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PSEUDOSYMMETRIC AND WEYL-PSEUDOSYMMETRIC 
\((\kappa, \mu)\)-CONTACT METRIC MANIFOLDS

N. Malekzadeh, E. Abedi, and U.C. De

Abstract. In this paper we classify pseudosymmetric and Ricci-pseudo-
symmetric \((\kappa, \mu)\)-contact metric manifolds in the sense of Deszcz. Next we
characterize Weyl-pseudosymmetric \((\kappa, \mu)\)-contact metric manifolds.

1. Introduction

Chaki [5] and Deszcz [11] introduced two different concept of a pseudosym-
mometric manifold. In both senses various properties of pseudosymmetric mani-
folds have been studied ([5]–[10]). We shall study properties of pseudosymmetric,
Ricci-pseudosymmetric and Weyl-pseudostymmetric manifolds in the sense of
Deszcz.

A Riemannian manifold is called semisymmetric if \(R(X,Y) \cdot R = 0\) where
\(X, Y \in \chi(M)\), [24]. Deszcz [11] generalized the concept of semisymmetry and
introduced pseudosymmetric manifolds. Let \((M^n, g), n \geq 3\) be a Riemannian
manifold. We denote by \(\nabla, R\) and \(\tau\) the Levi–Civita connection, the curvature
tensor and the scalar curvature of \((M, g)\), respectively. We define endomorphism
\(X \wedge Y\) for arbitrary vector field \(Z\), \((0, k)\)-tensor \(T\) and \((1, k)\)-tensor \(T_1, k \geq 1\), by
(1) \((X \wedge Y) Z = g(Y, Z)X - g(X, Z)Y\),
(2) \((X \wedge Y) \cdot T)(X_1, X_2, \ldots, X_k) = -T((X \wedge Y)X_1, X_2, \ldots, X_k)
- \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k),
and
(3) \((X \wedge Y) \cdot T_1)(X_1, X_2, \ldots, X_k) = (X \wedge Y)T_1(X_1, X_2, \ldots, X_k)
- T_1((X \wedge Y)X_1, X_2, \ldots, X_k)
- \cdots - T_1(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k),

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respectively. For a \((0, k)\)-tensor field \(T\), the \((0, k + 2)\) tensor fields \(R \cdot T\) and \(Q(g, T)\) are defined by (1, 11)

\[
(R \cdot T)(X_1, \ldots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \ldots, X_k)
\]

\[
= -T(R(X, Y)X_1, X_2, \ldots, X_k)
\]

\[
- \cdots - T(X_1, \ldots, X_{k-1}, R(X, Y)X_k),
\]

and

\[
Q(g, T)(X_1, \ldots, X_k; X, Y) = -T((X \wedge Y)X_1, X_2, \ldots, X_k)
\]

\[
- \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k).
\]

A Riemannian manifold \(M\) is said to be pseudosymmetric if the tensors \(R \cdot R\) and \(Q(g, R)\) are linearly dependent at every point of \(M\), i.e.

\[
R \cdot R = L_R Q(g, R).
\]

This is equivalent to

\[
(R(X, Y) \cdot R)(U, V, W) = L_R \left[ ((X \wedge Y) \cdot R)(U, V, W) \right]
\]

holding on the set \(U_R = \{ x \in M : Q(g, R) \neq 0 \text{ at } x \}\), where \(L_R\) is some function on \(U_R\), 11. The manifold \(M\) is called pseudosymmetric of constant type if \(L\) is constant. Particularly if \(L_R = 0\) then \(M\) is a semisymmetric manifold. The manifold \(M\) is said to be locally symmetric if \(\nabla R = 0\). Obviously locally symmetric spaces are semisymmetric, 25.

Let \(S\) denote the Ricci tensor of \(M^{2n+1}\). The Ricci operator \(Q\) is the symmetric endomorphism on the tangent space given by

\[
S(X, Y) = g(QX, Y).
\]

If the tensors \(R \cdot S\) and \(Q(g, S)\) are linearly dependent at every point of \(M\), i.e.

\[
R \cdot S = L_S Q(g, S),
\]

then \(M\) is called Ricci-pseudosymmetric. This is equivalent to

\[
(R(X, Y) \cdot S)(Z, W) = L_S \left[ ((X \wedge Y) \cdot S)(Z, W) \right]
\]

holds on the set \(U_S = \{ x \in M : S - \frac{\tau}{n}g \neq 0 \text{ at } x \}\), for some function \(L_S\) on \(U_S\) (7, [19]). We note that \(U_S \subset U_R\) and on 3-dimensional Riemannian manifolds we have \(U_S = U_R\). Every pseudosymmetric manifold is Ricci-pseudosymmetric but the converse statement is not true.

The Weyl conformal curvature operator \(C\) is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1} \left\{ (X \wedge QY)Z + (QX \wedge Y)Z - \frac{\tau}{2n}(X \wedge Y)Z \right\}.
\]

If \(C = 0, n \geq 3\), then \(M\) is called conformally flat. If the tensors \(R \cdot C\) and \(Q(g, C)\) are linearly dependent, then \(M\) is called Weyl-pseudosymmetric. This is equivalent to the statement that

\[
(R \cdot C)(U, V, W, X, Y) = L_C \left[ ((X \wedge Y) \cdot C)(U, V)W \right]
\]
holds on the set $U_C = \{ x \in M : C \neq 0 \text{ at } x \}$, where $L_C$ is defined on $U_C$. If $R \cdot C = 0$, then $M$ is called Weyl-semisymmetric. If $\nabla C = 0$, then $M$ is called conformally symmetric \([21], [23]\).

3-dimensional pseudosymmetric spaces of constant type have been studied by Kowalski and Sekizawa \([16]–[17]\). Conformally flat pseudosymmetric spaces of constant type were classified by Hashimoto and Sekizawa for dimension three, \([14]\) and by Calvaruso for dimensions $> 2$, \([4]\). In dimension three, Cho and Inoguchi studied pseudosymmetric contact homogeneous manifolds, \([6]\). Cho et al. treated the conditions that 3-dimensional trans-Sasakians, non-Sasakian generalized $(\kappa, \mu)$-spaces and quasi-Sasakians manifolds be pseudosymmetric, \([1]\). Belkhelfa et al. obtained some results on pseudosymmetric Sasakian space forms, \([1]\). Finally some classes of pseudosymmetric contact metric 3-manifolds have been studied by Gouli-Andreou and Moutafi \([12], [13]\).

Papantoniou classified semisymmetric $(\kappa, \mu)$-contact metric manifolds \([22, \text{ Theorem 3.4}]\). As a generalization, in this paper, we study pseudosymmetric $(\kappa, \mu)$-contact metric manifolds.

This paper is organized as follows. After some preliminaries on $(\kappa, \mu)$-contact metric manifolds, in Section 3 we study pseudosymmetric and Ricci-pseudosymmetric $(\kappa, \mu)$-contact metric manifolds. Next in Section 4 we characterize Weyl-pseudosymmetric $(\kappa, \mu)$-contact metric manifolds.

2. Preliminaries

A contact manifold is an odd-dimensional $C^\infty$ manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field $\xi$ on $M^{2n+1}$ satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any $X \in \chi(M)$ is called the Reeb vector field or characteristic vector field of $\eta$. A Riemannian metric $g$ is said to be an associated metric if there exists a $(1,1)$ tensor field $\varphi$ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 = -I + \eta \otimes \xi.$$  

The structure $(\varphi, \xi, \eta, g)$ is called a contact metric structure and a manifold $M^{2n+1}$ with a contact metric structure is said to be a contact metric manifold. Given a contact metric structure $(\varphi, \xi, \eta, g)$, we define a $(1,1)$ tensor field $h$ by $h = (1/2)L_\xi \varphi$ where $L$ denotes the operator of Lie differentiation. A contact metric manifold for which $\xi$ is a Killing vector field is called a $K$-contact manifold. It is well known that a contact manifold is $K$-contact if and only if $h = 0$. A contact metric manifold is said to be a Sasakian manifold if

$$(\nabla_X \varphi) Y = g(X, Y) \xi - \eta(Y) X$$

in which case

$$R(X, Y) \xi = \eta(Y) X - \eta(X) Y.$$  

Note that a Sasakian manifold is $K$-contact, but the converse holds only if $\dim M = 3$.  


A contact manifold is said to be $\eta$-Einstein if the Ricci operator $Q$ satisfies the condition

\begin{equation}
Q = a \operatorname{Id} + b \eta \otimes \xi,
\end{equation}

where $a$ and $b$ are smooth functions on $M^{2n+1}$.

The sectional curvature $K(\xi, X)$ of a plane section spanned by $\xi$ and a vector $X$ orthogonal to $\xi$ is called a $\xi$-sectional curvature, while the sectional curvature $K(X, \varphi X)$ is called a $\varphi$-sectional curvature.

The $(\kappa, \mu)$-nullity distribution of a contact metric manifold $M(\varphi, \xi, \eta, g)$ is a distribution, \[ N(\kappa, \mu) \]

\begin{equation}
N(\kappa, \mu): p \rightarrow N_p(\kappa, \mu) = \{ W \in T_p M \mid R(X, Y)W = \kappa[g(Y, W)X - g(X, W)Y] + \mu[g(Y, W)hX - g(X, W)hY] \},
\end{equation}

where $\kappa, \mu$ are real constants. Hence if the characteristic vector field $\xi$ belongs to the $(\kappa, \mu)$-nullity distribution, then we have

\begin{equation}
R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(X)hX - \eta(Y)hY\}.
\end{equation}

A contact metric manifold satisfying (14) is called a $(\kappa, \mu)$-contact metric manifold. If $M$ be a $(\kappa, \mu)$-contact metric manifold, then the following relations hold, \[ 3 \]:

\begin{enumerate}
\item[(15)] $S(X, \xi) = 2nk\eta(X)$,
\item[(16)] $Q\xi = 2nk\xi$,
\item[(17)] $h^2 = (k - 1)\varphi^2$,
\item[(18)] $R(\xi, X)Y = \kappa\{g(X, Y)\xi - \eta(Y)X\} + \mu\{g(hX, Y)\xi - \eta(Y)hX\}$,
\item[(19)] $S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2\kappa + \mu)]\eta(X)\eta(Y)$,
\item[(20)] $\tau = 2n(2(n - 1) + \kappa - n\mu)$,
\item[(21)] $Q\varphi - \varphi Q = 2[2(n - 1) + \mu]h\varphi$.
\end{enumerate}

We note that if $M^{2n+1}$ be a $(\kappa, \mu)$-contact metric manifold, then $\kappa \leq 1$, \[ 3 \]. When $\kappa < 1$, the nonzero eigenvalues of $h$ are $\pm \sqrt{1 - \kappa}$ each with multiplicity $n$. Let $\lambda$ and $D$ denote the positive eigenvalue of $h$ and the distribution $\text{Ker} \eta$ respectively. Then $M^{2n+1}$ admits three mutually orthogonal and integrable distributions $D(0), D(\lambda)$ and $D(-\lambda)$ defined by the eigenspaces of $h$, \[ 26 \]. We easily check that Sasakian manifolds are contact $(\kappa, \mu)$-manifolds with $\kappa = 1$ and $h = 0$, \[ 3 \]. In particular, if $\mu = 0$, then we obtain the condition of $k$-nullity distribution introduced by Tanno, \[ 26 \].
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We know that if \(M^{2n+1}\) be a contact metric manifold and \(R_{XY}\xi = 0\) for all vector fields \(X\) and \(Y\), then \(M^{2n+1}\) is locally isometric to the Riemannian product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of positive constant curvature 4.

In Blair et al. studied the condition of \((\kappa, \mu)\)-nullity distribution on a contact manifold and obtained the following theorem.

**Theorem 1.** Let \(M^{2n+1}(\varphi, \xi, \eta, g)\) be a contact manifold with \(\xi\) belonging to the \((\kappa, \mu)\)-nullity distribution. If \(\kappa < 1\), then for any \(X\) orthogonal to \(\xi\) the following formulas hold:

1. The \(\xi\)-sectional curvature \(K(X, \xi)\) is given by

\[
K(X, \xi) = \kappa + \mu g(hX, X) = \begin{cases} 
\kappa + \lambda \mu & \text{if } X \in D(\lambda) \\
\kappa + \lambda \mu & \text{if } X \in D(-\lambda)
\end{cases}
\]

2. The sectional curvature of a plan section \(\{X, Y\}\) normal to \(\xi\) is given by

\[
K(X, Y) = \begin{cases} 
2(1 + \lambda) - \mu & \text{if } X, Y \in D(\lambda) \\
-(\kappa + \mu)[g(X, \varphi Y)]^2 & \text{for any unit vectors} \\
2(1 - \lambda) - \mu & \text{if } X, Y \in D(-\lambda)
\end{cases}
\]

Pseudosymmetric contact 3-manifold were studied in and following result obtained.

**Theorem 2.** Contact Riemannian 3-manifolds such that \(Q\varphi = \varphi Q\) are pseudosymmetric. In particular, every Sasakian 3-manifold is a pseudosymmetric space of constant type.

Firstly we give the following propositions.

**Proposition 1.** Let \(M^{2n+1}\) be a \((\kappa, \mu)\)-contact metric pseudosymmetric manifold. Then for any unit vector fields \(X, Y \in \chi(M)\) orthogonal to \(\xi\) and such that \(g(X, Y) = 0\) we have:

\[
\{(\kappa - L_R)g(X, R(X, Y)Y) + \mu g(hX, R(X, Y)Y) - \kappa(\kappa - L_R)
- \mu(\kappa - L_R)g(hY, Y) - \kappa \mu g(hX, X) - \mu^2 g(hX, X)g(hY, Y)
+ \mu^2 g^2(hX, Y))\}\xi
- (\kappa - L_R)g(R(X, Y)Y, \xi)X - \mu g(R(X, Y)Y, \xi)hX = 0.
\]

**Proof.** Since \(M\) is pseudosymmetric then

\[
(R(\xi, X) \cdot R)(U, V)W = L_R[(\xi \wedge X) \cdot R](U, V)W.
\]
Putting $U = X$ and $V = W = Y$ in (24) and using (3) and (4), we get

$$R(\xi, X) \cdot R(X, Y) = R(R_\xi X, Y) - R(R_\xi Y, X) - R(X, R_\xi Y) Y - R(Y, R_\xi X) Y$$

$$= L_R \{(\xi \wedge X) \cdot R(X, Y) - R((\xi \wedge X), X) Y\} - R(X, (\xi \wedge X) Y) - R(Y, R(\xi, X)) Y.$$  

(25)

From (1) and (18) one can easily get the result. \hfill \Box

**Proposition 2.** Every pseudosymmetric Sasakian manifold with $L_R \neq 1$ is of constant curvature 1.

**Proof.** Let $X$ and $Y$ be tangent vectors such that $\eta(X) = \eta(Y) = 0$ and $g(X, Y) = 0$. Since $M$ is Sasakian then $\kappa = 1$ and $\mu = 0$. Using (12) and (18) in equation (25) and direct computations we get

$$(1 - L_R)\{\eta(R(X, Y) Y - g(X, R(X, Y) Y) \xi + g(X, X) g(Y, Y) Y\} = 0.$$  

Since $L_R \neq 1$ then

(26) \hspace{1cm} \eta(R(X, Y) Y - g(X, R(X, Y) Y) \xi + g(X, X) g(Y, Y) Y = 0. $$

Taking the inner product with $\xi$ gives

(27) \hspace{1cm} g(X, R(X, Y) Y) = g(X, X) g(Y, Y).$$

Then $(M^{2n+1}, g)$ is of constant $\varphi$-sectional curvature 1 and hence it is of constant curvature 1. \hfill \Box

**Theorem 3.** Let $M^{2n+1}$, $n > 1$ be a $(\kappa, \mu)$-contact metric pseudosymmetric manifold. Then $M^{2n+1}$ is either

1) A Sasakian manifold of constant sectional curvature 1 if $L_R \neq 1$ or

2) Locally isometric to the product of a flat $(n + 1)$-dimensional Euclidean manifold and an $n$-dimensional manifold of constant curvature 4.

**Proof.** If $\kappa = 1$ then $M$ is a Sasakian manifold and result get from Proposition 2.

Let $\kappa < 1$ and $X$, $Y$ are orthonormal vectors of the distribution $D(\lambda)$. Applying the relation (23) for $hX = \lambda X$, $hY = \lambda Y$ we get

$$\{(\kappa - L_R + \mu \lambda) g(X, R(X, Y) Y) - \kappa (\kappa - L_R) - \mu \lambda (\kappa - L_R) - \kappa \mu \lambda - \mu^2 \lambda^2 \} \xi$$

(28) \hspace{1cm} - (\kappa - L_R + \mu \lambda) g(R(X, Y) Y, \xi) X = 0. $$

Considering $\xi$-component of (28) gives

(29) \hspace{1cm} i) \ K(X, Y) = \kappa + \mu \lambda \hspace{1cm} \text{or} \hspace{1cm} ii) \ \kappa = -\lambda \mu + L_R.$$  

Comparing part (i) of equations (22) and (29) gives

(30) \hspace{1cm} \mu = 1 + \lambda.$$  

Let $X, Y \in D(-\lambda)$ and $g(X, Y) = 0$. Putting $hX = -\lambda X$ and $hY = -\lambda Y$ in (23) and taking the inner product with $\xi$ we get

(31) \hspace{1cm} i) \ K(X, Y) = \kappa - \lambda \mu \hspace{1cm} \text{or} \hspace{1cm} ii) \ \kappa = \lambda \mu + L_R.$$  

Comparing the equations (22)(iii) and (31)(i) we have

(32) \hspace{1cm} i) \ \mu = 1 - \lambda \hspace{1cm} \text{or} \hspace{1cm} ii) \ \lambda = 1.$$  

\hfill \Box
In the case \( X \in D(\lambda) \) and \( Y \in D(-\lambda) \) equation (23) is reduced to
\[
\{(\kappa - L_R + \mu \lambda)g(X, R(X,Y)Y) - \kappa(\kappa - L_R) + \mu \lambda(\kappa - L_R) - \kappa \mu \lambda + \mu^2 \lambda^2\} \xi
\]
(33) \( - (\kappa - L_R + \mu \lambda)g(R(X,Y)Y, \xi)X = 0, \)
from which taking the inner products with \( \xi \) we have
\[
(34) \quad i) \ K(X,Y) = \kappa - \lambda \mu \quad \text{or} \quad \kappa = -\lambda \mu + L_R,
\]
while if \( X \in D(-\lambda) \) and \( Y \in D(\lambda) \) we similarly prove that
\[
(35) \quad i) \ K(X,Y) = \kappa + \lambda \mu \quad \text{or} \quad \kappa = \lambda \mu + L_R.
\]
By the combination now of the equation (29)(ii), (30), (31)(ii), (32), (34) and (35) we establish the following nine systems among the unknowns \( \kappa, \lambda, \mu \) and \( L_R. \)

1) \( \{\mu = 1 - \lambda, \ \mu = 1 + \lambda, \ \lambda = 0\} \)
2) \( \{\kappa = -\lambda \mu + L_R, \ \kappa = \lambda \mu + L_R, \ \mu = 0, \ \lambda > 0\} \)
3) \( \{\kappa = -\lambda \mu + L_R, \ \lambda = 1, \ \mu = 0\} \)
4) \( \{\kappa = -\lambda \mu + L_R, \ \lambda = 1, \ \mu = L_R\} \)
5) \( \{K(X,Y) = \kappa + \lambda \mu, \ K(X,Y) = \kappa - \lambda \mu, \ \mu = 1 - \lambda, \ \kappa = -\lambda \mu + L_R\} \)
6) \( \{\mu = 1 + \lambda, \ \kappa = 1, \ \lambda = 1, \ \mu = L_R\} \)
7) \( \{\mu = 1 + \lambda, \ K(X,Y) = \kappa - \lambda \mu, \ K(X,Y) = \kappa + \lambda \mu\} \)
8) \( \{\mu = -\lambda \mu + L_R, \ \mu = 1 - \lambda, \ K(X,Y) = \kappa + \lambda \mu\} \)
9) \( \{\mu = 1 + \lambda, \ \kappa = \lambda \mu + L_R, \ K(X,Y) = \kappa - \lambda \mu\} \)

From the first system we get easily \( \mu = 1 \) and since \( \lambda^2 = 1 - \kappa \) we have \( \kappa = 1 \), which is a contradiction, since we required that \( \kappa < 1 \).

The systems 2, 3, 4 and 5 have as the only solution \( \kappa = 0, \ \mu = 0, \ \lambda = 1, \ \lambda = 0. \) Then \( R_{XY} \xi = 0 \) for any \( X, Y \in \chi(M) \) and \( M \) is locally isometric to the product \( E^{n+1}(0) \times S^n(4), \ [2] \). We show that remainder systems can not occur.

In system 6, from \( \lambda = 1 \) we have \( \mu = 0 \) and \( \kappa = 0. \) Using equation (34) (or (35)) and (22)(ii) we have \( [g(X, \varphi Y)]^2 = -1 \) and this is a contradiction.

From system 7, one can get easily \( \lambda \mu = 0. \) But \( \lambda \neq 0 \) (since \( \kappa < 1 \)) and then \( \mu = 0. \) Therefore \( \lambda = \mu - 1 = -1 \) and this is a contradiction with \( \lambda > 0. \)

In two last systems for all \( X, Y \in \chi(M) \) we have
\[
(36) \quad K(X,Y) = L_R.
\]
Let \( Y = \varphi X \) in (36) and comparing it with equation (22)(ii) we get
\[
(37) \quad L_R = -(\kappa + \mu),
\]
Replacing \( \kappa \) and \( \mu \) of two last systems in (37) we get two equation
\[
(38) \quad (1 - \lambda)^2 = -2L_R,
\]
and
\[
(39) \quad (1 + \lambda)^2 = -2L_R,
\]
respectively. Then in systems 8 and 9 \( L_R \leq 0. \)

In system 8, by virtue of \( \kappa = -\lambda \mu + L_R \) and \( \kappa = 1 - \lambda^2 \), we have
\[
2\lambda^2 - \lambda + (L_R - 1) = 0.
\]
This quadratic equation has two roots \( \lambda = 1 \pm \sqrt{9 - 8L_R} \). If \( \lambda = 1 + \sqrt{9 - 8L_R} \) and replacing it in (38) we get \( L_R = 1.5 \) and if \( \lambda = 1 - \sqrt{9 - 8L_R} \), since \( \lambda \) is positive, we get \( L_R > 1 \). Then in the both case we get contradiction wht \( L_R \leq 0 \). The roots of equation (39) in last system are \( \lambda = -1 \pm \sqrt{-2L_R} \) and since \( \lambda > 0 \) then \( \lambda = -1 + \sqrt{-2L_R} \) and hence \( \mu = \sqrt{-2L_R} \). Substituting \( \lambda \) and \( \mu \) in \( \kappa = \lambda \mu + L_R \) and \( \kappa = 1 - \lambda^2 \) we get \( L_R = -2 \) and then \( \lambda = 1, \mu = 2 \) and \( \kappa = 0 \) which are not acceptable since from (34) (or (35)) we get a contradiction from (22)(ii) and this complete the proof.

**Theorem 4.** Every 3-dimensional \((\kappa, \mu)\)-contact metric manifold is pseudosymmetric manifold.

**Proof.** From the combination of the equations (34) and (35) we get four systems with respect to the \( \kappa, \lambda, \mu, L_R \) and the sectional curvature \( K(X,Y) \), from which we have the following possibilities:

1) \[ K(X,Y) = \kappa \], \[ \lambda \mu = 0 \],

2) \[ \kappa = L_R \], \[ \lambda \mu = 0 \],

3) \[ \kappa = \lambda \mu + L_R \] or \[ \kappa = \lambda \mu - L_R \] and \( K(X,Y) = L_R \).

In two first cases we have \( \lambda \mu = 0 \). If \( \mu = 0 \) then equation (21) leads to \( Q \phi = \phi Q \) and result get from Theorem 2 if \( \lambda = 0 \) then \( M^3 \) being a Sasakian manifold and from Theorem 2 every Sasakian 3-manifold is a pseudosymmetric space of constant type.

In the last case, let \( Y = \phi X \) then \( K(X, \phi X) = L_R \). On the other hand, from (22)(ii) \( K(X, \phi X) = -(\kappa + \mu) \). Then \( L_R = -(\kappa + \mu) \) and manifold is of constant sectional curvature. Every Riemannian manifold of constant sectional curvature is locally symmetric (20 page 221) and then pseudosymmetric. Thus \( M^3 \) is pseudosymmetric manifold of constant type.

**Theorem 5.** Let \( M^{2n+1} \) be a Ricci-pseudosymmetric \((\kappa, \mu)\)-contact metric manifold. Then \( M^{2n+1} \) is either

(i) locally isometric to \( E^{n+1} \times S^n(4) \), or

(ii) an Einstein-Sasakian manifold if \( \kappa \neq L_S \), or

(iii) an \( \eta \)-Einstein manifold provided

\[
2n\kappa \mu - (\kappa - L_S)[2(n-1) + \mu] - \mu[2(n-1) - n\mu] \neq 0.
\]

**Proof.** (i) If \( \kappa = 0, \mu = 0 \) then we have \( R_{XY}\xi = 0 \) for any tangent vector fields \( X \), \( Y \) and hence \( M \) is locally isometric to \( E^{n+1} \times S^n(4) \), [2].

(ii) Let \( \kappa \neq 0 \).

Since \( M \) is a Ricci-pseudosymmetric \((\kappa, \mu)\)-contact metric manifold for any \( X, Y, U, V \in \chi(M) \) we have

\[
(R(X,Y) \cdot S)(U,V) = L_S Q(g,S)(U,V;X,Y).
\]

Then from (4) and (5) we can write

\[
-S(R(\xi,Y)Z) - S(Y,R(\xi,X)Z) = L_S \left[ -S((\xi \wedge X)Y,Z) - S(Y, (\xi \wedge X)Z) \right].
\]

Replacing \( Z \) with \( \xi \) and using (1), (15) and (14) one can get

\[
-2n\kappa (\kappa - L_S) g(X,Y) - 2n\kappa \mu g(hX,Y) + (\kappa - L_S) S(X,Y) + \mu S(hX,Y) = 0.
\]
If \( \mu = 0 \) then since \( \kappa \neq 0, L_S \), we get that the manifold is Einstein and then \( M \) is a Sasakian manifold \([26]\) Theorem 5.2).

(iii) Suppose now that \( \kappa \neq 0, \mu \neq 0 \). Then, using the equation (19) and (17), \( \kappa \leq 1 \), we have

\[
S(hX, Y) = [2(n-1) - n\mu]g(hX, Y) - (\kappa - 1)[2(n-1) + \mu]g(X, Y) \\
+ (\kappa - 1)[2(n-1) + \mu]\eta(X)\eta(Y).
\]

Replacing equation (43) in equation (42) gives

\[
\{2n\kappa\mu - (\kappa - L_S)[2(n-1) + \mu] - \mu[2(n-1) - n\mu]\} g(hX, Y)
\]

\[
= \{-2n\kappa(\kappa - L_S) + (\kappa - L_S)[2(n-1) - n\mu] - \mu(\kappa - 1)[2(n-1) + \mu]\} g(X, Y)
\]

\[
+ \{(\kappa - L_S)[2(1-n) + n(2\kappa + \mu)] + \mu(\kappa - 1)[2(n-1) + \mu]\} \eta(X)\eta(Y).
\]

From (19) and (44), we get

\[
S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)
\]

where

\[
\alpha = \frac{[2(n-1) + \mu][-2n\kappa(\kappa - L_S) + (\kappa - L_S)[2(n-1) - n\mu] - \mu(\kappa - 1)(2(n-1) + \mu)]}{2n\kappa\mu - (\kappa - L_S)[2(n-1) + \mu] - \mu[2(n-1) - n\mu]}
\]

\[
+ [2(n-1) - n\mu].
\]

\[
\beta = \frac{[2(n-1) + \mu][((\kappa - L_S)[2(1-n) + n(2\kappa + \mu)] + \mu(\kappa - 1)(2(n-1) + \mu)]}{2n\kappa\mu - (\kappa - L_S)[2(n-1) + \mu] - \mu[2(n-1) - n\mu]}
\]

\[
+ [2(1-n) + n(2\kappa + \mu)].
\]

So, the manifold is an \( \eta \)-Einstein manifold with constant coefficients and the proof is complete. \( \square \)

4. WEYL-PSEUDOSYMMETRIC \((\kappa, \mu)\)-CONTACT METRIC MANIFOLDS

In the present section our aim is to find the characterization of \((\kappa, \mu)\)-contact metric manifolds satisfying the condition \( R \cdot C = L_CQ(g, C) \).

**Theorem 6.** Let \( M^{2n+1}, n > 1 \) be a non-Sasakian \((\kappa, \mu)\)-contact metric manifold. If \( M \) is Weyl-pseudosymmetric manifold then either \( \mu = 0 \) and then \( L_C = \kappa \) or \( \mu = \frac{2n-1}{2n+1} \) holds on \( M \).

**Proof.** Since \( M \) is a Weyl-pseudosymmetric then

\[
(R(X, Y) \cdot C)(U, V, W) = L_CQ(g, C)(U, V, W; X, Y).
\]
Using (4) and (5) in (45) we can write
\[
- C(U, V)R(X, Y)W \\
= L_C[(X \land Y)C(U, V)W - C((X \land Y)U, V)W \\
- C(U, (X \land Y)V)W - C(U, V)(X \land Y)W].
\]
Replacing \( X \) with \( \xi \) and \( Y \) with \( U \) in (46) we have
\[
R(\xi, U)C(U, V)W - C(R(\xi, U)U, V)W - C(U, R(\xi, U)V)W \\
- C(U, V)R(\xi, U)W \\
= L_C[(\xi \land U)C(U, V)W - C((\xi \land U)U, V)W \\
- C(U, (\xi \land U)V)W - C(U, V)(\xi \land U)W].
\]
Substituting (1) and (18) in (47) and taking the inner product with \( \xi \), we get
\[
(\kappa - L_C)g(U, C(U, V)W) + \mu g(hU, C(U, V)W) - (\kappa - L_C)g(U, g(C(\xi, V)W, \xi)] \\
- \mu g(hU, g(C(\xi, V)W, \xi) + \mu g(U, g(C(\xi, V)W, \xi) \\
- (\kappa - L_C)g(U, g(C(U, \xi)W, \xi) - \mu g(hU, g(C(U, \xi)W, \xi) \\
+ \mu g(V, g(C(U, hU)W, \xi) + (\kappa - L_C)g(W, g(C(U, U)V)W, \xi) \\
+ \mu g(W, g(C(U, V)hU, \xi) = 0.
\]
Let \( U \in D(\lambda) \) and contraction of (48) with respect to \( U \) we have
\[
- 2n\kappa + (1 - 2n)\lambda\mu + 2nL_C)g(C(\xi, V)W, \xi) = 0.
\]
Similarity for \( U \in D(-\lambda) \) and contraction of (48) with respect to \( U \) we get
\[
- 2n\kappa - (1 - 2n)\lambda\mu + 2nL_C)g(C(\xi, V)W, \xi) = 0.
\]
Suppose \( \mu = 0 \). Then from the equation (49) we obtain
\[
(L_C - \kappa)g(C(\xi, V)W, \xi) = 0.
\]
If \( g(C(\xi, V)W, \xi) = 0 \). Using (20), (11) and straightforward computation, we have
\[
S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1)\mu]g(hX, Y) \\
+ [2(1 - n) + n(2\kappa + \mu)]\eta(X)\eta(Y).
\]
Comparing equation (52) with (19) one can get
\[
\mu = \frac{2n - 1}{2n - 2}
\]
and this is a contradiction. Then \( \kappa = L_C \).

Suppose now that \( \mu \neq 0 \) and substracting equations (49) and (50), we get
\[
\lambda \mu g(C(\xi, V)W, \xi) = 0.
\]
But \( \lambda \mu \neq 0 \) since \( \kappa < 1 \) and \( \mu \neq 0 \). Hence \( g(C(\xi, V)W, \xi) = 0 \) and then
\[
\mu = \frac{2n - 1}{2n - 2}.
\]
Therefore we have the following corollary.

**Corollary 1.** If $M$ be a Weyl-pseudosymmetric Sasakian manifold then either $L_C = 1$ or $\mu = \frac{2n-1}{2n-2}$ holds on $M$.

**Proof.** Since $M$ is Sasakian then $\kappa = 1$ and $\lambda = 0$. From equation (49) one can easily get the results. □

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**References**


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