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BOOTSTRAP METHOD FOR CENTRAL AND INTERMEDIATE ORDER STATISTICS UNDER POWER NORMALIZATION

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It has been known for a long time that for bootstrapping the distribution of the extremes under the traditional linear normalization of a sample consistently, the bootstrap sample size needs to be of smaller order than the original sample size. In this paper, we show that the same is true if we use the bootstrap for estimating a central, or an intermediate quantile under power normalization. A simulation study illustrates and corroborates theoretical results.

Keywords: bootstrap technique, power normalization, weak consistency, central order statistics, intermediate order statistics

Classification: 62G32, 62F40

1. INTRODUCTION

The bootstrap technique, which is a data driven method, was initiated by Efron [10]. The basic idea of the bootstrap technique lies in using the data of a sample study as a surrogate population to approximate the sampling distribution of the statistic under study. This will be done by re-sampling (with replacement) from the sample data at hand to create a large number of dummy samples known as bootstrap samples. After that, for each of these bootstrap samples we compute the sample summary, e. g., the average or the maximum likelihood estimate of any unknown parameter. In addition, a histogram of the set of these computed values is referred to as the bootstrap distribution of the statistic. By this way, the bootstrap technique provides estimates of standard errors for complex estimators of complex parameters of the distribution function (df) under study. The theoretical idea behind this technique is that the obtained dummy samples are actually real samples drawn from the empirical df, which in turn is theoretically too close to the df under study.

One of the desired properties of the bootstrapping method is the consistency, which guarantees that the limit of the bootstrap distribution is the same as that of the distribution of the given statistic. It has been known for a long time that for the bootstrap distribution of the maximum of a sample to be consistent, the bootstrap sample size needs to be of smaller order than the original sample size. Actually, Athreya and

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Fukuchi [2] showed that by employing a sub-sample bootstrap, where the re-samples have a size of an order of magnitude smaller than the size of the original sample, the bootstrap distribution of maximum order statistics converges to one of Gnedenko's extreme value distributions. The inconsistency, weak consistency and strong consistency of bootstrapping maximum order statistics under linear normalization are investigated by Athreya and Fukuchi [1], while for maximum order statistics under power normalization this study is extended by Nigm [12].

During the last two decades E. Pancheva and her collaborators developed the limit theory for extremes under nonlinear but monotone increasing normalizing mappings, such as the power normalization $G_n(x) = d_n|x|^{c_n}\operatorname{sign}(x), c_n, d_n > 0$, with $G_n^{-1}(x) = |\frac{x}{d_n}|^{\frac{1}{c_n}}\operatorname{sign}(x)$. Pancheva [13] derived all the possible limit df's of the maximum order statistics subjected to the power normalization. These limit df's are usually called the power max stable df's. Mohan and Ravi [11] showed that the power max stable df's (six power types of df's) attract more than linear stable df's. Therefore, using the power normalization, we get a wider class of limit df's which can be used in solving approximation problems. Another reason for using nonlinear normalization concerns the problem of refining the accuracy of approximation in the limit theorems using relatively non difficult monotone mappings in certain cases that can achieve a better rate of convergence (see Barakat et al. [6]). Recently, Barakat et al. [7], have tackled the problem of the mathematical modeling of extremes under power normalization. Barakat and Omar [4], [5] proved that the possible nondegenerate weak limits of any central order statistics with regular rank under the traditionally linear normalization and under the power normalization are the same. Moreover, they derived the class of all possible weak limits for lower and upper intermediate order statistics under power normalization from the corresponding weak limits of extremes under power normalization. Barakat et al. [8] studied the inconsistency, weak consistency and strong consistency of bootstrapping central and intermediate order statistics under linear normalization for an appropriate choice of re-sample size. In this paper, the consistency property of the bootstrapping central and intermediate order statistics under power normalization are investigated.

2. BOOTSTRAPPING CENTRAL ORDER STATISTICS UNDER POWER NORMALIZATION

Let $X_1, X_2, \cdots, X_n$ be iid random variables with common df $F(x) = P(X \leq x)$, and let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding order statistics. When the rank sequence r_n of the central order statistic $X_{r_n:n}$ is assumed to satisfy the regular condition $\sqrt{n}(\frac{r_n}{n} - \lambda) \to 0,\ 0 < \lambda < 1$, as $n \to \infty$, and

$$\Phi_{\lambda:n}(c_n x + d_n) = P(X_{r_n:n} \leq c_n x + d_n) = I_{F(c_n x + d_n)}(r_n, n - r_n + 1) \xrightarrow[n]{w} G(x), \quad (2.1)$$

where $I_x(a, b)$ denotes the usual incomplete ratio beta function, " $\xrightarrow[n]{w}$ "denotes the weak convergence, as $n \to \infty$, $G(x)$ is a nondegenerate df and $c_n > 0$ and d_n are suitable normalizing constants, Smirnov [14]) has shown that the df $G(x)$ must have one and only one of the types

(i) $\Phi_1(x; c, \alpha) = \Phi(cx^{\alpha}) I_{[0,\infty)}(x),\ c, \alpha > 0;$

(ii) $\Phi_2(x;c,\alpha)=\Phi(-c(-x)^{\alpha})I_{(-\infty,0)}(x)+I_{[0,\infty)}(x),\ c,\alpha>0;$

(iii) $\Phi_3(x;c_1,c_2,\alpha)=\Phi(-c_1(-x)^{\alpha})I_{(-\infty,0)}(x)+\Phi(c_2x^{\alpha})I_{[0,\infty)}(x),\ c_1,c_2,\alpha>0;$

(iv) $\Phi_4(x;1,1)$,

where $\Phi_4(x;A,B)=\frac{1}{2}I_{[-A,B)}(x)+I_{[B,\infty)}(x)$ and $\Phi(.)$ is the standard normal df. Barakat and Omar [4] considered the weak convergence of the power normalized central order statistic $\left|\frac{X_{r_n:n}}{a_n}\right|^{\frac{1}{b_n}}\operatorname{sign}(X_{r_n:n}), a_n, b_n>0,$

$$P\left(\left|\frac{X_{r_n:n}}{a_n}\right|^{\frac{1}{b_n}}\operatorname{sign}(X_{r_n:n})\le x\right)=I_{F(a_n|x|^{b_n}\operatorname{sign}(x))}(r_n,n-r_n+1)\ \xrightarrow[n]{w}\ \Psi(x).\quad(2.2)$$

Barakat and Omar [4] showed that the class of possible limit distributions of Ψ is $\{\Psi_1(x)=\Phi_1(x;1,1);\Psi_2(x)=\Phi_2(x;1,1);\Psi_3(x)=\Phi_3(x;c_1,c_2,1);\Psi_4(x)=\Phi_4(x,A,B)\}$ where $\Psi_4(x)$ has the six power types $\Phi_4(x;A,A),\ A>0;\ \Phi_4(x;A,B),\ B>A>0;$ $\Phi_4(x;A,0),\ A>0;\Phi_4(x;0,A),\ A>0;\Phi_4(x;-A,B),\ B>A>0$ and $\Phi_4(x;A,-B),\ A>B>0$. In this case we say that F belongs to the $\lambda-$normal domain of attraction of the limit df Ψ. Moreover, (2.2) is satisfied with $\Psi_i(x),\ i\in\{1,2,3,4\}$, if and only if

$$\sqrt{n}\frac{F(a_n|x|^{b_n}\operatorname{sign}(x)))-\lambda}{\sqrt{\lambda(1-\lambda)}}\to\Phi^{-1}(\Psi_i(x))=W_i(x),\ \text{as}\ n\to\infty,\quad(2.3)$$

Remark 2.1. Note that under the power normalization the function $c|x|^{\alpha}$ has the same type as $|x|$, while $\Phi_3(x;c_1,c_2,1)$ represents a family of two power types $c_1\neq c_2$ and $c_1=c_2$.

Although, the convergence in (2.2), as well as in (2.1), does not yield in general continuity types, but the following lemma (c.f., Lemma 2.1 in [3]), which will be be needed in our study, shows that this convergence is uniform.

Lemma 2.1. Under the condition $\sqrt{n}(\frac{r}{n}-\lambda)\to 0$, as $n\to\infty$, and for any arbitrary df F, we have

$$\Phi_{\lambda:n}(x)=\Phi\left(\sqrt{n}\frac{F(x)-\lambda}{\sqrt{\lambda(1-\lambda)}}\right)+R_n(x),\ \text{for large}\ n,$$

where $R_n(x)\to 0$, as $n\to\infty$, uniformly with respect to $x\in\Re$.

Remark 2.2. Actually, in the proof of Lemma 2.1 in [3], the remainder $R_n(x)$ depends on x, only within $F(x)$. Therefore, we can write $R_n(x)=R^{\star}(F(x))$. Moreover, the proof of the relation $R_n(x)\to 0$ depends only on the fact that $0\le F(x)\le 1$. Therefore, a quick check of this proof shows that $R_n^{\star}(\ell)\to 0$, uniformly with respect to ℓ, where $0\le\ell\le 1$. This shows that the remainder term R_n converges to zero uniformly over the set of the df's. Consequently, this fact enables us to replace $F(x)$ in the relation given in Lemma 2.1 by the normalized df $F(c_nx+d_n)$ to get the relation (2.1) (if the condition $\sqrt{n}\frac{F(c_nx+d_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}$ converges to any function of the types (i)−(iv), and in this case $F(c_n x + d_n) \to \lambda$) or to replace $F(x)$ in the relation given in Lemma 2.1 by the normalized df $F(a_n|x|^{b_n}\mathrm{sign}(x))$ to get the relation (2.2) (if the condition (2.3) is satisfied and in this case $F(a_n|x|^{b_n}\mathrm{sign}(x)) \to \lambda$) or to replace $F(x)$ in the relation given in Lemma 2.1 by any sequence $0 \le \ell_n \le 1$, whenever $\ell_n \to \ell, 0 \le \ell \le 1$.

Now, assume $Y_j,\ j = 1, 2, \dots, m$, where $m = m(n) \to \infty$, as $n \to \infty$, are conditionally iid random variables with

$$P(Y_1 = X_j|\underset{\sim}{X}_n) = \frac{1}{n},\ j = 1, 2, \dots, n,$$

where $\underset{\sim}{X}_n = (X_1, X_2, \dots, X_n)$ is a random sample of size n from an unknown df F. Hence $Y_1, \dots, Y_m$ is a re-sample of size m from the empirical df

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x)}(X_i) = \frac{1}{n}S_n(x), \tag{2.4}$$

where $I_A(x)$ is the usual indicator function and $S_n(x)$ is a random variable distributed as a binomial distribution $\mathrm{B}(n, F)$. Furthermore, let

$$H_{\lambda,n,m}(a_m\,|x|^{b_m}\,\mathrm{sign}(x)) = P\left(\left|\frac{X_{r_m:m}}{a_m}\right|^{\frac{1}{b_m}}\mathrm{sign}(X_{r_m:m}) \le x|\underset{\sim}{X}_n\right)$$

$$= I_{F(a_m|x|^{b_m}\mathrm{sign}(x))}(r_m, m - r_m + 1)$$

be the bootstrap df of $\left|\frac{X_{r_m:m}}{a_m}\right|^{\frac{1}{b_m}}\mathrm{sign}(X_{r_m:m})$.

A full-sample bootstrap is the case when $m = n$. In contrast, a sub-sample bootstrap is the case when $m < n$. The following theorem determines the asymptotic behavior of the bootstrap distribution $H_{\lambda,n,m}(a_m\,|x|^{b_m}\,\mathrm{sign}(x)) = P(X_{r_m:m} \le a_m\,|x|^{b_m}\,\mathrm{sign}(x)|\underset{\sim}{X}_n)$ of the central order statistic $X_{r_n:n}$ of $\underset{\sim}{X}_n$.

Theorem 2.2. Let (2.2) be satisfied with $\Psi(x) = \Phi(W_i(x)),\ i \in \{1, 2, 3, 4\}$. Then

$$H_{\lambda,n,n}(a_n\,|x|^{b_n}\,\mathrm{sign}(x)) \ \xrightarrow[n]{d}\ \Phi(Z(x)), \tag{2.5}$$

where $Z(x)$ has a normal distribution with mean $W_i(x)$ and unit variance, i. e., $P(Z(x) \le z) = \Phi(z - W_i(x))$, where " $\xrightarrow[n]{d}$ " stands for convergence in distribution, as $n \to \infty$ (i. e., weak convergence). Moreover, if $m = \circ(n) \to \infty$, as $n \to \infty$, then

$$\sup_{x\in\Re} \mid H_{\lambda,n,m}(a_m\,|x|^{b_m}\,\mathrm{sign}(x)) - \Phi(W_i(x)) \mid \ \xrightarrow[n]{p}\ 0, \tag{2.6}$$

where " $\xrightarrow[n]{p}$ "stands for convergence in probability, as $n \to \infty$.

Theorem 2.2 shows that if $m = n$, $H_{\lambda,n,m}(a_m\,|x|^{b_m}\,\mathrm{sign}(x))$ has a random limit and thus the naive bootstrap fails to approximate $\Phi_{\lambda:n}(a_m\,|x|^{b_m}\,\mathrm{sign}(x))$. In other words the naive bootstrap of the r_nth central order statistic, when $m = n$, fails to be consistent estimator for the limit df $\Phi(W_i(x))$, while the relation (2.6) shows that this bootstrap will be consistent if $m = \circ(n)$.

P r o o f. By applying Lemma 2.1, we get

$$H_{\lambda,n,m}(a_m\,|x|^{b_m}\operatorname{sign}(x)) = \Phi\left(\mathcal{T}_{\lambda,n,m}(x)\right) + R_m, \tag{2.7}$$

where $\mathcal{T}_{\lambda,n,m}(x) = \sqrt{m}\frac{F_n(a_m|x|^{b_m}\operatorname{sign}(x))-\lambda}{\sqrt{\lambda(1-\lambda)}}$ and $R_m \to 0$, as $n \to \infty$ (by using the fact given in Remark 2.2). Now assume that the condition $m = n$ is satisfied, then in view of (2.4) and by applying the central limit theorem we get

$$\frac{S_n(a_n\,|x|^{b_n}\operatorname{sign}(x)) - nF(a_n\,|x|^{b_n}\operatorname{sign}(x))}{\sqrt{nF(a_n\,|x|^{b_n}\operatorname{sign}(x))(1-F(a_n\,|x|^{b_n}\operatorname{sign}(x)))}} \xrightarrow[n]{d} Z, \tag{2.8}$$

where Z is the standard normal random variable. On the other hand, under the condition of Theorem 2.2, the relation (2.3) is satisfied. Thus, we get $F(a_n\,|x|^{b_n}\operatorname{sign}(x)) \to \lambda$, as $n \to \infty$, for all x such that $W_i(x) < \infty$. Therefore, as $n \to \infty$, we get

$$\frac{\sqrt{nF(a_n\,|x|^{b_n}\operatorname{sign}(x))(1-F(a_n\,|x|^{b_n}\operatorname{sign}(x)))}}{\sqrt{n}\sqrt{\lambda(1-\lambda)}} \to 1$$

and

$$\frac{nF(a_n\,|x|^{b_n}\operatorname{sign}(x)) - n\lambda}{\sqrt{n}\sqrt{\lambda(1-\lambda)}} \to W_i(x).$$

The above two limit relations enable us to apply the modified Khinchin's type theorem (cf., [4]) on the relation (2.8) to get

$$\mathcal{T}_{\lambda,n,n}(x) = \frac{S_n(a_n\,|x|^{b_n}\operatorname{sign}(x)) - n\lambda}{\sqrt{n}\sqrt{\lambda(1-\lambda)}} \xrightarrow[n]{d} Z + W_i(x) = Z(x). \tag{2.9}$$

Combining the relations (2.7) and (2.9) we get (2.5). To prove the relation (2.6), we first notice that (2.3) and (2.4) imply that

$$\mathrm{E}(\mathcal{T}_{\lambda,n,m}(x)) = \sqrt{m}\frac{F(a_m\,|x|^{b_m}\operatorname{sign}(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \to W_i(x), \text{ as } n \to \infty. \tag{2.10}$$

Moreover, in view of (2.3), (2.4) and the condition $m = \circ(n)$, we get

$$\begin{aligned} \operatorname{Var}(\mathcal{T}_{\lambda,n,m}(x)) &= \frac{m}{\lambda(1-\lambda)}\operatorname{Var}(F_n(a_m\,|x|^{b_m}\operatorname{sign}(x))) \\ &= \frac{m}{n^2\lambda(1-\lambda)}nF(a_m\,|x|^{b_m}\operatorname{sign}(x))(1-F(a_m\,|x|^{b_m}\operatorname{sign}(x))) \\ &= \frac{mF(a_m\,|x|^{b_m}\operatorname{sign}(x))(1-F(a_m\,|x|^{b_m}\operatorname{sign}(x)))}{n\lambda(1-\lambda)} \to 0, \text{ as } n \to \infty. \end{aligned} \tag{2.11}$$

The relation (2.6) follows immediately by combining (2.10) and (2.11). This completes the proof of Theorem 2.2. □

3. BOOTSTRAPPING INTERMEDIATE ORDER STATISTICS UNDER POWER NORMALIZATION

The intermediate order statistics have many applications. For example intermediate order statistics can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to available sample size (see [5]). The sequence $\{X_{r_n:n}\}$ is referred to the r_nth intermediate order statistic, if $r_n \to \infty$ as $n \to \infty$ and $\frac{r_n}{n} \to 0$ (the lower intermediate case) or $\frac{r_n}{n} \to 1$ (the upper intermediate case). Chibisov [9] studied a wide class of intermediate order statistics, where $r_n = \ell^2 n^\alpha(1+\circ(1)),\ 0<\alpha<1,\ \ell^2>0$. Namely, when the intermediate rank sequence $\{r_n\}$ satisfies the limit relation $\lim_{n\to\infty}(\sqrt{r_{n+z_n(\nu)}}-\sqrt{r_n}) = \frac{\alpha\nu\ell}{2}$, which is known as Chibisov's condition, where $\{z_n(\nu)\}$ is any sequence of integer values, for which $\frac{z_n(\nu)}{n^{1-\frac{\alpha}{2}}} \to \nu$, as $n\to\infty$, $0<\alpha<1,\ \ell>0$ and ν is any real number, Chibisov [9] showed that if there are normalizing constants $\alpha_n>0$ and β_n such that

$$\Phi_{r_n:n}(\alpha_n x+\beta_n) = P(X_{r_n:n} \le \alpha_n x+\beta_n) \xrightarrow[n]{w} N(x),$$

where $N(x)$ is a nondegenerate df, then $N(x)$ must have one and only one of the types $\Phi(V_i(x)),\ i=1,2,3,$ where $V_1(x)=x,\ \forall x,$

$$V_2(x) = \begin{cases} -\beta\log|\,x\,|, & x\le 0,\\ \infty, & x>0,\end{cases} \qquad V_3(x) = \begin{cases} -\infty, & x\le 0,\\ \beta\log|\,x\,|, & x>0,\end{cases}$$

and β is some positive constant.

Barakat and Omar [5] extended the work of Chibisov to the power normalization case by considering the limit relation

$$\Phi_{r_n}(\alpha_n\,|x|^{\beta_n}\operatorname{sign}(x)) = P\left(\left|\frac{X_{r_n:n}}{\alpha_n}\right|^{\frac{1}{\beta_n}}\operatorname{sign}(X_{r_n:n}) \le x\right) \xrightarrow[n]{w} L(x), \tag{3.1}$$

where $\alpha_n,\beta_n>0$ and $L(x)$ is a nondegenerate df. Barakat and Omar [5] proved that the class of possible limit distributions of $L(x)$ is $\{L_{i,\beta}(x),\ i=1,2,\ldots,6\}$, where

(i) $L_{1;\beta}(x) = I_{[-1,\infty)}(x) + \Phi(-\beta\log(\log|\,x\,|))I_{(-\infty,-1)}(x)$;

(ii) $L_{2;\beta}(x) = I_{[0,\infty)}(x) + \Phi(\beta\log(-\log|\,x\,|))I_{[-1,0)}(x)$;

(iii) $L_{3;\beta}(x) = I_{[1,\infty)}(x) + \Phi(\beta\log(-\log x))I_{[0,1)}(x)$;

(iv) $L_{4;\beta}(x) = \Phi(\beta\log(\log x))I_{[1,\infty)}(x)$;

(v) $L_{5;\beta}(x) = L_5(x) = I_{[0,\infty)}(x) + \Phi(-\log|\,x\,|)I_{(-\infty,0)}(x)$;

(vi) $L_{6;\beta}(x) = L_6(x) = \Phi(\log x)I_{[0,\infty)}(x)$.

Moreover, (3.1) is satisfied with $L(x)=L_{i;\beta}(x),\ i\in\{1,2,\ldots,6\}$, if and only if

$$\frac{nF(\alpha_n\,|x|^{\beta_n}\operatorname{sign}(x)) - r_n}{\sqrt{r_n}} = \sqrt{n}\frac{F(\alpha_n\,|x|^{\beta_n}\operatorname{sign}(x)) - \bar{r}_n}{\sqrt{\bar{r}_n}} \to \Phi^{-1}(L_{i;\beta}(x)) = \mathcal{U}_i(x), \tag{3.2}$$

as $n \to \infty$, where $\bar{r}_n = \frac{r_n}{n}$. Since, all the limit types in (3.1) are continuous, then the convergence in (3.1) is uniform with respect to $x \in \Re$. Therefore,

$$\Phi_{r_n}(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) = \Phi\left(\sqrt{n}\frac{F(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) - \bar{r}_n}{\sqrt{\bar{r}_n}}\right) + \rho_n(x), \text{ for large } n, \quad (3.3)$$

where $\rho_n(x) \to 0$, as $n \to \infty$, uniformly with respect to $x \in \Re$.

The following theorem determines the asymptotic behavior of the bootstrap distribution $H_{r_m,n,m}(\alpha_m|x|^{\beta_m}\operatorname{sign}(x)) = P(X_{r_m:m} \le \alpha_m|x|^{\beta_m}\operatorname{sign}(x)|\underline{X}_n)$ of the intermediate order statistic $X_{r_n:n}$ of $\underline{X}_n$.

Theorem 3.1. Let (3.1) be satisfied with $L(x) = \Phi(\mathcal{U}_i(x)),\ i \in \{1, 2, \ldots, 6\}$. Then

$$H_{r_n,n,n}(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) \to \Phi(\xi(x)), \quad (3.4)$$

where $\xi(x)$ has a normal distribution with mean $\mathcal{U}_i(x)$ and unit variance, i. e., $P(\xi(x) \le z) = \Phi(z - \mathcal{U}_i(x))$. Moreover, if $m = \circ(n)$, then

$$\sup_{x\in\Re} \mid H_{r_m,n,m}(\alpha_m |x|^{\beta_m} \operatorname{sign}(x)) - \Phi(\mathcal{U}_i(x)) \mid \ \xrightarrow[n]{p}\ 0. \quad (3.5)$$

Theorem 3.1 shows that if $m = n$, the naive bootstrap of the r_nth intermediate order statistic fails to be consistent estimator for the limit df $\Phi(\mathcal{U}_i(x))$, while the relation (3.5) shows that this bootstrap will be consistent if $m = \circ(n)$.

P r o o f. In view of (3.3), we get

$$H_{r_m,n,m}(\alpha_m |x|^{\beta_m} \operatorname{sign}(x)) = \Phi\left(\mathcal{T}_{r_m,n,m}(x)\right) + \rho_m, \quad (3.6)$$

where $\mathcal{T}_{r_m,n,m}(x) = \sqrt{m}\frac{F_n(\alpha_m|x|^{\beta_m}\operatorname{sign}(x)) - \bar{r}_m}{\sqrt{\bar{r}_m}}$ and $\rho_m \to 0$, as $n \to \infty$, uniformly in x. Now assume that the condition $m = n$ is satisfied, then in view of (2.4) and by applying the central limit theorem we get

$$\frac{S_n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) - nF(\alpha_n |x|^{\beta_n} \operatorname{sign}(x))}{\sqrt{nF(\alpha_n |x|^{\beta_n} \operatorname{sign}(x))(1 - F(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)))}} \ \xrightarrow[n]{d}\ Z. \quad (3.7)$$

On the other hand, under the condition of Theorem 3.1, the relation (3.2) is satisfied. Thus, we get $F(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) \sim \bar{r}_n \to 0$, as $n \to \infty$, for all x such that $\mathcal{U}_i(x) < \infty$. Therefore, as $n \to \infty$, we get

$$\frac{\sqrt{nF(\alpha_n |x|^{\beta_n} \operatorname{sign}(x))(1 - F(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)))}}{r_n} \sim \sqrt{\frac{n\bar{r}_n}{r_n}} = 1$$

and

$$\frac{nF(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) - n\bar{r}_n}{\sqrt{r_n}} = \frac{nF(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) - r_n}{\sqrt{r_n}} \to \mathcal{U}_i(x).$$

The above two limit relations enable us to apply the modified Khinchin's type theorem (cf., [4]) on the relation (3.7) to get

$$\mathcal{T}_{r_n,n,n}(x) = \frac{S_n(\alpha_n\,|x|^{\beta_n}\,\mathrm{sign}(x)) - r_n}{\sqrt{r_n}} \;\xrightarrow[n]{d}\; Z + \mathcal{U}_i(x). \tag{3.8}$$

Thus, by combining the relations (3.6) and (3.8), we get (3.4). Turning now to prove the relation (3.5). In view of (2.4) and (3.2) we get

$$\mathrm{E}(\mathcal{T}_{r_m,n,m}(x)) = \sqrt{m}\,\frac{F(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x)) - \bar{r}_m}{\sqrt{\bar{r}_m}} \to \mathcal{U}_i(x), \text{ as } n \to \infty. \tag{3.9}$$

Moreover, in view of (2.4), (3.2) and the condition $m = \circ(n)$, we get

$$\mathrm{Var}(\mathcal{T}_{r_m,n,m}(x)) = \frac{m}{\bar{r}_m}\mathrm{Var}(F_n(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x)))$$

$$= \frac{m}{n^2\bar{r}_m} nF(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x))(1 - F(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x)))$$

$$= \frac{mF(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x))(1 - F(\alpha_m\,|x|^{\beta_m}\,\mathrm{sign}(x))}{n\bar{r}_m} \sim \frac{m}{n} \sim \circ(1) \to 0, \text{ as } n \to \infty. \tag{3.10}$$

The relation (3.5) follows immediately by combining (3.9) and (3.10). This completes the proof of Theorem 3.1. □

4. SIMULATION STUDY

In this section, by a simulation study, we show that the sub-sample bootstrap technique suggests an efficient technique for modeling the quantile values such as the median. We consider the uniform df $F^\star(x) = \frac{1}{2}(x+1), \quad -1 \le x \le 1$. Let $a_n = \frac{1}{\sqrt{n}}$ and $b_n = K > 0$. Therefore, for any $\lambda \in (0,1)$, we get $\sqrt{n}\frac{F^\star(a_n|x|^{b_n}\mathrm{sign}(x))-\lambda}{\sqrt{\lambda(1-\lambda)}} = |x|^K\mathrm{sign}(x)$, $-\sqrt{n} \le x \le \sqrt{n}$, which implies that $\Phi_{\lambda:n}(a_n|x|^{b_n}\mathrm{sign}(x)) \;\xrightarrow[n]{w}\; \Phi(|x|^K\mathrm{sign}(x))$ (c.f, [4]). For this limit df, we have μ = mean = median = 0. Our aim is to apply the suggested technique given in Theorem 2.2 to estimate the limit df of the sample median $X_{r_n:n}$, $r_n = [\lambda n + 1] = [\frac{1}{2}n + 1]$, where $[\theta]$ denotes the integer part of θ, for two values $K = 1, \frac{1}{2}$. For $K = 1$, we first generate a random sample of size $n = 20000$ from the df $F^\star(x)$. We apply the sub-sample technique to get the estimated model and then check the compatibility of this estimate with the theoretical model. We firstly, choose a suitable value of the size of the bootstrap replicates m. Theorem 2.2 shows that this value should be small enough to satisfy the condition $m = \circ(n)$ and at the same time should be large enough to satisfy the condition $m \to \infty$, as $n \to \infty$. The simple way to determine a suitable value of m is to put n in the form $a(10)^b + c$, where a, b and c are integers such that $1 \le a < 10$, $0 \le c \le (10)^{b-1}$. Thus, in our case $a = 2, b = 4$ and $c = 0$. Moreover, in view of the two conditions $m = \circ(n)$ and $m \to \infty$, as $n \to \infty$, we can preliminary take two possible values of m such that $m = 2 \times (10)^3 = 2000$ and $m = 2\times(10)^2 = 200$. After that, we can differentiate between the two these values based on the accuracy of the estimate value of the median. This estimate, denoted by $\hat{\mu}$, is obtained by withdrawing from the original sample a large number of bootstrap replicates or blocks,

m, normal$(\hat{\mu},\hat{\sigma}) = \Phi(\hat{\mu},\hat{\sigma})$	H	P	KSSTAT	CV	Decision
$m = 300,\ \Phi(0.0011, 0.0578)$	0	0.1676	0.0297	0.0385	accept H_0
$m = 250,\ \Phi(-0.0010, 0.0624)$	0	0.1058	0.0333	0.0385	accept H_0
$m = 200,\ \Phi(0.0003, 0.0702)$	0	0.2936	0.0246	0.0385	accept H_0
$m = 150,\ \Phi(0.0013, 0.0799)$	0	0.1940	0.0285	0.0385	accept H_0
$m = 100,\ \Phi(0.0006, 0.0987)$	0	0.6890	0.0135	0.0385	accept H_0

Tab. 1. Simulation study for $K = 1$.

$m,\ \hat{\sigma}$	H	P	KSSTAT	CV	Decision
$m = 300,\ \hat{\sigma} = 295.2$	0	0.4645	0.0284	0.0385	accept H_0
$m = 250,\ \hat{\sigma} = 256.3$	0	0.4128	0.0277	0.0385	accept H_0
$m = 200,\ \hat{\sigma} = 203.3$	0	0.5324	0.0242	0.0385	accept H_0
$m = 150,\ \hat{\sigma} = 152$	0	0.4419	0.0250	0.0385	accept H_0
$m = 100,\ \hat{\sigma} = 101.3$	0	0.6004	0.0296	0.0385	accept H_0

Tab. 2. Simulation study for $K = 2$.

namely 1000 blocks, each of which has size m and determine the sample median of each block. Then, these medians are used as a sample drawn from a normal distribution to get the estimates for its mean and standard deviation, denoted by $\hat{\sigma}$, by using the ML method. We found that the value $m = 200$ gives the best estimate. To get more accurate value of m, we consider an appropriate discrete neighborhood of $m = 200$, namely $m = 100, 150, 200, 250, 300$ (Table 1). Moreover, we repeat the preceding procedures for all values of this neighborhood to select a value, which gives the best estimate for $\mu = 0$. Table 1 presents the results for these values and the corresponding estimates for μ, as well as the estimated standard deviations. All the obtained estimates for these values are close to the true value of the median $\mu = 0$ and the best of them is $m = 200$. Finally, the sub-samples corresponding to these values were fitted by using the K-S test. For the values $K = \frac{1}{2}$ (see Table 2) similar procedure is applied except we generate the original sample from the df $F^\star(\sqrt{\mid x \mid}\text{sign}(x))$ and we choose the values $m = 100, 150, 200, 250, 300$ (we can differentiate between these values by the corresponding values of $KSSTAT$, i. e., the best value of m, which is $m = 200$, has a minimum value of $KSSTAT$). Moreover, the sub-samples corresponding to these values were fitted by using the K-S test to the df $\Phi(\sqrt{\hat{\sigma} \mid x \mid}\text{sign}(x))$, where here $\hat{\sigma}$ is the ML estimate of the scale parameter. In this study, all computations are achieved by the Matlab package, where we have four functions $[H,\ P,\ KSSTAT,\ CV]$, $H = 0$, or $H = 1$, P is the $p-$value, $KSSTAT$ is the maximum difference between the data (i. e., the empirical df) and fitting curve and CV is a critical value. We accept H_0, if $H = 0$, $KSSTAT\ \leq\ CV$ and $P >$ level of significance, otherwise, we reject H_0. Tables 1 and 2 show that the estimated models, for all values of m are compatible with the theoretical models.

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