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SYNCHRONIZATION OF FRACTIONAL-ORDER CHAOTIC SYSTEMS WITH MULTIPLE DELAYS BY A NEW APPROACH

Jianbing Hu, Hua Wei and Lingdong Zhao

In this paper, we propose a new approach of designing a controller and an update rule of unknown parameters for synchronizing fractional-order system with multiple delays and prove the correctness of the approach according to the fractional Lyapunov stable theorem. Based on the proposed approach, synchronizing fractional delayed chaotic system with and without unknown parameters is realized. Numerical simulations are carried out to confirm the effectiveness of the approach.

Keywords: fractional-order, multiple delays, Lyapunov stable theorem, synchronization, unknown parameters

Classification: 34H10, 34C15, 34D06

1. INTRODUCTION

The concept of fractional-order calculus has been known since the contribution of Leibniz and Hospital in 1695 [7, 16], but its applications to engineering, physics and mathematical biology are just recent topics of interest [4, 6]. In fact, many systems can be described by fractional differential equations, for example, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, visco-elastic systems, quantitative finance, bioengineering, diffusion wave and nuclear magnetic resonance [5, 10, 13, 14, 19, 30, 31, 32].

Chaotic synchronization has been applied in different fields, including biological and physical systems, structural engineering, and ecological models [9]. The synchronization of fractional-order chaotic systems has also attracted considerable research attention [29]. In the recent years, a variety of approaches have been proposed for the synchronization of fractional chaotic systems with known and unknown parameters such as lag-synchronization [33], projective synchronization [3], sliding synchronization [22], generalized synchronization (GS) [1, 11, 27], etc.

From the viewpoint of engineering applications and characteristics of channel, time delays are inherent due to the finite propagation velocity of information [8, 20], from the latency of feedback loops, the finite switching times, etc. In the real world there

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could be several channels of information exchange, several switching mechanism, two or more feedbacks, etc. [12]. In other words, in comparison with single time-delay systems, multiple time-delays systems are often more realistic models of interacting complex systems. The systems with time-delays are difficult to achieve satisfying performance. The stability issue of time-delay systems is of practical importance [15, 17, 23, 24, 28].

Although some progresses have been made in analyzing the stability of fractional-order time-delay systems, it is very difficult to design a controller to control a fractional-order time-delay system based on these fruits. For the most readers, they would pay more attention to mastering the approach of designing a controller to realize controlling a system. Aimed at this problem, we propose a novel stability theorem for fractional delayed system and extend a simple approach based on a special matrix for designing controller to synchronize fractional chaotic system with multiple delays. Synchronizing fractional multi-time delayed chaotic system with known and unknown parameters is realized based on the proposed approach.

This paper is organized as follows: In Sections 2, Some definitions, Lemmas, and properties about fractional calculus are introduced; In Sections 3, we introduce the new approach and prove it; In Section 4, we show the synchronizing of fractional multi-time delayed chaotic system with and without unknown parameters as examples to explain how to use the proposed approach; Finally, a conclusion is made in Section 5.

2. FRACTIONAL CALCULUS

There are some definitions for fractional derivatives. The commonly used definitions are Grunwald–Letnikov(GL), Riemann–Liouville(RL) and Caputo(C) definitions. The advantage of Caputo approach is that the initial conditions for fractional differential equations take on the same form as those for integer-order differentiation, which have well understood physical meanings. In this paper, we adopt the Caputo definition for fractional derivative. The Caputo definition can be expressed as [18]:

$$ {}_0^C D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \times \int_0^t (t-\tau)^{-\alpha+n-1} x^{(n)}(\tau)\,\mathrm{d}\tau \tag{1} $$

where n is the first integer which is not less than α, i.e. $n-1 \le \alpha \le n$ and $\Gamma(\cdot)$ is gamma function.

Property 1. Let $\alpha \in (0,1)$, then

$$ {}_0^C D_t^\alpha x(t) = \frac{\mathrm{d}\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\tau)^{-\alpha} x(\tau)\mathrm{d}\tau}{\mathrm{d}t} - \frac{x(0)t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2} $$

Lemma 1. (Fractional Comparison Principle 1) (Slotine and Li [19]) Let $x(0) = y(0)$ and ${}_a^C D_t^\alpha x(t) \ge {}_a^C D_t^\alpha y(t)$, where $\alpha \in (0,1)$. Then $x(t) \ge y(t)$.

Lemma 2. (Integer Comparison Principle) Let $x(0) = y(0)$ and $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \ge \frac{\mathrm{d}y(t)}{\mathrm{d}t}$. Then $x(t) \ge y(t)$.

Proof.

$$\begin{aligned} &x(t)-y(t)-(x(0)-y(0))\\ &=\int_0^t \frac{\mathrm{d}x(t)}{\mathrm{d}t}-\frac{\mathrm{d}y(t)}{\mathrm{d}t}\,\mathrm{d}t \geq 0. \end{aligned} \tag{3}$$

As $x(0)=y(0)$, the conclusion

$$x(t)-y(t) \geq 0 \tag{4}$$

can be drawn. □

Lemma 3. (Duarte-Mermoud et al. [2]) For any positive definite matrix P,

$$2x^T(t)P{}_a^C D_t^\alpha x(t) \geq {}_a^C D_t^\alpha (x^T(t)Px(t)). \tag{5}$$

Lemma 4. (Slotine and Li [19]) Fractional system (1) if there is a positive definition function V satisfying that ${}_a^C D_t^\alpha V$ is negative definite, that is ${}_a^C D_t^\alpha V < 0$ for all time $t \geq 0$ and ${}_a^C D_t^\alpha V = 0$ if and only if $x(t)=0$, fractional system (1) is asymptotically stable.

3. MAIN RESULT

3.1. Stability theorem about fractional delayed system

A common fractional nonlinear delayed system can be usually depicted as:

$${}_a^C D_t^\alpha x(t) = f(x(t), x(t-\tau)) \tag{6}$$

where $\alpha \in R$ is fractional order, $x(t) \in R^n$ is state variable, and $f(\cdot)$ is a nonlinear function and satisfying Lipschiz condition.

Theorem 1. When fractional order $\alpha \in (0,1]$, if there is a positive matrix P and a semi positive matrix Q satisfying:

$$x^T(t)P{}_0^C D_t^\alpha x(t) + \frac{\mathrm{d}\int_{t-\tau}^t x^T(\xi)Qx(\xi)\,\mathrm{d}\xi}{\mathrm{d}t} \leq 0 \tag{7}$$

fractional delayed system (6) is asymptotically stable.

Proof. According to Lemma 3 and formula (7), we can get:

$$\begin{aligned} &\frac{1}{2}{}_0^C D_t^\alpha (x^T(t)Px(t)) + \frac{\mathrm{d}\int_{t-\tau}^t x^T(\xi)Qx(\xi)\mathrm{d}\xi}{\mathrm{d}t}\\ &\leq x^T(t)P{}_a^C D_t^\alpha x(t) + \frac{\mathrm{d}\int_{t-\tau}^t x^T(\xi)Qx(\xi)\mathrm{d}\xi}{\mathrm{d}t}\\ &\leq 0. \end{aligned} \tag{8}$$

According to Caputo fractional definition in function (1) and Property 1:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \times \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi + \frac{\mathrm{d}\int_{t-\tau}^t x^T(\xi)Qx(\xi)\mathrm{d}\xi}{\mathrm{d}t} \\
&\le \frac{(x^T(0)Px(0))t^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned} \tag{9}
$$

It obviously:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \lim_{t\to 0} \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi + \int_{t-\tau}^t x^T(\xi)Qx(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \lim_{t\to 0} \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi + 0 \\
&= \frac{1}{\Gamma(1-\alpha)} \lim_{t\to 0} \int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi.
\end{aligned} \tag{10}
$$

According to Lemma 2, calculate integer integral of function (9) and get:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi + \int_{t-\tau}^t x^T(\xi)Qx(\xi)\,\mathrm{d}\xi \\
&\le \frac{1}{\Gamma(1-\alpha)} \int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi.
\end{aligned} \tag{11}
$$

As $\int_{t-\tau}^t x^T(\xi)Qx(\xi)\,\mathrm{d}\xi \ge 0$, then

$$
\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi \le \frac{1}{\Gamma(1-\alpha)} \int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi. \tag{12}
$$

There must exist a nonnegative function $m(t)$ satisfying:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi + \int_0^t m(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi.
\end{aligned} \tag{13}
$$

Take integer order differential of function (13) and get:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi - \frac{1}{\Gamma(1-\alpha)} \int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi\Big) \\
&= -\frac{\mathrm{d}\int_0^t m(\xi)\mathrm{d}\xi}{\mathrm{d}t}.
\end{aligned} \tag{14}
$$

As $m(t) \geq 0$ for any time t, $\frac{\mathrm{d}\int_0^t m(\xi)\,\mathrm{d}xi}{\mathrm{d}t} = m(\varepsilon) \geq 0$, where $(\varepsilon) \in [0, t]$. Then we can get:

$$\begin{aligned}
&{}_0^C D_t^\alpha (x^T(t)Px(t)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\Gamma(1-\alpha)} \times \int_0^t (t-\xi)^{-\alpha}(x^T(\xi)Px(\xi))\,\mathrm{d}\xi - \frac{1}{\Gamma(1-\alpha)}\int_0^t (x^T(0)Px(0))\xi^{-\alpha}\,\mathrm{d}\xi\Big) \\
&= -\frac{\mathrm{d}\int_0^t m(\xi)\,\mathrm{d}\xi}{\mathrm{d}t} \leq 0.
\end{aligned} \tag{15}$$

According to Lemma 4, fractional delayed system (6) is asymptotically stable. The proof of Theorem 1 is completed. □

3.2. A novel approach controlling fractional delayed system

A nonlinear system with multiple time delays consisting of unknown parameters can generally be described as follows:

$$ {}_{t_0}^C D_t^\alpha x(t) = f(x(t)) + \eta(x(t)) + \sum_{i=1}^k g_i(x(t-\tau_i)) \tag{16}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ are the state variables of the nonlinear system at time t, the positive constants $\tau_1, \tau_2, \ldots, \tau_k (\tau_i > 0, i = 1, 2, \ldots, k)$ are the time delays, and function $f(x(t)), \eta(x(t)), g_1(x(t-\tau_1)), g_2(x(t-\tau_2)), \ldots, g_k(x(t-\tau_k))$ are real-valued continuous functions satisfying the Lipschitz condition and function $\eta(x(t))$ is a function includes unknown parameters.

We can transform system (16) as:

$$ {}_{t_0}^C D_t^\alpha(t) = A(x(t))x(t) + \eta(x(t)) + \sum_{i=1}^k g_i x(t-\tau_i) \tag{17}$$

where $A(x(t)) \in R^{n\times n}$, $G \in R^{n\times n}$, $f(x(t)) = A(x(t))x(t), g(x(t-\tau)) = Gx(t-\tau)$. Define the unknown parameter vector as $p(t) = [p_1(t), p_2(t), \ldots, p_k(t)]^T$ and rewrite $\eta(x(t))$ as:

$$ \eta(x(t)) = \Psi(x(t))p(t) \tag{18}$$

where $\Psi(x(t)) \in R^{n\times k}$. The key problem is how to design the controller $u(t)$ and the update rules of unknown parameters to make system (16) asymptotically stable.

Define the estimate value of unknown parameter vector as: $\tilde{p}(t) = [\tilde{p_1}(t), \tilde{p_2}(t), \ldots, \tilde{p_k}(t)]^T$ and the parameter error as $e_p = \tilde{p}(t) - p(t)$. We can express $\eta(x(t))$ as:

$$ \eta(x(t)) = \Psi(x(t))(\tilde{p}(t) - e_p) \tag{19}$$

where $\Psi(x(t)) \in R^{n\times k}$.

We define the update rule of unknown parameter ${}_{t_0}^C D_t^\alpha \tilde{p}(t)$ as:

$$ {}_{t_0}^C D_t^\alpha \tilde{p}(t) = {}_{t_0}^C D_t^\alpha(\tilde{p}(t) - p(t)) = {}_{t_0}^C D_t^\alpha e_p(t) \tag{20}$$

and

$$\Psi(x(t))\tilde{p}(t) = M(\tilde{p}(t))x(t) \tag{21}$$

where $M(\tilde{p}(t)) \in R^{n\times n}$.

Design a controller $u(t)$ and get:

$${}_{t_0}^{C}D_t^{\alpha}x(t) = A(x(t))x(t) + M(\tilde{p}(t))x(t) - \Psi(x(t))e_p + \sum_{i=1}^{k} g_i(x(t-\tau_i)) + u(t). \tag{22}$$

The key problem is how to design the controller $u(t)$ and the update rules of the unknown parameters. We suppose the controller as $u(t) = D(x(t))x(t)$ and the update rules as ${}_{t_0}^{C}D_t^{\alpha}e_p = \Theta x(t)$, where $D(x(t)) \in R^{n\times n}$ and $\Theta \in R^{k\times n}$. If we can design the matrix $D(x(t))$ and the matrix Θ, the controller $u(t)$ and the update rule ${}_{t_0}^{C}D_t^{\alpha}e_p$ can also be realized.

Define $C(x(t)) = A(x(t)) + M(\tilde{p}(t)) + D(x(t))$ and get:

$$\begin{aligned}\begin{bmatrix} {}_{t_0}^{C}D_t^{\alpha}x(t) \\ {}_{t_0}^{C}D_t^{\alpha}e_p \end{bmatrix} &= \begin{bmatrix} C(x(t)) & | & -\Psi(x(t)) \\ -- & & -- \\ \Theta & | & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ e_p \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(t-\tau_1) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} G_2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(t-\tau_2) \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} G_k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(t-\tau_k) \\ 0 \end{bmatrix}\end{aligned} \tag{23}$$

where:

$$C(x(t)) = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{24}$$

and

$$g_i(x(t-\tau_i)) = G_i x(t-\tau_i) = \begin{bmatrix} g_{i,11} & g_{i,12} & \cdots & g_{i,1n} \\ g_{i,21} & g_{i,22} & \cdots & g_{i,2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i,n1} & g_{i,n2} & \cdots & g_{i,nn} \end{bmatrix} x(t-\tau_i), \quad i = 1, 2, \cdots, k. \tag{25}$$

We study how to design the matrix $C(x(t))$ and the matrix Θ to solve the key problem of designing the controller $u(t)$ and the update rule ${}_{t_0}^{C}D_t^{\alpha}e_p$. For this purpose, we propose an approach based on a special matrix as follows:

Theorem 2. If the matrix $C(x(t))$ and the matrix Θ in formula (24) satisfy:

(1) $\Theta = \Psi^T(x(t))$

(2) $c_{mj} = -c_{jm} \quad (m \neq j)$

(3) $c_{mm} + \frac{\sum\limits_{l=1}^{k}\sum\limits_{j=1}^{n}(|g_{l,mj}|+|g_{l,jm}|)}{2} \leq 0, \quad m = 1, 2, \ldots, n.$

(Not all $c_{ii} + \frac{\sum\limits_{l=1}^{k}\sum\limits_{j=1}^{n}(|g_{l,mj}|+|g_{l,jm}|)}{2}$ equal to zero.)

The controlled system (16) is stable to zero and the unknown parameter can be recognized.

P r o o f. Define positive matrix $P = I$ and semi positive matrix Q_i as:

$$Q_i = \frac{1}{2}\begin{bmatrix} \sum_{m=1}^{n}|g_{i,1m}| & 0 & \cdots & 0 \\ 0 & \sum_{m=1}^{n}|g_{i,2m}| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{m=1}^{n}|g_{i,nm}| \end{bmatrix} \quad i = 1, 2, \ldots, k. \tag{26}$$

We can get:

$$\begin{aligned}
&x^T(t)P{}_0^C D_t^{\alpha}x(t) + e^T(t)P{}_0^C D_t^{\alpha}e(t) + \sum_{i=1}^{k}\frac{\mathrm{d}\int_{t-\tau_i}^{t}x^T(\xi)Q_i x(\xi)\,\mathrm{d}\xi}{\mathrm{d}t} \\
&= x^T(t)C(x(t))x(t) - x^T(t)\Psi(x(t))e_p(t) + e_p^T(t)\Psi(x(t))x(t) \\
&+ \sum_{i=1}^{k}x^T(t)G_i x(t-\tau_i) + \sum_{i=1}^{k}x^T(t)Q_i x(t) - \sum_{i=1}^{k}x^T(t-\tau_i)Q_i x(t-\tau_i) \\
&= \sum_{j=1}^{n}c_{jj}x_j^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}\sum_{m=1}^{n}g_{i,mj}(x_m(t)x_j(t-\tau_i)) + \sum_{i=1}^{k}x^T(t)Q_i x(t) \\
&- \sum_{i=1}^{k}x^T(t-\tau_i)Q_i x(t-\tau_i) \\
&\leq \sum_{j=1}^{n}c_{jj}x_j^2 + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n}\sum_{m=1}^{n}|g_{i,mj}|(x_m^2(t) + x_j^2(t-\tau_i)) + \sum_{i=1}^{k}x^T(t)Q_i x(t) \\
&- \sum_{i=1}^{k}x^T(t-\tau_i)Q_i x(t-\tau_i) \\
&= \sum_{j=1}^{n}c_{jj}x_j^2 + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{n}|g_{i,mj}|(x_m^2(t) + x_j^2(t-\tau_i)) \\
&+ \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{n}|g_{i,jm}|x_j^2(t) - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{k}\sum_{m=1}^{n}|g_{i,jm}|x_j^2(t-\tau_i) \\
&= \sum_{j=1}^{n}((c_{jj} + \frac{1}{2}\sum_{i=1}^{k}\sum_{m=1}^{n}(|g_{i,mj}| + |g_{i,jm}|))x_j^2(t)) \leq 0
\end{aligned} \tag{27}$$

when $(c_{jj} + \frac{1}{2}\sum_{i=1}^{k}\sum\limits_{m=1}^{n}(|g_{i,mj}| + |g_{i,jm}|))x_j^2(t)) \leq 0$ where $j = 1, 2, \ldots, n$.

According to the Theorem 1, the fractional multi-time delays system (16) is asymptotically stable. The proof of Theorem 2 is completed. □

4. SYNCHRONIZING FRACTIONAL MULTI-TIME DELAYS CHAOTIC SYSTEM WITHOUT AND WITH UNKNOWN PARAMETERS

4.1. Synchronizing fractional multi-time delays hyperchaotic Lorenz system without unknown parameters

The fractional hyperchaotic Lorenz system with time delays can be described as [25]:

$$\begin{aligned}
{}_a^C D_t^\alpha x_1(t) &= 9x_2(t) - 10x_1(t) + x_2(t-\tau_1)\\
{}_a^C D_t^\alpha x_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t) + 0.3x_2(t-\tau_2)\\
{}_a^C D_t^\alpha x_3(t) &= x_1(t)x_2(t) - \frac{8}{3}x_3(t)\\
{}_a^C D_t^\alpha x_4(t) &= x_2(t)x_3(t) - x_4(t) + 0.5x_2(t-\tau_3).
\end{aligned} \tag{28}$$

Where $\tau_1, \tau_2, \tau_3 > 0$ are the delay times. When $\tau_1 = 0, \tau_2 = 0, \tau_3 = 0$ and fractional order $\alpha = 0.97$, system (29) has chaotic attractor shown as Figure 1. For convenience, we call it fractional hyperchaotic Lorenz system.

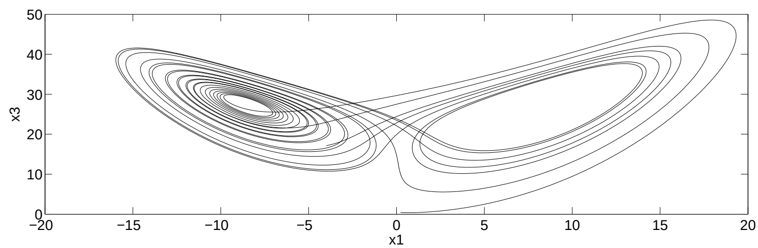

Fig. 1. The chaotic attractor of hyperchaotic Lorenz system in system (29).

The hyperchaotic Lorenz delayed system (28) is chosen as the drive system, and the response system is defined as:

$$\begin{aligned}
{}_a^C D_t^\alpha y_1(t) &= 9y_2(t) - 10y_1(t) + y_2(t-\tau_1) + u_1(t)\\
{}_a^C D_t^\alpha y_2(t) &= 28y_1(t) - y_2(t) - y_1(t)y_3(t) + 0.3y_2(t-\tau_2) + u_2(t)\\
{}_a^C D_t^\alpha y_3(t) &= y_1(t)y_2(t) - \frac{8}{3}y_3(t) + u_3(t)\\
{}_a^C D_t^\alpha y_4(t) &= y_2(t)y_3(t) - y_4(t) + 0.5y_2(t-\tau_3) + u_4(t)
\end{aligned} \tag{29}$$

where $u(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ is the controller to be constructed. Subtract the drive system (28) from the response system (29) and get:

$$
\begin{aligned}
{}_a^C D_t^\alpha e_1(t) &= 9e_2(t) - 10e_1(t) + e_2(t-\tau_1) + u_1(t) \\
{}_a^C D_t^\alpha e_2(t) &= 28e_1(t) - e_2(t) - e_1(t)y_3(t) - e_3(t)x_1(t) + 0.3e_2(t-\tau_2) + u_2(t) \\
{}_a^C D_t^\alpha e_3(t) &= e_1(t)y_2(t) + e_2(t)x_1(t) - \frac{8}{3}e_3(t) + u_3(t) \\
{}_a^C D_t^\alpha e_4(t) &= e_2(t)y_3(t) + e_3(t)x_2(t) - e_4(t) + 0.5e_2(t-\tau_3) + u_4(t)
\end{aligned}
\tag{30}
$$

where $e_1(t) = y_1(t) - x_1(t), e_2(t) = y_2(t) - x_2(t), e_3(t) = y_3(t) - x_3(t), e_4(t) = y_4(t) - x_4(t)$. We can transform the error system (30) as:

$$
\begin{aligned}
&\begin{bmatrix} {}_a^C D_t^\alpha e_1(t) \\ {}_a^C D_t^\alpha e_2(t) \\ {}_a^C D_t^\alpha e_3(t) \\ {}_a^C D_t^\alpha e_4(t) \end{bmatrix} = \begin{bmatrix} -10 & 9 & 0 & 0 \\ 28 - y_3 & -1 & -x_1(t) & 0 \\ y_2(t) & x_1(t) & -\frac{8}{3} & 0 \\ 0 & y_3(t) & x_2(t) & -1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_1) \\ e_2(t-\tau_1) \\ e_3(t-\tau_1) \\ e_4(t-\tau_1) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_2) \\ e_2(t-\tau_2) \\ e_3(t-\tau_2) \\ e_4(t-\tau_2) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_3) \\ e_2(t-\tau_3) \\ e_3(t-\tau_3) \\ e_4(t-\tau_3) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}.
\end{aligned}
\tag{31}
$$

We design the $C(e(t))$ according to Theorem 2:

$$
C(e(t)) = \begin{bmatrix} -10 + k_1 & y_3(t) & -y_2(t) & 0 \\ -y_3(t) & -1 + k_2 & -x_1(t) & -y_3(t) \\ y_2(t) & x_1(t) & -\frac{8}{3} + k_3 & -x_2(t) \\ 0 & y_3(t) & x_2(t) & -1 + k_4 \end{bmatrix}.
\tag{32}
$$

Then, we can get:

$$
\begin{aligned}
\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} &= \begin{bmatrix} -10 + k_1 & y_3(t) & -y_2(t) & 0 \\ -y_3(t) & -1 + k_2 & -x_1(t) & -y_3(t) \\ y_2(t) & x_1(t) & -\frac{8}{3} + k_3 & -x_2(t) \\ 0 & y_3(t) & x_2(t) & -1 + k_4 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} \\
&- \begin{bmatrix} -10 & 9 & 0 & 0 \\ 28 - y_3(t) & -1 & -x_1(t) & 0 \\ y_2(t) & x_1(t) & -\frac{8}{3} & 0 \\ 0 & y_3(t) & x_2(t) & -1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \end{bmatrix} \\
&= \begin{bmatrix} (y_3(t) - 9)e_2(t) - y_2(t)e_3(t) + k_1e_1(t) \\ -28e_1(t) - 2e_2(t) - y_3(t)e_4(t) + k_2e_2(t) \\ -x_2e_4(t) + k_3e_3(t) \\ +k_4e_4(t) \end{bmatrix}.
\end{aligned}
\tag{33}
$$

According to Theorem 2, $-10+k_1+\frac{\tau_1}{2}\le 0$, $-1+k_2+\frac{\tau_1}{2}+\frac{0.3\tau_2}{2}+\frac{0.3\tau_2}{2}+\frac{0.5\tau_3}{2}\le 0$, $-\frac{8}{3}+k_3\le 0$,$-1+k_4+\frac{0.5\tau_3}{2}\le 0$, the synchronizing error system (30) is asymptotically stable. In numerical simulations, the time delays are chosen as $\tau_1=0.1, \tau_2=0.13, \tau_3=0.21$, and $k_1=0$,$k_2=0.2$,$k_3=0$,$k_4=0$. The initial states of the drive system and the response system are $x_1(0)=0.13, x_2(0)=0.11, x_3(0)=0.13, x_4(0)=0.12, y_1(0)=0.38, y_2(0)=0.55, y_3(0)=0.46, y_4(0)=0.32$. The simulation results of synchronizing errors $e_1(t), e_2(t)$ are shown in Figure 2. As expected, all the errors exponentially converge to zero with time t.

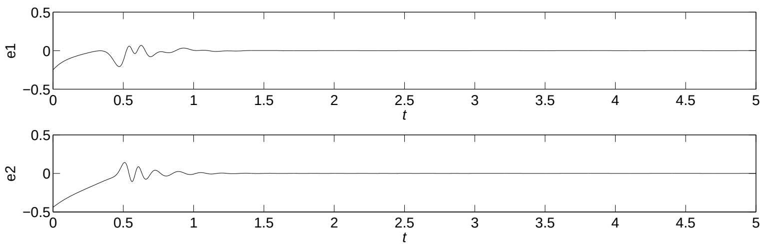

Fig. 2. The synchronizing error signals $e_1(t)$ and $e_2(t)$ in the error system (31) with time t.

4.2. Synchronizing fractional multi-time delays chaotic system with unknown parameters

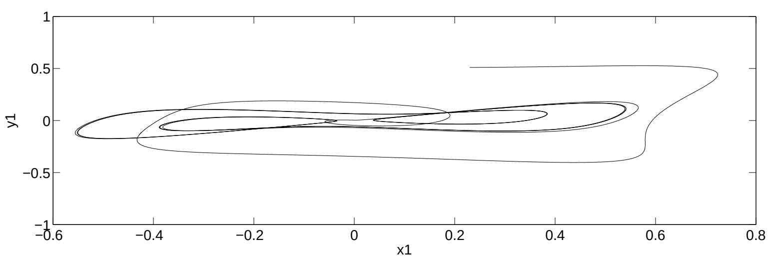

Fig. 3. The chaotic attractor of fractional-order Newton–Leipnik chaotic system.

The fractional-order Newton–Leipnik chaotic system is proposed in [26], which can be depicted as:

$$\begin{cases} {}_a^C D_t^\alpha x_1(t) = -ax_1(t) + y_1(t) + 10y_1(t)z_1(t) \\ {}_a^C D_t^\alpha y_1(t) = -x_1(t) - 0.4y_1(t) + 5x_1(t)z_1(t) \\ {}_a^C D_t^\alpha z_1(t) = bz_1(t) - 5x_1(t)y_1(t) \end{cases} \tag{34}$$

where α is fractional order. It has been shown that the system will exhibit chaotic behavior when $0.94 \le \alpha < 1$ and $a = 0.4, b = 0.175$. The attractor is shown as Figure 3. The multiple delayed fractional-order Newton–Leipnik system can be expressed as:

$$\begin{cases} {}_a^C D_t^\alpha x_1(t) = -ax_1(t) + 0.5y_1(t) + 10y_1(t)z_1(t) + 0.5y_1(t-\tau_1) \\ {}_a^C D_t^\alpha y_1(t) = -x_1(t) - 0.4y_1(t-\tau_2) + 5x_1(t)z_1(t) \\ {}_a^C D_t^\alpha z_1(t) = bz_1(t) - 5x_1(t)y_1(t) \end{cases} \tag{35}$$

where $\tau_1, \tau_2 > 0$ are the delay time, a and b are the parameters of system (35).

In this section, we consider how to synchronize the system (35) when the parameters a and b are unknown. Define the fractional delayed system (35) as the drive system, and the response system is defined as:

$$\begin{cases} {}_a^C D_t^\alpha x_2(t) = -\hat{a}x_2(t) + 0.5y_2(t) + 10y_2(t)z_2(t) + 0.5y_2(t-\tau_1)) + u_1(t) \\ {}_a^C D_t^\alpha y_2(t) = -x_2(t) - 0.4y_2(t-\tau_2) + 5x_2(t)z_2(t) + u_2(t) \\ {}_a^C D_t^\alpha z_2(t) = \hat{b}z_2(t) - 5x_2(t)y_2(t) + u_3(t) \end{cases} \tag{36}$$

where the parameters $\hat{a}$ and $\hat{b}$ are the estimate values of the unknown parameters a and b. We define the synchronizing errors and the estimate errors of unknown parameters respectively as:

$$\begin{cases} e_1(t) = x_2(t) - x_1(t) \\ e_2(t) = y_2(t) - y_1(t) \\ e_3(t) = z_2(t) - z_1(t) \end{cases} \tag{37}$$

$$\begin{cases} e_a(t) = \hat{a} - a \\ e_b(t) = \hat{b} - b. \end{cases} \tag{38}$$

Then, we can get the synchronizing errors as:

$$\begin{aligned} {}_a^C D_t^\alpha e_1(t) &= -\hat{a}e_1(t) + 0.5e_2(t) + 10y_2(t)e_3(t) + 10z_1(t)e_2(t) + 0.5e_2(t-\tau_1) - e_a x_1 + u_1(t) \\ {}_a^C D_t^\alpha e_2(t) &= -e_1(t) - 0.4e_2(t-\tau_2) + 5x_1(t)e_3(t) + 5z_2(t)e_1(t) + u_2(t) \\ {}_a^C D_t^\alpha e_3(t) &= \hat{b}e_3(t) - 5x_1(t)e_2(t) - 5y_2(t)e_1(t) + e_b z_1 + u_3(t). \end{aligned} \tag{39}$$

We can transform the error system (39) as:

$$\begin{bmatrix} {}_a^C D_t^\alpha e_1(t) \\ {}_a^C D_t^\alpha e_2(t) \\ {}_a^C D_t^\alpha e_3(t) \\ {}_a^C D_t^\alpha e_a(t) \\ {}_a^C D_t^\alpha e_b(t) \end{bmatrix} = \begin{bmatrix} -\hat{a} & 0.5+10z_1(t) & 10y_2(t) & -x_1 & 0 \\ -1+5z_2(t) & 0 & 5x_1(t) & 0 & 0 \\ -5y_2(t) & -5x_1(t) & \hat{b} & 0 & y_1(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_a(t) \\ e_b(t) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_1) \\ e_2(t-\tau_1) \\ e_3(t-\tau_1) \\ 0 \\ 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_2) \\ e_2(t-\tau_2) \\ e_3(t-\tau_2) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ 0 \\ 0 \end{bmatrix}. \tag{40}$$

According to Theorem 1, we can design the controller update rules of unknown parameters and matrix $C(e(t))$ respectively as:

$$C(e(t)) = \begin{bmatrix} -2+k_1 & 0.5+10z_1(t) & 10y_2(t) \\ -0.5-10z_1(t) & -2+k_2 & 5x_1(t) \\ -10y_2(t) & -5x_1(t) & k_3 \end{bmatrix} \tag{41}$$

and

$$\begin{bmatrix} \dot{e_a}(t) \\ \dot{e_a}(t) \end{bmatrix} = \begin{bmatrix} x_1 e_1 \\ -z_1 e_3 \end{bmatrix}. \tag{42}$$

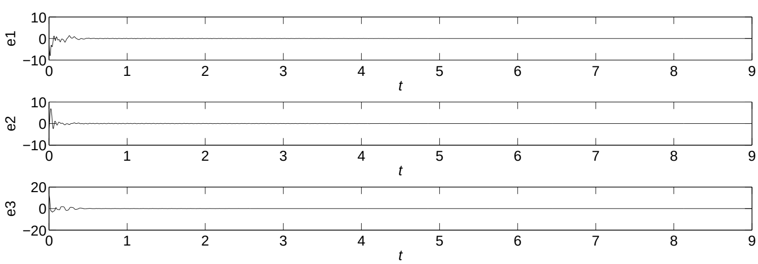

Fig. 4. The synchronizing error signals $e_1(t)$, $e_2(t)$ and $e_3(t)$ in the error system (41) with time t.

Then, we can get:

$$\begin{bmatrix} {}_a^C D_t^\alpha e_1(t) \\ {}_a^C D_t^\alpha e_2(t) \\ {}_a^C D_t^\alpha e_3(t) \\ {}_a^C D_t^\alpha e_a(t) \\ {}_a^C D_t^\alpha e_b(t) \end{bmatrix} = \left[\begin{array}{ccc|cc} -2+k_1 & 0.5+10z_1(t) & 10y_2(t) & -x_1(t) & 0 \\ -0.5-10z_1(t) & -2+k_2 & 5x_1(t) & 0 & 0 \\ -10y_2(t) & -5x_1(t) & k_3 & 0 & z_1(t) \\ \hline x_1(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & -z_1(t) & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_a(t) \\ e_b(t) \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_1) \\ e_2(t-\tau_1) \\ e_3(t-\tau_1) \\ 0 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t-\tau_2) \\ e_2(t-\tau_2) \\ e_3(t-\tau_2) \\ 0 \\ 0 \end{bmatrix} \tag{43}$$

and

$$\begin{aligned} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} &= \begin{bmatrix} -2+k_1 & 0.5+10z_1(t) & 10y_2(t) \\ -0.5-10z_1(t) & -2+k_2 & 5x_1(t) \\ -10y_2(t) & -5x_1(t) & k_3 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} -\hat{a} & 0.5+10z_1(t) & 10y_2(t) \\ -1+5z_2(t) & 0 & 5x_1(t) \\ -5y_2(t) & -5x_1(t) & \hat{b} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &= \begin{bmatrix} (\hat{a}-2+k_1)e_1(t) \\ (0.5-15z_1(t))e_1(t)-(2-k_2)e_2(t) \\ -5y_2e_1(t)+(k3-\hat{b})e_3(t) \end{bmatrix}. \end{aligned} \tag{44}$$

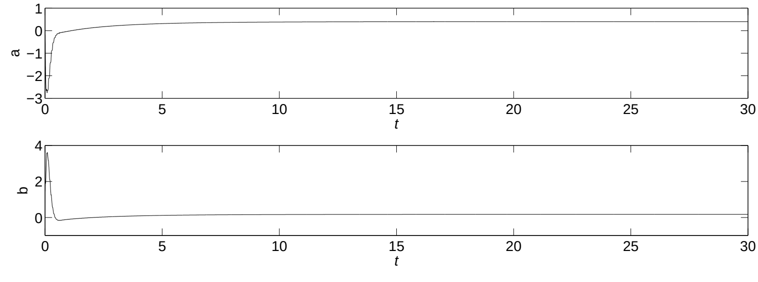

Fig. 5. The identification process of unknown parameters a and b.

According to Theorem 2, $-2+k_1+\frac{0.5\tau_1}{2}\le 0$, $-2+k_2+\frac{0.5\tau_1}{2}+\frac{0.4\tau_2}{2}+\frac{0.4\tau_2}{2}\le 0$, the synchronizing error system (39) is asymptotically stable. In numerical simulations, the time delays are chosen as $\tau_1=0.12, \tau_2=0.13$ $k_1=0, k_2=0$ and the unknown parameters as $a=0.4, b=0.175$. The simulation results of synchronizing errors $e_1(t), e_2(t)$ and the identification process of unknown parameters a, b are shown in Figure 4 and Figure 5 respectively. All the errors exponentially converge to zero with time t.

5. CONCLUSION

In this paper, we propose a new approach for synchronizing fractional chaotic systems with multiple delays that is by the integer derivative intermediate process, rather than by the result of calculating the integer derivate of function V. We design controllers and adaptive update rule of unknown parameters according to the proposed matrix configuration approach and realize synchronizing multi-delay fractional chaotic systems with and without unknown parameters. The proposed approach is a general and simple approach.

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