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A VARIATIONAL APPROACH TO BIFURCATION POINTS OF A REACTION-DIFFUSION SYSTEM WITH OBSTACLES AND NEUMANN BOUNDARY CONDITIONS

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Abstract. Given a reaction-diffusion system which exhibits Turing's diffusion-driven instability, the influence of unilateral obstacles of opposite sign (source and sink) on bifurcation and critical points is studied. In particular, in some cases it is shown that spatially nonhomogeneous stationary solutions (spatial patterns) bifurcate from a basic spatially homogeneous steady state for an arbitrarily small ratio of diffusions of inhibitor and activator, while a sufficiently large ratio is necessary in the classical case without unilateral obstacles. The study is based on a variational approach to a non-variational problem which even after transformation to a variational one has an unusual structure for which usual variational methods do not apply.

Keywords: reaction-diffusion system; unilateral condition; variational inequality; local bifurcation; variational approach; spatial patterns; Turing instability

MSC 2010: 35B32, 35K57, 35J50, 35J57, 47J20

1. Introduction

Let us consider a system

$$\begin{aligned} u_t &= d_1 \Delta u + f_1(u,v) \\ v_t &= d_2 \Delta v + f_2(u,v) \end{aligned} \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitzian boundary $\partial\Omega$ and f_i are differentiable functions, $f_i(0,0) = 0$. We are interested in existence and displacement of bifurcation points of nontrivial stationary solutions of the system (1.1) with Neumann boundary conditions and some unilateral obstacles for v. An example are boundary conditions

$$\left\{\begin{aligned} &\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Gamma_N, \\ &\pm v \geqslant 0, \quad \pm\frac{\partial v}{\partial n} \geqslant 0, \quad v \cdot \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_\pm, \end{aligned}\right. \tag{1.2}$$

where Γ_+, Γ_-, Γ_N are pairwise disjoint subsets of $\partial\Omega$,

$$\operatorname{mes}\Gamma_+, \operatorname{mes}\Gamma_- > 0, \quad \operatorname{mes}(\partial\Omega \setminus (\Gamma_- \cup \Gamma_+ \cup \Gamma_N)) = 0 \tag{1.3}$$

(the $(d-1)$-dimensional Lebesgue measure). Clearly, $(0,0)$ is a solution of (1.1) with pure Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

as well as with (1.2), and also with the other unilateral obstacles we will consider. We will use a certain nondirect variational approach, which will force us to deal in fact with the particular case

$$\left.\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v = 0 \\ &d_2\Delta v + b_{21}u + b_{22}v + n(v) = 0 \end{aligned}\right. \quad \text{in } \Omega, \tag{1.5}$$

where

$$n(0) = n'(0) = 0, \tag{1.6}$$

or even with the linearized stationary system

$$\left.\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v = 0 \\ &d_2\Delta v + b_{21}u + b_{22}v = 0 \end{aligned}\right. \quad \text{in } \Omega. \tag{1.7}$$

However, in order to explain the meaning of our results, let us recall some facts concerning the general reaction-diffusion system (1.1) and its relations to (1.7) and (1.5). We will denote $b_{ij} = \partial f_i/\partial u_j(0,0)$ and assume that

$$\begin{gathered} b_{11} + b_{22} < 0, \quad \det B := b_{11}b_{22} - b_{12}b_{21} > 0, \\ b_{11} > 0 > b_{22}, \quad b_{12}b_{21} < 0. \end{gathered} \tag{1.8}$$

The first line in (1.8) guarantees that the equilibrium $(0,0)$ is asymptotically stable as a solution of the corresponding system of ODEs without any diffusion ($d_1 = d_2 = 0$). If also the second line is fulfilled, then $(0,0)$ as a solution of the whole system (1.1) with Neumann conditions (1.4) is linearly stable only for values (d_1, d_2) from a certain open domain $D_S \subseteq \mathbb{R}^2_+$, but linearly unstable for (d_1, d_2) from the interior of the complement $D_U := \mathbb{R}^2_+ \setminus \overline{D}_S$. For (d_1^0, d_2^0) from the boundary C_E between D_S and D_U it usually happens that there is a bifurcation of spatially nonconstant stationary solutions, that is, each neighborhood of $(d_1^0, d_2^0, 0, 0)$ in $\mathbb{R}^2 \times (W^{1,2}(\Omega))^2$ contains stationary solutions (d_1, d_2, u, v) of (1.1), (1.4) with spatially nonconstant (u, v), see e.g. [18], [20]. Such solutions can describe Turing's spatial patterns having interpretation in biology, see e.g. [4], [19], [13].

Let us note that standard linearization and compactness arguments imply that such a bifurcation point $(d_1, d_2) = (d_1^0, d_2^0)$ must necessarily be a critical point of (1.1), (1.4), that is, the system (1.7), (1.4) has a nontrivial solution (u, v) which in view of $\det B \neq 0$ is necessarily spatially nonconstant.

If the system under consideration describes a chemical reaction, then the second line in (1.8) means that our system is of activator-inhibitor type (the case $b_{12} < 0 < b_{21}$) or of positive feedback (substrate-depletion) type. See e.g. [4], [19], [13]. In the first case, u and v are related to the concentration of the activator and inhibitor, respectively. In fact, in applications u and v typically describe the *difference* of the concentration of some chemicals to some spatially constant equilibrium $(\bar{u}, \overline{v})$ so, after variable substitution in an original model, it is no loss of generality to assume $(\bar{u}, \overline{v}) = (0, 0)$, and also negative values of u and v have a natural physical interpretation (they correspond to concentrations under the equilibrium threshold).

The unilateral condition (1.2) can describe a source on Γ_+ which prevents a decrease of the value v below zero and a sink on Γ_- which prevents an increase of v above zero. The last line in (1.2) means that the source or the sink is not active in those points of Γ_+ or Γ_- where $v > 0$ or $v < 0$, respectively.

The set D_S contains, in particular, all points $(d_1, d_2) \in \mathbb{R}^2_+$ with $d_1 > b_{11}/\kappa_1$ where κ_1 is the first positive eigenvalue of $-\Delta$ with Neumann boundary conditions so that bifurcations of stationary solutions to (1.1), (1.4) do not occur with $d_1 > b_{11}/\kappa_1$.

The influence of unilateral obstacles to the bifurcation of spatially nonconstant stationary solutions of system (1.1) was studied already in the past, but usually for the case when also a Dirichlet condition is imposed in some part of the boundary (e.g. [2], [21], [15], [6], [9], [23], [24], [10]). It was shown that if a unilateral condition is prescribed for v, then there are bifurcation points also in D_S. However, also in all these results a bifurcation in fact cannot occur if $d_1 > b_{11}/\kappa_1$.

A surprisingly different situation occurs if no Dirichlet boundary data are prescribed and if unilateral conditions of only one sign are imposed for v, e.g., unilateral boundary conditions (1.2) are considered and one of the two sets Γ_+ or Γ_- is empty. It has been shown in [17] that in this case for every sufficiently large d_1, in particular for some $d_1 > b_{11}/\kappa_1$ (in dimension $d = 1$ even for *every* $d_1 > 0$, see [11]), there is some $d_2 > 0$ such that there is a bifurcation of stationary spatially nonconstant solutions of (1.1) with unilateral obstacles at (d_1, d_2). In fact, there are bifurcation points with d_1/d_2 arbitrarily large. By standard arguments (see e.g. [17]) one obtains again that each bifurcation point $(d_1, d_2) \in \mathbb{R}^2_+$ of (1.1) with unilateral conditions (e.g. with (1.2)) is necessarily a critical point, that is, (1.7) with unilateral conditions has a nontrivial solution.

However, the methods used in the cited papers [17], [11] break down if unilateral conditions of opposite sign are given on different parts of the boundary or of the interior, that is, if simultaneously there are unilateral sources and sinks for v, e.g., if both of the sets Γ_+ and Γ_- are nonempty in (1.2). In the current paper, we will show that in this case there are bifurcation (hence critical) points $(d_1, d_2) \in \mathbb{R}^2_+$ of (1.5), (1.2) with any $d_1 > b_{11}/\kappa_1$, but that for obstacles in the interior of Ω, which is modeled in (4.4), it might also happen that there are no such critical points, that is, that (1.7) with unilateral obstacles has only the trivial solution $(u, v) = (0, 0)$ in $W^{1,2}$ for all $d_2 > 0$, $d_1 > b_{11}/\kappa_1$. In fact, using a variational approach, we will be able to give a *necessary and sufficient* criterion for the existence of such critical points. This criterion will relate in a rather implicit manner the geometry and location of the unilateral obstacles with the values of the Jacobi matrix $B := (b_{ij}) = (D_j f_i(0, 0))$.

We emphasize that, although (1.7) is linear, unilateral obstacles are of an inherently nonlinear nature so one cannot expect to use any tools from linear theory or linearization methods. We use variational methods in spite of the fact that the matrix B is non-symmetric because of (1.8), and thus the original problem has no potential. We apply a modification of a trick which was used in a primitive form already in [14], [15], and then for more detailed study of systems with unilateral conditions in [1]. We will work with a weak formulation written as a system of an operator equation and a variational inequality in $W^{1,2}(\Omega)$, we fix an arbitrary $d_1 = d_1^0$ and consider only d_2 as a parameter. Expressing u from the equation and substituting it into the inequality, we get a single variational inequality for v with a potential operator and a parameter d_2. By a variational approach we obtain the maximal bifurcation point d_2^0 of this variational inequality, which is simultaneously the maximal eigenvalue of the inequality with the linearized operator, and consequently $[d_1^0, d_2^0]$ is a critical and simultaneously bifurcation point of the system (1.5) with unilateral conditions. However, in the lack of a Dirichlet condition considered in [14], [15] and [1], this inequality has a structure for which "standard" variational methods for inequalities do not apply, and therefore the situation is more complicated.

Unfortunately, analogously as in [1], the approach mentioned cannot be used for the proof of bifurcation in the case when a nonlinearity appears also in the first equation or if n in the second equation of (1.5) depends also on u. In these cases, even if it were possible to express u from the first equation, the potentiality of the operator obtained would not be clear. So, in general the question whether the critical point obtained by our procedure is simultaneously a bifurcation point of the full system (1.1) with both nonlinear f_1 and f_2 remains open. However, in some particular situations it is known that an eigenvalue of a variational inequality is also a bifurcation point (see [22], [8]) and that a critical point of a unilateral problem for (1.7) is also a bifurcation point of the unilateral problem for (1.1), see [16]. (Sometimes it is also possible to determine the direction of the bifurcation branch, see [7].) In concrete examples discussed in all these papers, a Dirichlet boundary condition on a part of the boundary is considered, which simplifies the situation. However, it seems that also in our case of purely Neumann conditions, the results of the current paper give in fact an information about bifurcations for the general system (1.1), at least for nonlocal (integral) unilateral conditions as in [16] or for the one-dimensional case $d = 1$.

The authors want to thank the referee for valuable suggestions which improved the application enormously. In fact, the result that for unilateral conditions of type (1.2) one has bifurcation points (d_1, d_2) even for every $d_1 \neq b_{11}/\kappa_k$, $k = 1, 2, \ldots$, without any additional condition, uses the observations of the referee.

2. Abstract formulation

Let us assume that n is a continuous function satisfying (1.6) and that there exists $c \in \mathbb{R}$ such that

$$|n(u)| \leqslant c(1 + |u|)^{q-1} \tag{2.1}$$

with some $q > 2$ or $2 < q < 2d/(d-2)$ in the case $d = 2$ or $d > 2$, respectively (in the case $d = 1$, we do not need the hypothesis (2.1) and put $q = \infty$ in the following). We equip the (real) Hilbert space $\mathbb{H} = W^{1,2}(\Omega)$ with the usual scalar product

$$\langle u, \varphi\rangle = \int_\Omega (\nabla u(x) \cdot \nabla \varphi(x) + u(x)\varphi(x))\,\mathrm{d}x \quad \forall\, u, \varphi \in \mathbb{H}, \tag{2.2}$$

and the corresponding norm $\|\varphi\|^2 = \langle \varphi, \varphi\rangle$ and define operators $A, N\colon\ \mathbb{H} \to \mathbb{H}$ by

$$\langle Au, \varphi\rangle = \int_\Omega u(x)\varphi(x)\,\mathrm{d}x \quad \forall\, u, \varphi \in \mathbb{H}, \tag{2.3}$$

$$\langle N(u), \varphi\rangle = \int_\Omega n(u(x))\varphi(x)\,\mathrm{d}x \quad \forall\, u, \varphi \in \mathbb{H}. \tag{2.4}$$

It follows from the compactness of the embedding $\mathbb{H} \hookrightarrow\hookrightarrow L^q(\Omega)$ and the continuity of the Nemyckij operator of $L^q(\Omega)$ into $L^{q^*}(\Omega)$, $1/q + 1/q^* = 1$ (see e.g. [12]) that under the assumption (2.1)

$$\text{(2.5)} \qquad A \text{ is linear, symmetric, positive and compact with the largest simple eigenvalue } 1,$$

$$\text{(2.6)} \qquad N \text{ is nonlinear, continuous and compact.}$$

Furthermore, under the conditions (1.6), (2.1)

$$\text{(2.7)} \qquad N \text{ is Fréchet differentiable at } 0,\ N(0) = 0,\ N'(0) = 0,$$

see e.g. [3]. Moreover, let us introduce the functional $G_N\colon \mathbb{H} \to \mathbb{R}$ by

$$G_N(u) = \int_\Omega \int_0^{u(x)} n(s)\,\mathrm{d}s\,\mathrm{d}x.$$

Under the assumption (2.1), this functional is well defined, Fréchet differentiable, and we have

$$\text{(2.8)} \qquad G'_N(u) = N(u),$$

i.e., G_N is a potential of the operator N.

It is natural to define (weak) solutions of (1.7), (1.4) or (1.5), (1.4) as pairs (u, v) satisfying

$$\text{(2.9)} \qquad \begin{cases} u, v \in \mathbb{H}, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ d_2 v - d_2 Av - b_{21} Au - b_{22} Av = 0 \end{cases}$$

or

$$\text{(2.10)} \qquad \begin{cases} u, v \in \mathbb{H}, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ d_2 v - d_2 Av - b_{21} Au - b_{22} Av - N(v) = 0, \end{cases}$$

respectively. In order to treat the unilateral conditions (1.2), we define the cone

$$\text{(2.11)} \qquad K := \{v \in \mathbb{H}\colon\ v|_{\Gamma_+} \geqslant 0 \text{ and } v|_{\Gamma_-} \leqslant 0\},$$

where the inequalities are understood in the sense of traces. We define correspondingly solutions of the problems (1.7), (1.2) or (1.5), (1.2) as pairs (u, v) satisfying the variational inequalities

$$\begin{cases} u \in \mathbb{H},\ v \in K, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ \langle d_2 v - d_2 Av - b_{21} Au - b_{22} Av, \varphi - v \rangle \geqslant 0 \quad \forall \varphi \in K \end{cases} \tag{2.12}$$

or

$$\begin{cases} u \in \mathbb{H},\ v \in K, \\ d_1 u - d_1 Au - b_{11} Au - b_{12} Av = 0, \\ \langle d_2 v - d_2 Av - b_{21} Au - b_{22} Av - N(v), \varphi - v \rangle \geqslant 0 \quad \forall \varphi \in K, \end{cases} \tag{2.13}$$

respectively. We will actually obtain bifurcation of (2.13) "with fixed d_1" in the following sense:

Definition 2.1. A parameter d_2 is a bifurcation point of (2.13) with fixed d_1 if in any neighborhood of $(d_2, 0, 0)$ in $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$ there is $(\widetilde{d}_2, u, v)$ with $(u, v) \neq (0, 0)$ such that $(d_1, \widetilde{d}_2, u, v)$ satisfies (2.13). We call d_2 a critical point of (2.12) (with fixed d_1) if (2.12) has a solution $(u, v) \neq (0, 0)$.

R e m a r k 2.1. Every bifurcation point (with fixed d_1) is a critical point, see e.g. [2].

Notation 2.1. Let us denote by $0 = \kappa_0 < \kappa_1 \leqslant \kappa_2 \leqslant \ldots$ the eigenvalues of $-\Delta$ with Neumann boundary conditions, counted according to multiplicity, and let e_k $(k = 0, 1, \ldots)$ be a corresponding orthonormal system of eigenvectors in $\mathbb{H}$. With each κ_k $(k = 1, 2, \ldots)$, we associate the hyperbola segment

$$C_k := \Big\{ d = (d_1, d_2) \in \mathbb{R}^2_+ \colon\ d_2 = \frac{b_{12} b_{21} / \kappa_k^2}{d_1 - b_{11}/\kappa_k} + \frac{b_{22}}{\kappa_k} \Big\}.$$

We denote by C_E the envelope of C_k $(k = 1, 2, \ldots)$ and define the domain of stability

$$D_S := \{ d \in \mathbb{R}^2_+ \colon\ d \text{ lies to the right of } C_E, \text{ i.e., of all } C_k,\ k = 1, 2, \ldots \}$$

and the domain of instability

$$D_U := \{ d \in \mathbb{R}^2_+ \colon\ d \text{ lies to the left of } C_E, \text{ i.e., of at least one } C_k \}$$

(see Figure 1).

For any $k = 1, 2, \ldots$, we will denote by $a_k := b_{11}/\kappa_k$ the d_1-coordinate of the vertical asymptote of C_k.

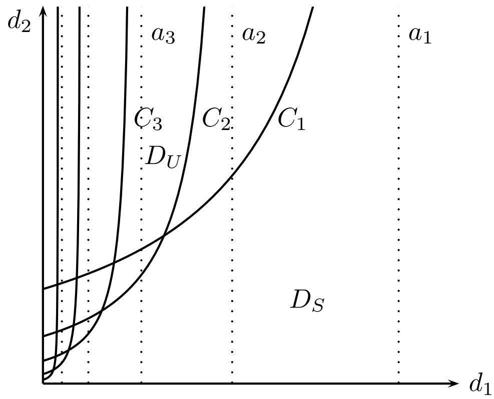

Figure 1. The system of hyperbolas C_k, their asymptotes a_k, domains of stability D_S (to the right of the envelope C_E) and instability D_U (to the left from C_E).

R e m a r k 2.2. The above definition of the domains D_S and D_U of stability and instability indeed corresponds to the domains for which $(0,0)$ is a linearly stable or unstable, respectively, solution of (1.1), (1.4). Actually, for $(d_1,d_2) \in D_S$, the solution $(0,0)$ of (1.1), (1.4) is even exponentially asymptotically stable in $\mathbb{H} \times \mathbb{H}$, see e.g. [25].

R e m a r k 2.3. The hyperbolas C_k have a natural interpretation. They consist exactly of those points $(d_1,d_2) \in \mathbb{R}^2_+$ for which (2.9) has a nontrivial solution $(u,v) \neq (0,0)$. More precisely, if $V(d_1,d_2)$ denotes the space of all linear combinations of e_k where k is such that $(d_1,d_2) \in C_k$, then for each $v \in V(d_1,d_2)$ there is some $u \in V(d_1,d_2)$ such that (u,v) is a solution of (2.9), and conversely, all solutions of (2.9) have such a form, see e.g. [9].

R e m a r k 2.4. The eigenvalues of the operator A from (2.3) are of the form $\lambda_k = 1/(1+\kappa_k)$ for $k = 0,1,\ldots$, and the corresponding eigenspaces are the eigenspaces of $-\Delta$ with Neumann boundary conditions to the eigenvalues κ_k.

3. Main results

In this section we will consider a general Hilbert space $\mathbb{H}$ with the scalar product $\langle\cdot,\cdot\rangle$ and a closed convex cone K with its vertex at the origin in $\mathbb{H}$. We will discuss the variational inequalities (2.12) and (2.13) with general operators $A, N\colon \mathbb{H} \to \mathbb{H}$ satisfying (2.5), (2.6), (2.7) with N having a potential G_N, i.e., (2.8) holds. The condition (1.8) will be always assumed.

Let $1 = \lambda_0 > \lambda_1 \geqslant \ldots > 0$ be the eigenvalues of A, counted according to multiplicity, and let $e_0, e_1, \ldots$ be a corresponding orthonormal system of eigenfunctions. In accordance with Remark 2.4, we use the notation $\kappa_k := \lambda_k^{-1} - 1$. For d_1 from

$$D_1 := \{d_1 > 0\colon\ d_1 \neq a_k = b_{11}/\kappa_k \quad \forall\, k = 1, 2, \ldots\}$$

let us define the auxiliary functions

$$c_k(d_1) := \frac{1}{1+\kappa_k}\Big(\frac{b_{12}b_{21}}{\kappa_k d_1 - b_{11}} + b_{22}\Big) = \frac{b_{22}\kappa_k d_1 - \det B}{(1+\kappa_k)(\kappa_k d_1 - b_{11})}. \tag{3.1}$$

Remark 3.1. Clearly, $c_0(d_1) = \det B/b_{11} > 0$ is actually independent of d_1. There holds $c_k(d_1) < 0$ if $d_1 > a_k$, and $c_k(d_1) > 0$ if $d_1 < a_k$.

Theorem 3.1. *Let $e_0 \notin K \cup (-K)$, and $d_1 \in D_1$. Then* (2.12) *has a critical point $d_2 > 0$ (with fixed d_1) if and only if*

$$\text{there is } v \in K \text{ with } \sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2 > 0. \tag{3.2}$$

Moreover, in this case $\langle (I-A)v, v\rangle > 0$ for every $v \in K \setminus \{0\}$, and

$$\begin{aligned} d_2^{\max} &:= \max_{v \in K\setminus\{0\}} \sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2 / \langle (I-A)v, v\rangle \\ &= \max_{\substack{v\in K \\ \langle (I-A)v,v\rangle = 1}} \sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2 \in (0, \infty) \end{aligned} \tag{3.3}$$

is the maximal critical point of (2.12) *and simultaneously the maximal bifurcation point of* (2.13) *with fixed d_1.*

Let us note that $d_2^{\max}$ is in fact $\max\langle Sv, v\rangle$ over all $v \in K$ with $\langle (I-A)v, v\rangle = 1$, where S is a symmetric operator which we will use to reduce our problem to a variational setting (see Lemma 5.1).

We postpone the proof of Theorem 3.1 and of the subsequent Propositions 3.1 and 3.2 to Section 5.

Proposition 3.1. *Suppose $e_0 \notin K \cup (-K)$ and $d_1 \in D_1$.*

If (3.2) *holds and v is a corresponding maximizer of $d_2 = d_2^{\max} > 0$ in* (3.3)*, then there is a uniquely determined u such that (u, v) is a nontrivial solution of* (2.12).

Conversely, if there is a positive value $d_2 > 0$ such that there is a nontrivial solution (u, v) of (2.12)*, then $d_2^{\max}$ is maximal such value. If (u, v) is a nontrivial solution of* (2.12) *with $d_2 = d_2^{\max}$, then $v \neq 0$ is a maximizer of* (3.3) *(after appropriate scaling).*

Proposition 3.2. *Let $e_0 \notin K \cup (-K)$. Then the set $D_{1,0}$ of all $d_1 \in D_1$ satisfying* (3.2) *is open, and the quantity $d_2^{\max}$ from* (3.3) *depends continuously on $d_1 \in D_{1,0}$ and tends to* 0 *if d_1 tends to some element from $D_1 \setminus D_{1,0}$.*

Remark 3.2. Suppose $d_1 \in D_1$, $d_1 < a_1$, and $e_0 \notin K \cup (-K)$. Let $d_2 > 0$ be such that (d_1, d_2) belongs to at least one of the hyperbolas C_k. Let $V(d_1, d_2)$ denote the corresponding set from Remark 2.3, and suppose that there are $v \in V(d_1, d_2)$ and $\lambda \in \mathbb{R}$ with $v + \lambda e_0 \in K \setminus \{0\}$. Then $(d_1, d_2^{\max})$ cannot lie below the hyperbola C_k, more precisely,

$$(3.4)\quad d_2^{\max} \geqslant d_2 + \sup\Big\{\frac{\lambda^2 \det B}{b_{11}\langle (I-A)v, v\rangle} \colon\ v \in V(d_1, d_2),\ \lambda \in \mathbb{R},\ v + \lambda e_0 \in K \setminus \{0\}\Big\},$$

where the fraction is automatically defined and nonnegative for every corresponding (v, λ).

Indeed, let $v + \lambda e_0 \in K \setminus \{0\}$. Since $e_0 \notin K \cup (-K)$, we have $v \neq 0$. Applying Theorem 3.1 and the second part of Proposition 3.1 with K replaced by $K_0 = V(d_1, d_2)$, we find that $\langle (I-A)v, v\rangle > 0$ and that v is a maximizer of (3.3) (with K replaced by K_0), that is

$$d_2 = \frac{\sum\limits_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2}{\langle (I-A)v, v\rangle}.$$

If we replace v on the right-hand side by $\widetilde{v} = v + \lambda e_0$, we have in view of $Ae_0 = e_0$ that $\langle (I-A)\widetilde{v}, \widetilde{v}\rangle = \langle (I-A)v, v\rangle$, and in view of $c_0(d_1) = \det B / b_{11} > 0$ the sum increases by $c_0(d_1)\lambda^2$, which shows (3.4).

Remark 3.3. In the case $d_1 > a_1$, the points $(d_1, d_2^{\max})$ obtained from Theorem 3.1 automatically belong to the set D_S, in which the corresponding classical linear problem (2.9) cannot have a nontrivial solution and the nonlinear problem (2.10) cannot have a bifurcation.

In contrast, if $d_1 \in D_1$ satisfies $d_1 < a_1$, in view of (3.4), it happens for many cones from applications (cf. Section 4, depending on the location of the obstacles) that the point $(d_1, d_2^{\max})$ which one obtains from Theorem 3.1 satisfies $d_2^{\max} \geqslant d_2$ or even $d_2^{\max} > d_2$ with some $(d_1, d_2) \in C_k$, and so $(d_1, d_2^{\max}) \notin D_S$. This does not mean that there cannot be any bifurcation point in D_S with $d_1 < a_1$. It just means that Theorem 3.1 typically cannot be used to find these points, because by its very nature Theorem 3.1 only gives the point with the largest d_2-coordinate.

For this reason, we are mainly interested in the case $d_1 > a_1$ in the subsequent discussion and examples.

Remark 3.4. It is remarkable that for the case when we assume a Dirichlet condition (instead of a Neumann condition) on some parts of the boundary, one has an opposite situation compared to Remark 3.3 and (3.4): In this case, the maximal bifurcation point $(d_1, d_2^{\max})$ of the problem with unilateral condition satisfies always $d_2^{\max} \leqslant d_2$ where d_2 is the maximal value satisfying $(d_1, d_2) \in C_k$ with some k, independently of the cone K, see [1]. Moreover, in this case, and if $V(d_1, d_2) \cap K = \{0\}$, the inequality is even strict [1]. The explanation for this difference is that in our case a special role is played by e_0 for which there is no analogue in the Dirichlet case.

For $d_1 \in D_1$, we denote by $m(d_1)$ the largest integer $k \geqslant 1$ such that $d_1 < a_k = b_{11}/\kappa_k$. If no such integer exists, that is, if $d_1 > a_1$, we put $m(d_1) := 0$.

Proposition 3.3. *Let $d_1 \in D_1$. If there is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0\rangle = 0$ such that*

$$\sum_{k=m(d_1)+1}^{\infty} |c_k(d_1)||\langle u, e_k\rangle|^2 < c_0(d_1) = \frac{\det B}{b_{11}}, \tag{3.5}$$

then the condition (3.2) *is satisfied. In the case $d_1 > a_1$, the existence of such u is also necessary for* (3.2).

Proof. Recall that by Remark 3.1, we have $c_k(d_1) < 0 < c_j(d_1)$ for all $k > m(d_1) \geqslant j$. Hence, putting $v := u + e_0$ or $v := u - e_0$ (choosing the sign such that $v \in K$), we obtain

$$\sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2 \geqslant c_0(d_1) - \sum_{k=m(d_1)+1}^{\infty} |c_k(d_1)||\langle u, e_k\rangle|^2, \tag{3.6}$$

and the latter is positive if and only if (3.5) holds. In particular, (3.5) implies (3.2). In the case $m(d_1) = 0$, we have equality in (3.6), and the only positive summand in (3.2) can be the first. Hence, if (3.2) holds and $m(d_1) = 0$, we have necessarily $\langle v, e_0\rangle \neq 0$, by scaling without loss of generality $|\langle v, e_0\rangle| = 1$, and so v has the form $v = u + e_0$ or $v = u - e_0$ with $\langle u, e_0\rangle = 0$. Now the above calculation shows that u satisfies (3.5). ☐

Due to Theorem 3.1, we are only interested in the case $e_0 \notin K \cup (-K)$. In this case, we cannot choose $u = 0$ in Proposition 3.3, and it is a question of the interplay of the geometry of K and of the matrix $B = (b_{ij})$ (which determines the values $c_j(d_1)$) whether a fixed parameter $d_1 \in D_1$ satisfies (3.2). We will see in Sections 4.1 and 4.2 that there are indeed examples in which (3.2) can hold or be violated for some or all $d_1 > a_1$, respectively. The case $d_1 > a_1$ is here of a particular interest to us for the reasons described in Remark 3.3.

Corollary 3.1. *Suppose that $e_0 \notin K \cup (-K)$. Then the set of all $d_1 > a_1$ for which* (3.2) *holds is either empty or an interval of the form $(d_{1,0}, \infty)$ with $d_{1,0} \in [a_1, \infty)$. The value $d_2^{\max}$ from* (3.3) *is a strictly increasing continuous function of $d_1 \in (d_{1,0}, \infty)$, and*

$$d_2^{\max} \leqslant c_0(d_1) \sup_{v \in K \setminus \{0\}} \frac{|\langle v, e_0\rangle|^2}{\langle (I-A)v, v\rangle} = \frac{\det B}{b_{11}} \sup_{\substack{v \in K \\ \langle (I-A)v, v\rangle = 1}} |\langle v, e_0\rangle|^2 < \infty \tag{3.7}$$

for every $d_1 > d_{1,0}$. In the case $d_{1,0} > a_1$, we have $d_2^{\max} \to 0$ as $d_1 \to d_{1,0}$.

P r o o f. The functions $c_k(d_1) < 0$ for $k \geqslant 1$ are strictly increasing with respect to d_1 on (a_1, ∞), and $c_0(d_1) > 0$ is independent of d_1. Hence, if (3.5) holds with $d_1 > a_1$, it holds for all larger values of d_1 as well. Moreover, if v denotes a fixed corresponding maximizer of (3.3), then, since e_n form a complete orthonormal system of $\mathbb{H}$ and v is not a multiple of e_0, we must have $\langle v, e_k\rangle \neq 0$ for some $k \geqslant 1$. With this fixed v, it follows that the maximum in (3.3) must be strictly greater if d_1 is replaced by $\widetilde{d_1} \in (a_1, d_1)$.

In view of Proposition 3.3, we conclude that if (3.2) holds with $d_1 > a_1$, then it holds for all larger values of d_1 as well, and $d_2^{\max}$ is strictly increasing as a function of d_1. The estimate (3.7) follows from $c_k(d_1) < 0$ for $k \geqslant 1$. The finiteness of (3.7) will be shown in Lemma 5.3.

The remaining assertions follow in view of Proposition 3.2. □

The following result gives an exhaustive answer to the question whether the interval from Corollary 3.1 is empty.

Proposition 3.4. *There is $d_1 > a_1$ satisfying* (3.2) *if and only if*

$$\inf_{\substack{u \in (K+e_0) \cup (K-e_0) \\ \langle u, e_0\rangle = 0}} \langle Au, u\rangle < \frac{b_{12}b_{21}}{b_{11}b_{22}} - 1 \quad \left(= \left|\frac{\det B}{b_{11}b_{22}}\right|\right). \tag{3.8}$$

P r o o f. Since (e_k) forms a complete orthonormal basis, we can write every $u \in \mathbb{H}$ as a series $u = \sum\limits_{k=0}^{\infty} \mu_k e_k$ with $\mu_k = \langle u, e_k\rangle$; using the fact that $Ae_k = \lambda_k e_k$, we obtain

$$\langle Au, u\rangle = \sum_{k=0}^{\infty} \lambda_k |\langle u, e_k\rangle|^2 = |\langle u, e_0\rangle|^2 + \sum_{k=1}^{\infty} \frac{1}{1+\kappa_k} |\langle u, e_k\rangle|^2. \tag{3.9}$$

Hence, (3.8) holds if and only if there is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0\rangle = 0$ and

$$\sum_{k=1}^{\infty} \frac{1}{1+\kappa_k} |\langle u, e_k\rangle|^2 < \frac{b_{12}b_{21}}{b_{11}b_{22}} - 1. \tag{3.10}$$

Let $d_1 > a_1$ satisfy (3.2), and let u be the function from Proposition 3.3. It follows from (3.1) and the assumption (1.8) that $|c_k(d_1)| > -b_{22}/(1+\kappa_k)$ for $k = 1, 2, \ldots$, inserting into (3.5) and dividing by $(-b_{22})$, we obtain (3.10). Thus (3.8) holds. Conversely, if (3.8) holds then there is $u \in (K+e_0) \cup (K-e_0)$ with $\langle u, e_0 \rangle = 0$ such that (3.10) holds. Since the difference

$$\left| |c_k(d_1)| - \frac{-b_{22}}{1+\kappa_k} \right| < \frac{1}{1+\kappa_1} \cdot \frac{|b_{12}b_{21}|}{|\kappa_1 d_1 - b_{11}|}$$

tends to zero as $d_1 \to \infty$ uniformly in $k = 1, 2, \ldots$, then also (3.5) holds for all large d_1 and the first part of Proposition 3.3 implies (3.2). $\square$

If we are interested in the existence of a bifurcation point for *every* $d_1 \in D_1$, we can use the following sufficient condition which was pointed out to us by the referee.

Proposition 3.5. *If $d_1 \in D_1$ is such that*

$$\text{(3.11)} \qquad \inf_{\substack{u \in (K+e_0) \cup (K-e_0) \\ \langle u, e_0 \rangle = 0}} \langle Au, u \rangle < \left(|b_{22}| + \left| \frac{b_{12}b_{21}}{\kappa_{m(d_1)+1} d_1 - b_{11}} \right| \right)^{-1} \frac{\det B}{b_{11}}$$
$$\left(= \left(|b_{11}b_{22}| + \left| \frac{b_{12}b_{21}}{a^{-1}_{m(d_1)+1} d_1 - 1} \right| \right)^{-1} \det B \right),$$

then condition (3.2) *is fulfilled. In particular, if*

$$\text{(3.12)} \qquad \inf_{\substack{u \in (K+e_0) \cup (K-e_0) \\ \langle u, e_0 \rangle = 0}} \langle Au, u \rangle = 0,$$

then the condition (3.2) *is satisfied for every* $d_1 \in D_1$.

P r o o f. Denote the term in the first brace on the right-hand side of (3.11) by C and note that $C > 0$. Then we have for every $k > m(d_1)$ that $|c_k(d_1)| \leqslant C/(1+\kappa_k)$. For any $u \in \mathbb{H}$ we obtain by using (3.9) that

$$C\langle Au, u \rangle \geqslant C \sum_{k=m(d_1)+1}^{\infty} \frac{1}{1+\kappa_k} |\langle u, e_k \rangle|^2 \geqslant \sum_{k=m(d_1)+1}^{\infty} |c_k(d_1)| |\langle u, e_k \rangle|^2,$$

and so (3.11) implies (3.5). Hence the assertion follows from Proposition 3.3. $\square$

4. Examples

4.1. Unilateral conditions on the boundary. We consider the problem (1.5), (1.2) of Section 1. Assume the sign condition (1.8) and let the nonlinearity n satisfy (1.6) and the growth condition (2.1). In fact we have in mind the weak formulation, that is, the variational inequality (2.13) with the operators (2.3), (2.4) and with the cone (2.11). Note that e_0 is the eigenfunction of $-\Delta$ to the eigenvalue $\kappa_0 = 0$, that is, a constant. Without loss of generality, we can assume that it is positive, that is,

$$e_0(x) = \frac{1}{\sqrt{\operatorname{mes} \Omega}}. \tag{4.1}$$

Hence, the hypothesis $e_0 \notin K \cup (-K)$ of Theorem 3.1 holds in view of (1.3).

Our theory implies that for every $d_1 > 0$, $d_1 \neq a_k$ $(k = 1, 2, \ldots)$ there is a bifurcation point $d_2 > 0$ for fixed d_1, the maximal such bifurcation point is $d_2 = d_2^{\max} > 0$ from Theorem 3.1, and this is simultaneously the maximal criticial point. This follows from Theorem 3.1 by using Lemma 4.1 proved below and Proposition 3.5. The values $d_2^{\max}$ are for $d_1 \in (a_1, \infty)$ continuous and strictly increasing with respect to d_1 by Corollary 3.1, and bounded from above by (3.7).

Let us emphasize that for $d_1 > a_1$ we have automatically $(d_1, d_2) \in D_S$ for all $d_2 > 0$. Hence we get a bifurcation point of the unilateral problem in the domain where bifurcation and critical points of the classical problem (1.5), (1.4) are excluded.

In the case $d_1 < a_1$, if there are $d_2 > 0$ with $(d_1, d_2) \in C_k$, $v \in V(d_1, d_2)$ (the set from Remark 2.3), $v \neq 0$, and $\lambda \in \mathbb{R}$ satisfying $v|_{\Gamma_+} \geqslant \lambda \geqslant v|_{\Gamma_-}$, then $(d_1, d_2^{\max}) \notin D_S$. This is a consequence of Remark 3.2. Moreover, if one can choose $\lambda \neq 0$, then $d_2^{\max} > d_2$ by (3.4). In particular, if $d_2 > 0$ is the maximal value with $(d_1, d_2) \in C_k$ for some k, then we can conclude that $(d_1, d_2^{\max}) \notin \bigcup\limits_{k=1}^{\infty} C_k$, and so in this case we obtain a bifurcation point of (1.5), (1.2) which is no bifurcation point of the classical problem (1.5), (1.4). Due to Proposition 3.2, the value $d_2^{\max}$ again depends continuously on d_1, $d_1 \neq a_k$ $(k = 1, 2, \ldots)$.

When one is interested in calculating or estimating $d_2^{\max}$ explicitly, it might be worth to note that

$$\langle (I - A)v, v\rangle = \int_\Omega |\nabla v|^2 \, \mathrm{d}x. \tag{4.2}$$

Lemma 4.1. *If K denotes the cone* (2.11) *and A the operator* (2.3), *then* (3.12) *holds.*

P r o o f. We note that our choice of A implies

$$\inf_{\substack{u\in(K+e_0)\cup(K-e_0)\\ \langle u,e_0\rangle=0}} \langle Au,u\rangle = \inf_{\substack{u\in(K+e_0)\cup(K-e_0)\\ \int_\Omega u(x)\,\mathrm{d}x=0}} \int_\Omega u(x)^2\,\mathrm{d}x. \tag{4.3}$$

Fix some $v\in\mathbb{H}$ with $\operatorname{supp} v\subseteq\Omega$ and $\int_\Omega v(x)\,\mathrm{d}x=1$. For any $\varepsilon>0$, there are an open set $\Omega_\varepsilon\subseteq\mathbb{R}^d$ with $\Gamma_+\subseteq\Omega_\varepsilon$, $\overline{\Omega}_\varepsilon\cap\operatorname{supp} v=\emptyset$, $\operatorname{mes}\overline{\Omega}_\varepsilon<\varepsilon$ and a function $u_\varepsilon\in\mathbb{H}$ such that

$$u_\varepsilon=e_0 \text{ on } \Gamma_+,\quad u_\varepsilon=0 \text{ on } \Omega\setminus\Omega_\varepsilon,\quad 0\leqslant u_\varepsilon\leqslant e_0 \text{ on } \Omega.$$

For instance, for sufficiently small $\delta>0$, one can let u_ε be a suitable standard mollification of a multiple of the characteristic function χ_δ of the set $\{x\in\mathbb{R}^d\colon \operatorname{dist}(x,\Gamma_+)<\delta\}$. More precisely, we choose a smooth function $\varphi_\delta\colon\mathbb{R}^d\to[0,\infty)$ with integral 1 and support in $\{x\colon \|x\|<\delta\}$ (a mollifier), and put

$$v_\delta(x):=\frac{1}{\sqrt{\operatorname{mes}\Omega}}\int_{\mathbb{R}^d}\varphi_\delta(x)\chi_\delta(x-y)\,\mathrm{d}y\quad \forall\, x\in\mathbb{R}^d.$$

Then we can take $u_\varepsilon\in\mathbb{H}$ as the restriction of v_δ and Ω_ε as a sufficiently small neighborhood of the support of v_δ.

Put $c:=\int_\Omega u_\varepsilon(x)\,\mathrm{d}x$. Then $u=u_\varepsilon-cv\in K+e_0$ satisfies $\int_\Omega u(x)\,\mathrm{d}x=0$ and

$$\int_\Omega u(x)^2\,\mathrm{d}x\leqslant\int_{\Omega\cap\overline{\Omega}_\varepsilon} e_0(x)^2\,\mathrm{d}x+c^2\int_{\operatorname{supp} v} v(x)^2\,\mathrm{d}x\leqslant\frac{\varepsilon}{\operatorname{mes}\Omega}+\frac{\varepsilon^2}{\operatorname{mes}\Omega}\int_\Omega v(x)^2\,\mathrm{d}x.$$

Since $v\in\mathbb{H}$ was fixed and $\varepsilon>0$ arbitrary, (3.12) follows. □

4.2. Unilateral conditions in the interior. Let us consider now unilateral obstacles describing sources and sinks in the interior of Ω. Let $\Omega_\pm\subseteq\Omega$ be nonempty open subsets such that $\overline{\Omega}_+\cap\overline{\Omega}_-=\emptyset$ and (for simplicity) $\overline{\Omega}_\pm\cap\partial\Omega=\emptyset$ and $\operatorname{mes}(\partial\Omega_\pm)=0$ (the d-dimensional Lebesgue measure). We consider now the problem

$$\begin{aligned}
&d_1\Delta u+b_{11}u+b_{12}v=0 && \text{in } \Omega,\\
&d_2\Delta v+b_{21}u+b_{22}v+n(v)=0 && \text{in } \Omega\setminus(\Omega_+\cup\Omega_-),\\
&\pm\,(d_2\Delta v+b_{21}u+b_{22}v+n(v))\leqslant 0,\quad \pm\, v\geqslant 0 && \text{in } \Omega_\pm,\\
&(d_2\Delta v+b_{21}u+b_{22}v+n(v))v=0 && \text{in } \Omega_\pm
\end{aligned}\tag{4.4}$$

with Neumann boundary conditions (1.4). It describes a situation when there is a source on Ω_+ which prevents a decrease of the value v below zero and a sink on Ω_- which prevents an increase of v above zero. The last line in (4.4) means that the source or the sink is not active in the points of Ω_+ or Ω_- where $v > 0$ or $v < 0$, respectively.

Assume that the sign condition (1.8) is fulfilled and that the nonlinearity n satisfies (1.6) and the growth condition (2.1). The weak formulation of (4.4), (1.4) is again (2.13) with the same operators as in Section 4.1, but with the cone

$$K := \{v \in \mathbb{H}\colon\ v|_{\Omega_+} \geqslant 0 \text{ and } v|_{\Omega_-} \leqslant 0\}. \tag{4.5}$$

Lemma 4.2. *Let K denote the cone* (4.5), *and A the operator* (2.3). *Then*

$$\inf_{\substack{u\in(K+e_0)\cup(K-e_0)\\ \langle u,e_0\rangle=0}} \langle Au, u\rangle = \min\Big\{\frac{\operatorname{mes}\Omega_+}{\operatorname{mes}(\Omega\setminus\Omega_+)}, \frac{\operatorname{mes}\Omega_-}{\operatorname{mes}(\Omega\setminus\Omega_-)}\Big\}. \tag{4.6}$$

The proof of Lemma 4.2 will be given later. First, we will summarize what our theory implies for the problem (4.4), (1.4).

It follows from Theorem 3.1 that the set $D_{1,0}$ from Proposition 3.2 coincides with the set of all $d_1 > 0$, $d_1 \neq a_k$ $(k = 1, 2, \ldots)$ for which there is a critical point $d_2 > 0$ (with fixed d_1) of (4.4), (1.4). For $d_1 \in D_{1,0}$, the maximal critical point is simultaneously the maximal bifurcation point and equals to $d_2^{\max}$ from Theorem 3.1. It follows from Proposition 3.2 that the set $D_{1,0}$ is open, and the function $d_2^{\max}$ depends continuously on $d_1 \in D_{1,0}$. The quantity $d_2^{\max}$ can be written in a more concrete form by using the formula (4.2).

Furthermore, let us show that if

$$\frac{\min\{\operatorname{mes}\Omega_+, \operatorname{mes}\Omega_-\}}{\operatorname{mes}\Omega} < \Big(1 + \Big|\frac{b_{11}b_{22}}{\det B}\Big|\Big)^{-1}, \tag{4.7}$$

then $D_{1,0} \cap (a_1, \infty) \neq \emptyset$, and conversely, if (4.7) is violated, then $D_{1,0} \cap (a_1, \infty) = \emptyset$. Indeed, the formula (4.6) of Lemma 4.2 shows after some calculation that (4.7) is equivalent to (3.8), so Proposition 3.4 implies that (4.7) is equivalent to $D_{1,0} \cap (a_1, \infty) \neq \emptyset$.

Let us emphasize that in the case (4.7), the corresponding bifurcation/critical points (d_1, d_2) with $d_1 > a_1$ necessarily belong to the set D_S, in which the trivial solution of the evolution problem (1.1) with Neumann conditions (1.4) is linearly stable.

It is remarkable that for any fixed Ω and $\Omega_\pm$ both of the cases ($D_{1,0} \cap (a_1, \infty)$ being nonempty or empty, that is, (4.7) being satisfied or violated) actually do occur for many matrices $B = (b_{ij})$.

If (4.7) is satisfied, i.e., $D_{1,0} \cap (a_1, \infty) \neq \emptyset$, we obtain from Corollary 3.1 that $D_{1,0} \cap (a_1, \infty) = (d_{1,0}, \infty)$ with some $d_{1,0} \in [a_1, \infty)$. In this case the function $d_2^{\max}$

is continuous and strictly monotone with respect to $d_1 \in (d_{1,0}, \infty)$ and is bounded by (3.7), and in the case $d_{1,0} > a_1$ we have $d_2^{\max} \to 0$ as $d_1 \to d_{1,0}$.

If $d_1 > 0$ satisfies $d_1 \neq a_k$ $(k = 1, 2, \dots)$ and

$$\frac{\min\{\operatorname{mes}\Omega_+, \operatorname{mes}\Omega_-\}}{\operatorname{mes}\Omega} < \left(1 + \frac{1}{\det B}\Big(|b_{11}b_{22}| + \Big|\frac{b_{12}b_{21}}{a_{m(d_1)+1}^{-1}d_1 - 1}\Big|\Big)\right)^{-1}, \tag{4.8}$$

then $d_1 \in D_{1,0}$ by Proposition 3.5 and Theorem 3.1. Indeed, the formula (4.6) of Lemma 4.2 shows after some calculation that (4.8) is equivalent to (3.11).

Whenever $d_1 \in D_{1,0}$ satisfies $d_1 < a_1$, Remark 3.2 implies the following assertion: If $d_2 > 0$ is such that (d_1, d_2) belongs to at least one of the hyperbolas C_k $(k = 1, 2, \dots)$ and if the set $V(d_1, d_2)$ from Remark 2.3 contains a function $v \neq 0$ satisfying $v|_{\Omega_+} \geqslant \lambda \geqslant v|_{\Omega_-}$ with some number $\lambda \in \mathbb{R}$, then the largest bifurcation/critical point $d_2^{\max}$ of (4.4), (1.4) satisfies $d_2^{\max} \geqslant d_2$, and even $d_2^{\max} > d_2$ if one can choose $\lambda \neq 0$.

P r o o f of Lemma 4.2. Note that (4.3) holds by our choice of A. If $u \in K + e_0$, then we have by (4.1) that

$$\int_{\Omega_+} u(x)^2 \, \mathrm{d}x \geqslant \int_{\Omega_+} e_0(x)^2 \, \mathrm{d}x \geqslant \frac{\operatorname{mes}\Omega_+}{\operatorname{mes}\Omega}.$$

If additionally $\int_\Omega u(x)\, \mathrm{d}x = 0$, then we have with $\Omega_0 := \Omega \setminus \Omega_+$ that

$$\frac{\operatorname{mes}\Omega_+}{(\operatorname{mes}\Omega)^{1/2}} \leqslant \int_{\Omega_+} u(x)\, \mathrm{d}x = -\int_{\Omega_0} u(x)\cdot 1 \, \mathrm{d}x \leqslant \left(\int_{\Omega_0} u(x)^2 \, \mathrm{d}x\right)^{1/2} (\operatorname{mes}\Omega_0)^{1/2}$$

by Hölder's inequality, and so

$$\langle Au, u\rangle = \int_{\Omega_+} u(x)^2 \, \mathrm{d}x + \int_{\Omega_0} u(x)^2 \, \mathrm{d}x \geqslant \frac{\operatorname{mes}\Omega_+}{\operatorname{mes}\Omega} + \frac{(\operatorname{mes}\Omega_+)^2}{\operatorname{mes}\Omega \operatorname{mes}\Omega_0} = \frac{\operatorname{mes}\Omega_+}{\operatorname{mes}\Omega_0}.$$

For $u \in K - e_0$, one obtains an analogous estimate with exchanged roles of Ω_+ and Ω_-. This proves "$\geqslant$" in (4.6). To prove the converse inequality, we assume first that the minimum in (4.6) is given by the first expression. Given $\varepsilon > 0$, we fix some $v \in \mathbb{H}$, $v \geqslant 0$ such that $\operatorname{supp} v \subseteq \Omega_0 := \Omega \setminus \overline{\Omega}_+$, $\operatorname{mes}(\Omega_0 \setminus \operatorname{supp} v)$ sufficiently small, v a positive constant in $\widetilde{\Omega}_0 \subseteq \Omega_0$, $\operatorname{mes}(\Omega_0 \setminus \widetilde{\Omega}_0)$ small, and

$$\int_\Omega v(x)\, \mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega v(x)^2 \, \mathrm{d}x \leqslant \frac{1+\varepsilon}{\operatorname{mes}\Omega_0}.$$

There is an open set $\Omega_\varepsilon \subseteq \Omega$ containing $\overline{\Omega}_+$ with $\operatorname{mes}\Omega_\varepsilon < \operatorname{mes}\Omega_+ + \varepsilon$ and $\overline{\Omega}_\varepsilon \cap \overline{(\Omega_- \cup \operatorname{supp} v)} = \emptyset$ and $u_\varepsilon \in \mathbb{H}$ with

$$u_\varepsilon = e_0 \text{ on } \Omega_+, \quad u_\varepsilon = 0 \text{ outside } \Omega_\varepsilon, \quad \text{and} \quad 0 \leqslant u_\varepsilon \leqslant e_0 \text{ on } \Omega.$$

Hence, $u_\varepsilon \in K + e_0$, $\operatorname{supp} u_\varepsilon \subseteq \Omega_\varepsilon$, and $|u_\varepsilon| \leqslant e_0$. Putting $c := \int_\Omega u_\varepsilon(x)\,\mathrm{d}x$ and $u := u_\varepsilon - cv$, we have then $u \in K + e_0$, $\int_\Omega u(x)\,\mathrm{d}x = 0$, and

$$\begin{aligned}\int_\Omega u(x)^2\,\mathrm{d}x &\leqslant \int_{\Omega_\varepsilon} e_0(x)^2\,\mathrm{d}x + c^2 \int_{\operatorname{supp} v} v(x)^2\,\mathrm{d}x \\ &\leqslant \frac{\operatorname{mes}\Omega_+ + \varepsilon}{\operatorname{mes}\Omega} + \frac{(\operatorname{mes}\Omega_+ + \varepsilon)^2}{\operatorname{mes}\Omega} \cdot \frac{1+\varepsilon}{\operatorname{mes}\Omega_0}.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain "$\leqslant$" in (4.6). For the case when the minimum in (4.6) is given by the second expression, the proof is analogous by exchanging the roles of Ω_+ and Ω_-, and by putting $u := u_\varepsilon + cv$. $\square$

Remark 4.1. Our proof shows that if we drop the hypothesis $\operatorname{mes}(\partial\Omega_\pm) = 0$, then the infimum in (4.6) remains bounded from below by the right-hand side, but it is bounded from above only by

$$\min\Big\{\frac{\operatorname{mes}\Omega_+}{\operatorname{mes}\Omega}\Big(1 + \frac{\operatorname{mes}\Omega_+}{\operatorname{mes}(\Omega \setminus \overline{\Omega}_+)}\Big), \frac{\operatorname{mes}\Omega_-}{\operatorname{mes}\Omega}\Big(1 + \frac{\operatorname{mes}\Omega_-}{\operatorname{mes}(\Omega \setminus \overline{\Omega}_-)}\Big)\Big\}.$$

Remark 4.2. The assertion of Remark 4.1 holds also in the case $\Omega_- = \emptyset$ or $\Omega_+ = \emptyset$. This can be used to strengthen the assertion of [17], Example 2.3, slightly if one assumes that the set Ω_0 (which takes there the role of one of our sets $\Omega_\pm$) is open. In this case, our proof shows that the hypothesis (2.10) from [17] can be relaxed to

$$0 < \frac{\operatorname{mes}\Omega_0}{\operatorname{mes}\Omega}\Big(1 + \frac{\operatorname{mes}\Omega_0}{\operatorname{mes}(\Omega \setminus \overline{\Omega}_0)}\Big) < \Big(\frac{b_{12}b_{21}}{b_{11}b_{22}} - 1\Big)^2,$$

which in the case $\operatorname{mes}(\partial\Omega_0) = 0$ simplifies after some calculation to

$$0 < \frac{\operatorname{mes}\Omega_0}{\operatorname{mes}\Omega} < \Big(1 + \frac{(b_{11}b_{22})^2}{(\det B)^2}\Big)^{-1}.$$

5. Proof of the main results

Let $\sigma(A)$ denote the spectrum of A. Since A is compact, $\sigma(A)$ consists of all eigenvalues of A and of the value 0. For fixed $d_1 \in D_1$, we define the auxiliary function $f\colon \sigma(A) \to \mathbb{R}$ by

$$f(\lambda) := \frac{b_{12}b_{21}\lambda^2}{d_1 - (b_{11} + d_1)\lambda} + b_{22}\lambda.$$

Note that we have

$$f(0) = 0 \quad \text{and} \quad f(\lambda_k) = c_k(d_1) \quad \text{for } k = 0, 1, \ldots \tag{5.1}$$

Since A is a symmetric operator in $\mathbb{H}$, we can define a selfadjoint operator $S := f(A)$ in the usual way by means of spectral calculus of symmetric operators.

Lemma 5.1. *For $d_1 \in D_1$ and*

$$S = f(A) = b_{12}b_{21}A(d_1(I + A) - b_{11}A)^{-1}A + b_{22}A \tag{5.2}$$

the variational inequality (2.13) *is equivalent to*

$$v \in K, \quad \langle d_2(I - A)v - Sv - N(v), \varphi - v\rangle \geqslant 0 \quad \forall\, \varphi \in K, \tag{5.3}$$

$$u = (d_1(I - A) - b_{11}A)^{-1}b_{12}Av. \tag{5.4}$$

Similarly, (2.12) *is equivalent to* (5.4) *and*

$$v \in K, \quad \langle d_2(I - A)v - Sv, \varphi - v\rangle \geqslant 0 \quad \forall\, \varphi \in K. \tag{5.5}$$

P r o o f. The condition $d_1 \in D_1$ means that the operator $d_1(I - A) - b_{11}A$ is invertible, and so for every $v \in \mathbb{H}$ the first equation of (2.13) has a unique solution given by (5.4). Inserting this formula into the inequality in (2.13), we obtain the assertion. □

For the rest of this section, we keep $d_1 \in D_1$ fixed and put $S = f(A)$ as above.

Lemma 5.2. *For every $v \in \mathbb{H}$ we have*

$$\langle Sv, v\rangle = \sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2.$$

P r o o f. Since e_k form a complete orthonormal system, we can write the Fourier expansion $v = \sum\limits_{k=0}^{\infty} \mu_k e_k$ with $\mu_k := \langle v, e_k\rangle$. The spectral calculus implies

$$\langle Sv, v\rangle = \sum_{k=0}^{\infty} \langle f(\lambda_k)\mu_k e_k, v\rangle = \sum_{k=0}^{\infty} f(\lambda_k)|\mu_k|^2,$$

so the assertion follows from (5.1). □

Lemma 5.3. *If $e_0 \notin K \cup (-K)$, then there is some $c > 0$ with*

$$c\|v\|^2 \leqslant \langle (I-A)v, v\rangle \leqslant \|v\|^2 \quad \forall\, v \in K, \tag{5.6}$$

and

$$\sup_{v \in K\setminus\{0\}} \frac{\langle Sv, v\rangle}{\langle (I-A)v, v\rangle} = \sup_{v \in K\setminus\{0\}} \frac{\sum\limits_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2}{\langle (I-A)v, v\rangle} < \infty. \tag{5.7}$$

Proof. Since $\sigma(A) \subseteq [0,1]$, we have for all $v \in \mathbb{H}$ with $\|v\| = 1$ that $\langle Av, v\rangle \in [0,1]$, and so $\langle (I-A)v, v\rangle \in [0,1]$. Hence, if (5.6) fails there is a sequence $v_n \in K$ with $\|v_n\| = 1$ and $1 - \langle Av_n, v_n\rangle = \langle (I-A)v_n, v_n\rangle \to 0$. Passing to a subsequence if necessary, we can assume $v_n \rightharpoonup v$. Then $Av_n \to Av$ and thus $\langle Av_n, v_n\rangle \to \langle Av, v\rangle$. In particular, $\langle Av, v\rangle = 1$, which implies $\|v\| \geqslant 1$. From $v_n \rightharpoonup v$ and $\|v_n\| = 1 \leqslant \|v\|$, we thus obtain by a standard Hilbert space argument that $v_n \to v$. Since $\langle Av, v\rangle = 1$ and $\|v\| = 1$, and since 1 is the largest eigenvalue of A with a simple eigenvector e_0, we obtain $v_n \to v \in \{\pm e_0\}$, which is a contradiction, because K is closed, $v_n \in K$, and $e_0 \notin K \cup (-K)$. Hence, (5.6) is established. The equality (5.7) follows from Lemma 5.2, and the finiteness of (5.7) follows from the boundedness of S and (5.6). $\square$

In the following, we identify $\mathbb{H}$ with its dual by means of the scalar product. In this sense, the derivative of a functional $\Phi\colon \mathbb{H} \to \mathbb{R}$ becomes a function $\Phi'\colon \mathbb{H} \to \mathbb{H}$.

The following proof uses some ideas from [27], Section 64.5. However, we cannot use the corresponding [27], Theorem 64.4, since the bilinear form $a(u,v) := \langle (I-A)u, v\rangle$ fails to be positive definite on $\mathbb{H}$ in our situation.

Replacing G_N by $G_N - G_N(0)$ if necessary, we assume from now on without loss of generality that $G_N(0) = 0$.

Lemma 5.4. *Let $e_0 \notin K \cup (-K)$, and suppose that the quantity from* (5.7) *is positive. Then the two suprema in* (5.7) *are maxima, hence, they are equal to $d_2^{\max}$ from Theorem* 3.1. *Moreover:*

(i) *For each sufficiently small $r > 0$ the maximum*

$$d_{2,r} := \frac{1}{r^2} \max_{\substack{v \in K \\ \langle (I-A)v,v\rangle = r^2}} \left(\langle Sv, v\rangle + G_N(v)\right) \tag{5.8}$$

exists, and $d_{2,r} \to d_2^{\max} > 0$ as $r \to 0^+$.

(ii) *If* v_r *is a maximizer of* (5.8), *then there is a unique* d_{2,r,v_r} *such that*

$$v_r \in K, \quad d_{2,r,v_r}\langle (I-A)v_r, \varphi - v_r\rangle \geqslant \langle Sv_r + N(v_r), \varphi - v_r\rangle \quad \forall\, \varphi \in K, \tag{5.9}$$

and $d_{2,r,v_r} \to d_2^{\max}$ *as* $r \to 0^+$ *(independent of the choice of* v_r*).*

Proof. The set $K_r := \{v \in K\colon \langle (I-A)v, v\rangle \leqslant r^2\}$ is convex, closed, and bounded in view of (5.6). The functionals $\Phi_1(v) = \langle Sv, v\rangle$ and $\Phi_2(v) = \Phi_1(v) + G_N(v)$ have compact Fréchet derivatives and thus are weakly sequentially continuous by e.g. [26], Corollary 41.9. Hence, the two maxima

$$m_{i,r} = \max_{v\in K_r} \Phi_i(v)$$

exist, see e.g. [26], Corollary 38.8 and 38.9. Let $v_{i,r}$ be a corresponding maximizer. Since (5.7) is positive, it follows that $\Phi_1(v_{1,r}) = \langle Sv_{1,r}, v_{1,r}\rangle > 0$, and thus by homogeneity of Φ_1, we have

$$v_{1,r} \in B_r := \{v \in K\colon \langle (I-A)v, v\rangle = r^2\}.$$

Hence, the maximum of the first term in (5.7) is attained at $v_{1,r}/r$, and $m_{1,r} = r^2 d_2^{\max}$.

Let us prove that

$$d_{2,r} = \frac{m_{2,r}}{r^2} \to d_2^{\max} \quad \text{as } r \to 0. \tag{5.10}$$

We note first that $N(0) = 0$ and $N'(0) = 0$ imply

$$\lim_{r\to 0} \sup_{\|v\|\leqslant r} \frac{\|N(v)\|}{r} = 0,$$

hence, it follows by using (5.6) that

$$\lim_{r\to 0} \sup_{v\in K_r} \frac{|\langle N(v), v\rangle|}{r^2} = 0. \tag{5.11}$$

Applying the classical mean value theorem to the function $t \mapsto G_N(tu)$ on $[0,1]$, we obtain in view of $G_N(0) = 0$, $G_N' = N$, and since K_r is convex with $0 \in K_r$, that

$$\lim_{r\to 0} \sup_{v\in K_r} \frac{|G_N(v)|}{r^2} = 0. \tag{5.12}$$

The definition of $v_{i,r}$ implies

$$\langle Sv_{1,r}, v_{1,r}\rangle = \Phi_1(v_{1,r}) = m_{1,r} \geqslant \Phi_1(v_{2,r}) = \langle Sv_{2,r}, v_{2,r}\rangle,$$

and

$$\langle Sv_{2,r}, v_{2,r}\rangle + G_N(v_{2,r}) = \Phi_2(v_{2,r}) = m_{2,r} \geqslant \Phi_2(v_{1,r}) = \langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{1,r}).$$

Adding the term $G_N(v_{2,r})$ to the first inequality and using the second one, we get

$$\langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{2,r}) \geqslant \langle Sv_{2,r}, v_{2,r}\rangle + G_N(v_{2,r}) \geqslant \langle Sv_{1,r}, v_{1,r}\rangle + G_N(v_{1,r}).$$

Since $m_{1,r} = r^2 d_2^{\max}$, we obtain in view of (5.12) that

$$\lim_{r\to 0} d_{2,r} = \lim_{r\to 0} \frac{m_{2,r}}{r^2} = \lim_{r\to 0} \frac{\langle Sv_{2,r}, v_{2,r}\rangle}{r^2} = \lim_{r\to 0} \frac{\langle Sv_{1,r}, v_{1,r}\rangle}{r^2} = \lim_{r\to 0} \frac{m_{1,r}}{r^2} = d_2^{\max}.$$

Hence, (5.10) is proved.

Let us show now that $v_{2,r} \in B_r$ if r is small enough. By using $\langle \Phi_2'(v), v\rangle = \langle Sv, v\rangle + \langle N(v), v\rangle$, we obtain in view of (5.11) also

$$\lim_{r\to 0} \frac{\langle \Phi_2'(v_{2,r}), v_{2,r}\rangle}{r^2} = d_2^{\max} > 0. \tag{5.13}$$

Assuming by contradiction that $v_{2,r} \notin B_r$ holds for infinitely many $r = r_n \to 0$, we find for each $r = r_n$ that $(1+t)v_{2,r} \in K_r$ for all small $t > 0$ and thus $\Phi_2((1+t)v_{2,r}) \leqslant m_{2,r} = \Phi_2(v_{2,r})$. Letting $t \to 0^+$, we obtain $\langle \Phi_2'(v_{2,r}), v_{2,r}\rangle \leqslant 0$ for every $r = r_n$, which in view of $r_n \to 0$ contradicts (5.13).

The first assertion of (ii) follows from the Lagrange multiplier rule on cones (see e.g. [27], Proposition 64.3 with $F(v) := \langle (I-A)v, v\rangle$ and $G(v) := \langle Sv, v\rangle + G_N(v)$ and the cone $C := K$). Setting $\varphi = 0$ and $\varphi = 2v_r$ in (5.9), we find

$$d_{2,r,v_r} = \frac{\langle Sv_r, v_r\rangle + \langle N(v_r), v_r\rangle}{\langle (I-A)v_r, v_r\rangle} = d_{2,r} + \frac{\langle N(v_r), v_r\rangle - G_N(v_r)}{r^2}.$$

Using (5.11), (5.12) and (5.10), we get indeed $d_{2,r,v_r} \to d_2^{\max}$ as $r \to 0$. □

P r o o f of Theorem 3.1. Let us note that the equality (5.7) in Lemma 5.3 together with Lemma 5.4 imply that (3.2) is equivalent to the assertion that the quantities in (5.7) are positive.

Assume first that (2.12) has a critical point $d_2 > 0$. By Lemma 5.1, we find some $v \neq 0$ satisfying (5.5). Setting $\varphi = 0$ in (5.5), we obtain $d_2^{\max} \geqslant d_2 > 0$, in particular, (3.2) is satisfied.

Conversely, if the quantities from (5.7) are positive, then Lemma 5.4 implies that they are equal to $d_2^{\max}$, and by the above argument $d_2^{\max} \geqslant d_2$ for any critical point d_2.

It remains to show that $d_2^{\max}$ is a bifurcation point with fixed d_1 (and thus a critical point). Due to Lemma 5.4, for any $r > 0$ small enough there are v_r, d_{2,r,v_r} satisfying (5.9) with $\langle (I-A)v_r, v_r\rangle = r^2$, $d_{2,r,v_r} \to d_2^{\max}$, and (5.6) in Lemma 5.3 gives $\|v_r\| \to 0$ as $r \to 0$. Lemma 5.1 implies that $(d_1, d_{2,r,v_r}, u_r, v_r)$ with u_r defined by (5.4) (with $v = v_r$) satisfies (2.13). Hence, $d_2^{\max}$ is a bifurcation point of (2.13) with fixed d_1. $\square$

P r o o f of Proposition 3.1. We will apply the previous results always with $N = 0$, hence $G_N = 0$. Thus, if v is a maximizer of (3.3), i.e., of (5.8), then $v = v_r$ and $d_{2,r} = d_{2,r,v_r} = d_2^{\max}$ in Lemma 5.4, and so v satisfies (5.5) with $d_2 = d_2^{\max}$. According to Lemma 5.1, the variational inequality (2.12) holds with this v if and only if u satisfies (5.4).

Conversely, let $(u, v) \neq (0, 0)$ satisfy (2.12) with some $d_2 > 0$. According to Lemma 5.1 we have then $v \neq 0$ in view of (5.4), and v satisfies (5.5). As in the above proof of Theorem 3.1, we obtain by the choice $\varphi = 0$ that $d_2 \leqslant d_2^{\max}$ and that (3.2) is satisfied. If $d_2 = d_2^{\max}$, the choice $\varphi = 0$ in (5.5) shows that v is a maximizer. $\square$

P r o o f of Proposition 3.2. For clarity, we denote the operator of Lemma 5.1 by $S(d_1)$. By Lemma 5.2, we thus have for every $d_1 \in D_1$

$$\langle S(d_1)v, v\rangle = \sum_{k=0}^{\infty} c_k(d_1)|\langle v, e_k\rangle|^2. \tag{5.14}$$

Note that (5.2) shows that $S(d_1)$ is a compact operator which depends on $d_1 \in D_1$ continuously in operator norm.

Let $d_1 \in D_{1,0}$ and $v \in K$ be a maximizer of (3.3). Then $d_2^{\max}$ is (5.14), and for every $\varepsilon > 0$, we have $\langle S(\widetilde{d}_1)v, v\rangle \geqslant d_2^{\max} - \varepsilon > 0$ if $\widetilde{d}_1$ is sufficiently close to d_1. Hence, $D_{1,0}$ is open, and (3.3) is lower semicontinuous.

Conversely, let $d_{1,n} \in D_{1,0}$ converge to $d_1 \in D_1$. Let $v_n \in K$ satisfying $\langle (I-A)v_n, v_n\rangle = 1$ be corresponding maximizers of (3.3) with $d_{1,n}$, and set $d_{2,n}^{\max} := \langle S(d_{1,n})v_n, v_n\rangle$. Let $l := \limsup d_{2,n}^{\max}$. It follows from (5.6) that the sequence v_n is bounded. Hence, passing to a subsequence, we can assume that $v_n \rightharpoonup v$ and $l = \langle S(d_1)v, v\rangle$. Since closed convex sets are weakly closed, it follows that $v \in K$ and $\langle (I-A)v, v\rangle \leqslant 1$. Using (5.6) once more, we find $\langle (I-A)v, v\rangle \in (0, 1]$ or $v = 0$. In the case $d_1 \in D_{1,0}$, we show now that

$$\limsup_{n\to\infty} d_{2,n}^{\max} = l = \langle S(d_1)v, v\rangle \leqslant \max_{\substack{\widetilde{v}\in K\\ \langle (I-A)\widetilde{v},\widetilde{v}\rangle = 1}} \langle S(d_1)\widetilde{v}, \widetilde{v}\rangle = d_2^{\max}.$$

Indeed, if $v \neq 0$, we can choose $\widetilde{v} := \langle (I-A)v, v\rangle^{-1} v$, and if $v = 0$, then $l = 0$, and we can choose $\widetilde{v}$ as any vector satisfying (3.2) (here we use the fact that $d_1 \in D_{1,0}$). This proves the upper semicontinuity of (3.3) at d_1. In the case $d_1 \notin D_{1,0}$, we have $\langle S(d_1)v, v\rangle \leqslant 0$ and thus again $\limsup d_{2,n}^{\max} = l = 0$. $\square$

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