_{j+1}^{j+1} m(r, A_{j+2}) + O(\log r) \\ &= \sum_{i=0}^{j+1} C_{j+1}^i m(r, A_{i+1}) + O(\log r). \end{aligned}$$

For the case $j \geq k$, we need just to remark that $m(r, A_{i+1}) = 0$ when $i \geq k-1$ and by using the same procedure as before we obtain

$$m\big(r, A_1^j\big) \leq \sum_{i=0}^{k-2} C_j^i m(r, A_{i+1}) + O(\log r). \qquad \square$$

Lemma 6. *Let $A_0, A_1, \ldots, A_{k-1}$ be the coefficients of (1). For any integer j, the following inequalities hold*

$$\sum_{p=0}^{j-1} m\big(r, A_1^p\big) \leq \begin{cases} \sum\limits_{i=0}^{j-1} \left(\sum\limits_{p=i}^{j-1} C_p^i \right) m(r, A_{i+1}) + O(\log r) & \text{if } j < k, \\ \sum\limits_{i=0}^{k-2} \left(\sum\limits_{p=i}^{j-1} C_p^i \right) m(r, A_{i+1}) + O(\log r) & \text{if } j \geq k. \end{cases} \tag{9}$$

Proof. We prove only the case $j < k$. By Lemma 5 we have

$$\sum_{p=0}^{j-1} m\big(r, A_1^p\big) \leq \sum_{p=0}^{j-1} \left(\sum_{i=0}^{p} C_p^i m(r, A_{i+1}) \right) + O(\log r). \tag{10}$$

The first term of the right hand of (10) can be expressed as

$$\begin{aligned} \sum_{p=0}^{j-1} \left(\sum_{i=0}^{p} C_p^i m(r, A_{i+1}) \right) &= C_0^0 m(r, A_1) + \big(C_1^0 m(r, A_1) + C_1^1 m(r, A_2)\big) \\ &\quad + \big(C_2^0 m(r, A_1) + C_2^1 m(r, A_2) + C_2^2 m(r, A_3)\big) + \cdots \\ &\quad + \big(C_{j-1}^0 m(r, A_1) + C_{j-1}^1 m(r, A_2) + \cdots + C_{j-1}^{j-1} m(r, A_j)\big) \end{aligned}$$

which we can write as

$$\begin{aligned} \sum_{p=0}^{j-1} \left(\sum_{i=0}^{p} C_p^i m(r, A_{i+1}) \right) &= \big(C_0^0 + C_1^0 + \cdots + C_{j-1}^0\big) m(r, A_1) \\ &\quad + \big(C_1^1 + C_2^1 + \cdots + C_{j-1}^1\big) m(r, A_2) + \cdots \\ &\quad + \big(C_{j-2}^{j-2} + C_{j-1}^{j-2}\big) m(r, A_{j-1}) + C_{j-1}^{j-1} m(r, A_j). \end{aligned}$$

Then

$$\begin{aligned}\sum_{p=0}^{j-1} m\big(r, A_1^p\big) &\le \sum_{p=0}^{j-1}\Bigg(\sum_{i=0}^{p} C_p^i m(r, A_{i+1})\Bigg) + O(\log r)\\ &= \sum_{i=0}^{j-1}\Bigg(\sum_{p=i}^{j-1} C_p^i\Bigg) m(r, A_{i+1}) + O(\log r).\end{aligned}$$

By using the same procedure as above we can prove the case $j \ge k$. □

Lemma 7. [18] *Let $\phi(z)$ be a nonconstant entire function and k be a positive integer. Then, with at most one exception, every solution F of the differential equation*

$$F^{(k)} - \mathrm{e}^{\phi(z)} F = 1$$

satisfies $\rho_2(F) = \rho(\mathrm{e}^{\phi})$.

Lemma 8. [4] *Let P be a nonconstant entire function, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} + \mathrm{e}^{P(z)} f = Q(z) \quad (k \in \mathbb{N}).$$

If P is polynomial, then f has an infinite order and its hyper-order $\rho_2(f)$ is a positive integer not exceeding the degree of P. If P is transcendental with order less than $\frac{1}{2}$, then the hyper-order of f is infinite.

Lemma 9. *Let f be a meromorphic function with $\rho(f) = \rho \ge 0$. Then, there exists a set $E_1 \subset [1, +\infty)$ with infinite logarithmic measure*

$$\mathrm{lm}(E_1) = \int_1^{+\infty} \frac{\chi_{E_1}(t)}{t}\, \mathrm{d}t = \infty\,,$$

where $\chi_{E_1}(t)$ is the characteristic function of the set E_1, such that

$$\lim_{\substack{r\to+\infty\\ r\in E_1}} \frac{\log T(r, f)}{\log r} = \rho.$$

Proof. Since $\rho(f) = \rho$, then there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to $+\infty$ satisfying $\big(1 + \frac{1}{n}\big) r_n < r_{n+1}$ and

$$\lim_{r_n\to+\infty} \frac{\log T(r_n, f)}{\log r_n} = \rho(f).$$

So, there exists an integer n_1 such that for all $n \ge n_1$, for any $r \in \big[r_n, \big(1 + \frac{1}{n}\big) r_n\big]$, we have

$$\frac{\log T(r_n, f)}{\log\big(1 + \frac{1}{n}\big) r_n} \le \frac{\log T(r, f)}{\log r} \le \frac{\log T\big(\big(1 + \frac{1}{n}\big) r_n, f\big)}{\log r_n}\,.$$

Set $E_1 = \bigcup_{n=n_1}^{\infty} \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, we obtain

$$\lim_{\substack{r\to+\infty\\ r\in E_1}} \frac{\log T(r,f)}{\log r} = \lim_{r_n\to+\infty} \frac{\log T(r_n,f)}{\log r_n},$$

and

$$\mathrm{lm}(E_1) = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{\mathrm{d}t}{t} = \sum_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty.$$

Thus, the proof of the lemma is completed. □

Lemma 10. *Let f_1, f_2 be meromorphic functions satisfying $\rho(f_1) > \rho(f_2)$. Then there exists a set $E_2 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have*

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

Proof. Set $\rho_1 = \rho(f_1)$, $\rho_2 = \rho(f_2)$, $(\rho_1 > \rho_2)$. By Lemma 9, there exists a set $E_2 \subset (1,+\infty)$ having infinite logarithmic measure such that for any given $0 < \varepsilon < \frac{\rho_1-\rho_2}{2}$ and all sufficiently large $r \in E_2$

$$T(r,f_1) > r^{\rho_1-\varepsilon}$$

and for all sufficiently large r, we have

$$T(r,f_2) < r^{\rho_2+\varepsilon}.$$

From this we can get

$$\frac{T(r,f_2)}{T(r,f_1)} < \frac{r^{\rho_2+\varepsilon}}{r^{\rho_1-\varepsilon}} = \frac{1}{r^{\rho_1-\rho_2-2\varepsilon}} \quad (r \in E_2).$$

Since $0 < \varepsilon < \frac{\rho_1-\rho_2}{2}$, then we obtain

$$\lim_{\substack{r\to+\infty\\ r\in E_2}} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$ □

3 Proofs of the Theorems and the Corollary

Proof of Theorem 2. For the proof, we use the principle of mathematical induction. Since $A_0(z) \not\equiv 0$ and $F(z) \not\equiv 0$, then by using Lemma 2 we have

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Dividing both sides of (1) by A_0, we obtain

$$\frac{A_k}{A_0} f^{(k)} + \frac{A_{k-1}}{A_0} f^{(k-1)} + \cdots + \frac{A_1}{A_0} f' + f = \frac{F}{A_0}. \tag{11}$$

Differentiating both sides of equation (11), we have

$$\frac{A_k}{A_0}f^{(k+1)}+\left(\left(\frac{A_k}{A_0}\right)'+\frac{A_{k-1}}{A_0}\right)f^{(k)}+\cdots+\left(\left(\frac{A_1}{A_0}\right)'+1\right)f'=\left(\frac{F}{A_0}\right)'. \quad (12)$$

Multiplying now (12) by A_0, we get

$$f^{(k+1)}+A^1_{k-1}(z)f^{(k)}+\cdots+A^1_0(z)f'=F^1(z), \quad (13)$$

where

$$\begin{aligned}A^1_i(z)&=A_0\left(\left(\frac{A_{i+1}(z)}{A_0(z)}\right)'+\frac{A_i(z)}{A_0(z)}\right)\\&=A_i(z)+A'_{i+1}(z)-A_{i+1}(z)\frac{A'_0(z)}{A_0(z)}\quad(i=0,\dots,k-1)\end{aligned}$$

and

$$F^1(z)=A_0(z)\left(\frac{F(z)}{A_0(z)}\right)'=F'(z)-F(z)\frac{A'_0(z)}{A_0(z)}.$$

Since $A^1_0(z)\not\equiv 0$ and $F^1(z)\not\equiv 0$ are meromorphic functions with finite order, then by using Lemma 2 we obtain

$$\overline{\lambda}(f')=\lambda(f')=\rho(f)=+\infty$$

and

$$\overline{\lambda}_2(f')=\lambda_2(f')=\rho_2(f)=\rho.$$

Dividing now both sides of (13) by A^1_0, we obtain

$$\frac{A^1_k}{A^1_0}f^{(k+1)}+\frac{A^1_{k-1}}{A^1_0}f^{(k)}+\cdots+\frac{A^1_1}{A^1_0}f''+f'=\frac{F^1}{A^1_0}. \quad (14)$$

Differentiating both sides of equation (14) and multiplying by A^1_0, we get

$$f^{(k+2)}+A^2_{k-1}(z)f^{(k+1)}+\cdots+A^2_0(z)f''=F^2(z), \quad (15)$$

where $A^2_0(z)\not\equiv 0$ and $F^2(z)\not\equiv 0$ are meromorphic functions defined in (3) and (4). By using Lemma 2, we obtain

$$\overline{\lambda}(f'')=\lambda(f'')=\rho(f)=+\infty$$

and

$$\overline{\lambda}_2(f'')=\lambda_2(f'')=\rho_2(f)=\rho.$$

Suppose now that

$$\overline{\lambda}_i\big(f^{(k)}\big)=\lambda_i\big(f^{(k)}\big)=\rho_i(f)\quad(i=1,2) \quad (16)$$

for all $k=0,1,2,\dots,j-1$, and we prove that (16) is true for $k=j$. With the same procedure as before, we can obtain

$$f^{(k+j)}+A^j_{k-1}(z)f^{(k-1+j)}+\cdots+A^j_0(z)f^{(j)}=F^j(z),$$

where $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$ are meromorphic functions defined in (3) and (4). By using Lemma 2, we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) = \rho.$$

For the case $\rho(f) > \max\limits_{i=0,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\}$ we use simply similar reasoning as above and by using Lemma 3, we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) \quad (j = 0, 1, 2, \dots).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. By Lemma 4, all nontrivial solutions of (1) are of infinite order with at most one exceptional solution f_0 of finite order. By (3) we have

$$\begin{aligned} A_0^j &= A_0^{j-1} + A_1^{j-1}\left(\frac{\big(A_1^{j-1}\big)'}{A_1^{j-1}} - \frac{\big(A_0^{j-1}\big)'}{A_0^{j-1}}\right) \\ &= A_0^{j-2} + A_1^{j-2}\left(\frac{\big(A_1^{j-2}\big)'}{A_1^{j-2}} - \frac{\big(A_0^{j-2}\big)'}{A_0^{j-2}}\right) + A_1^{j-1}\left(\frac{\big(A_1^{j-1}\big)'}{A_1^{j-1}} - \frac{\big(A_0^{j-1}\big)'}{A_0^{j-1}}\right) \\ &= A_0 + \sum_{p=0}^{j-1} A_1^p\left(\frac{\big(A_1^{p}\big)'}{A_1^{p}} - \frac{\big(A_0^{p}\big)'}{A_0^{p}}\right). \end{aligned} \tag{17}$$

Now, suppose that there exists $j \in \mathbb{N}$ such that $A_0^j(z) \equiv 0$. By (17) we obtain

$$-A_0 = \sum_{p=0}^{j-1} A_1^p\left(\frac{\big(A_1^{p}\big)'}{A_1^{p}} - \frac{\big(A_0^{p}\big)'}{A_0^{p}}\right). \tag{18}$$

Hence

$$m(r, A_0) \le \sum_{p=0}^{j-1} m\big(r, A_1^p\big) + O(\log r). \tag{19}$$

Using Lemma 6 and (19) we have

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le \begin{cases} \sum\limits_{i=0}^{j-1}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) m(r, A_{i+1}) + O(\log r) & \text{if } j < k \\ \sum\limits_{i=0}^{k-2}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) m(r, A_{i+1}) + O(\log r) & \text{if } j \ge k \end{cases} \\ &= \begin{cases} \sum\limits_{i=0}^{j-1}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) T(r, A_{i+1}) + O(\log r) & \text{if } j < k \\ \sum\limits_{i=0}^{k-2}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) T(r, A_{i+1}) + O(\log r) & \text{if } j \ge k \end{cases} \end{aligned} \tag{20}$$

which implies the contradiction

$$\rho(A_0) \le \max_{i=1,\dots,k-1} \rho(A_i),$$

and we can deduce that $A_0^j(z) \not\equiv 0$ for all $j \in \mathbb{N}$. Suppose now there exists $j \in \mathbb{N}$ which is the first index such that $F^j(z) \equiv 0$. From (4) we obtain

$$\big(F^{j-1}(z)\big)' - F^{j-1}(z)\frac{\big(A_0^{j-1}(z)\big)'}{A_0^{j-1}(z)} = 0$$

which implies

$$F^{j-1}(z) = cA_0^{j-1}(z), \tag{21}$$

where $c \in \mathbb{C} \setminus \{0\}$. By (17) and (21) we have

$$\frac{1}{c}F^{j-1} = A_0(z) + \sum_{p=0}^{j-2} A_1^p(z)\left(\frac{\big(A_1^p(z)\big)'}{A_1^p(z)} - \frac{\big(A_0^p(z)\big)'}{A_0^p(z)}\right). \tag{22}$$

On the other hand, we obtain from (4)

$$m(r, F^j) \le m(r, F) + O(\log r) \quad (j \in \mathbb{N}). \tag{23}$$

By (20), (22) and (23), we have

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le \sum_{p=0}^{j-2} m\big(r, A_1^p\big) + m(r, F^{j-1}) + O(\log r) \\ &\le \begin{cases} \sum\limits_{i=0}^{j-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) m(r, A_{i+1}) + m(r, F) + O(\log r) & \text{if } j-1 < k \\ \sum\limits_{i=0}^{k-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) m(r, A_{i+1}) + m(r, F) + O(\log r) & \text{if } j-1 \ge k \end{cases} \\ &= \begin{cases} \sum\limits_{i=0}^{j-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) T(r, A_{i+1}) + T(r, F) + O(\log r), & \text{if } j-1 < k \\ \sum\limits_{i=0}^{k-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) T(r, A_{i+1}) + T(r, F) + O(\log r), & \text{if } j-1 \ge k \end{cases} \end{aligned}$$

which implies the contradiction $\rho(A_0) \le \max_{i=1,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\}$. Since $A_0^j \not\equiv 0$ and $F^j \not\equiv 0$ $(j \in \mathbb{N})$, then by applying Theorem 2 and Lemma 4 we have

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one exceptional solution f_0 of finite order. Since

$$\rho(A_0) > \max_{i=1,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\},$$

then by (6) we obtain

$$\rho(f_0) \leq \max\{\rho(A_0), \overline{\lambda}(f_0)\}. \tag{24}$$

On the other hand by (1), we can write

$$A_0 = \frac{F}{f_0} - \left(\frac{f_0^{(k)}}{f_0} + A_{k-1}\frac{f_0^{(k-1)}}{f_0} + \cdots + A_1\frac{f_0'}{f_0}\right).$$

It follows that by Lemma 1

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\leq m\left(r, \frac{F}{f_0}\right) + \sum_{i=1}^{k-1} m(r, A_i) + O(\log r) \\ &\leq T(r, f_0) + T(r, F) + \sum_{i=1}^{k-1} T(r, A_i) + O(\log r), \end{aligned}$$

which implies

$$\rho(A_0) \leq \max_{i=1,\dots,k-1}\{\rho(f_0), \rho(A_i), \rho(F)\} = \rho(f_0). \tag{25}$$

Since $\overline{\lambda}(f_0) \leq \rho(f_0)$, then by using (24) and (25) we obtain

$$\rho(f_0) = \max\{\rho(A_0), \overline{\lambda}(f_0)\}.$$

If $\rho(A_0) \leq \frac{1}{2}$, then by the theorem of Hellerstein et al. (see [10]) every transcendental solution f of (1) is of infinite order without exceptions. So, by the same proof as before we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots).$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Using the same proof as Theorem 3, we obtain Theorem 4. □

Proof of Corollary 2.

1. If P is polynomial, since $A_0(z) = e^{P(z)}$, $A_i(z) \equiv 0$ $(i = 1, \dots, k-1)$ and $F(z) = Q(z)$, then

$$\rho(A_0) > \max_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\},$$

hence $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$, $j \in \mathbb{N}$. On the other hand, by Lemma 8 every solution f has an infinite order and its hyper-order $\rho_2(f)$ is a positive integer not exceeding the degree of P. So, by applying Theorem 2 we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = \infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, 2, \dots),$$

where ρ is positive integer not exceeding the degree of P.

2. Using the same reasoning as in 1. for the case P is transcendental with $\rho(P) < \frac{1}{2}$. □

Proof of Theorem 5. First, we prove that $A_0^n(z) \not\equiv 0$ for all $n = 1, 2, \ldots, j$. Suppose there exists $1 \le s \le j$ such that $A_0^s \equiv 0$. By (20), we have

$$T(r, A_0) = m(r, A_0) \le \begin{cases} \sum_{l=0}^{s-1} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s < k \\ \sum_{l=0}^{k-2} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s \ge k \end{cases}$$

$$= \begin{cases} \sum_{l \in I_s} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + \sum_{l \in \{0,1,\ldots,s-1\} - I_s} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s < k \\ \sum_{l \in I_k} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + \sum_{l \in \{0,1,\ldots,k-2\} - I_k} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s \ge k. \end{cases}$$

Then, by using Lemma 10, there exists a set $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have

$$T(r, A_0) = m(r, A_0) \le \begin{cases} \sum_{l \in I_s} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + o\big(T(r, A_0)\big) & \text{if } s < k, \\ \sum_{l \in I_k} \left(\sum_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + o\big(T(r, A_0)\big) & \text{if } s \ge k, \end{cases}$$

which implies the contradiction

$$\tau(A_0) \le \begin{cases} \sum_{l \in I_s} \beta_l \tau(A_{l+1}) & \text{if } s < k, \\ \sum_{l \in I_k} \beta_l \tau(A_{l+1}) & \text{if } s \ge k, \end{cases}$$

where $\beta_l = \sum_{p=l}^{s-1} C_p^l$. Hence $A_0^n(z) \not\equiv 0$ for all $n = 1, 2, \ldots, j$. By the same procedure we deduce that $F^n(z) \not\equiv 0$ for all $n = 1, 2, \ldots, j$. Then, by Theorem 2 we have

$$\overline{\lambda}\big(f^{(m)}\big) = \lambda\big(f^{(m)}\big) = +\infty \quad (m = 0, 1, \ldots, j)$$

and

$$\overline{\lambda_2}\big(f^{(m)}\big) = \lambda_2\big(f^{(m)}\big) = \rho \quad (m = 0, 1, \ldots, j). \qquad \square$$

Proof of Theorem 6. By using the same reasoning as in the proof of Theorem 3, we can prove Theorem 6. $\square$

Proof of Theorem 7. Since f and $f^{(k)}$ share the value a CM, then

$$\frac{f^{(k)} - a}{f - a} = \mathrm{e}^{Q(z)},$$

where Q is entire function. Set $G = \frac{f}{a} - 1$. Then G satisfies the following differential equation

$$G^{(k)} - \mathrm{e}^{Q(z)} G = 1. \tag{26}$$

(a) If Q is constant, by solving (26) we obtain $\rho(G) = \rho(f) = 1$.

(b) If Q is nonconstant, we know from Lemma 7 that $\rho_2(G) = \rho(\mathrm{e}^Q)$ with at most one exception, which means that G is of infinite order with one exception at most. On the other hand, $A_0(z) = -\mathrm{e}^{Q(z)}$, $A_i(z) \equiv 0$ $(i = 1, \dots, k-1)$ and $F(z) = 1$, then

$$\rho(A_0) > \max_{i=1,\dots,k-1} \big\{\rho(A_i), \rho(F)\big\}.$$

Hence $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$, $j \in \mathbb{N}$. So, by applying Theorem 2, we obtain

$$\overline{\lambda}\big(G^{(j)}\big) = \lambda\big(G^{(j)}\big) = \rho(G) = \infty \quad (j = 0, 1, 2, \dots)$$

with one exception at most. Since $G = \frac{f}{a} - 1$, we deduce

$$\overline{\lambda}(f - a) = \overline{\lambda}\big(f^{(j)}\big) = \infty \quad (j = 1, 2, \dots)$$

with one exception at most. □

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