Indrajit Lahiri; Amit Sarkar
Meromorphic function sharing a small function with a linear differential polynomial


Persistent URL: [http://dml.cz/dmlcz/144846](http://dml.cz/dmlcz/144846)

**Terms of use:**

© Institute of Mathematics AS CR, 2016

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
Abstract. The problem of uniqueness of an entire or a meromorphic function when it shares a value or a small function with its derivative became popular among the researchers after the work of Rubel and Yang (1977). Several authors extended the problem to higher order derivatives. Since a linear differential polynomial is a natural extension of a derivative, in the paper we study the uniqueness of a meromorphic function that shares one small function CM with a linear differential polynomial, and prove the following result: Let $f$ be a nonconstant meromorphic function and $L$ a nonconstant linear differential polynomial generated by $f$. Suppose that $a = a(z) (\neq 0, \infty)$ is a small function of $f$. If $f - a$ and $L - a$ share $0$ CM and

$$(k + 1)\mathbb{N}(r, \infty; f) + \mathbb{N}(r, 0; f') + N_k(r, 0; f') < \lambda T(r, f') + S(r, f')$$

for some real constant $\lambda \in (0, 1)$, then $f - a = (1 + c/a)(L - a)$, where $c$ is a constant and $1 + c/a \neq 0$.

Keywords: meromorphic function; differential polynomial; small function; sharing

MSC 2010: 30D35

1. INTRODUCTION, DEFINITIONS AND RESULTS

Let $f$, $g$ be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that $f$, $g$ share the value $a$ CM (counting multiplicities) if $f$, $g$ have the same $a$-points with the same multiplicities, and we say that $f$, $g$ share the value $a$ IM (ignoring multiplicities) if $f$, $g$ have the same $a$-points but the multiplicities are not taken into account.
We refer the reader to [6] for the standard notation and definitions of the value distribution theory. However, in the following we explain some notation used in the paper.

**Definition 1.1.** For a meromorphic function \( f \) and for \( a \in \mathbb{C} \cup \{\infty\} \) and for a positive integer \( k \)

(i) \( N_k(r,a;f) \) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not less than \( k \);  
(ii) \( N_k(r,a;f) \) denotes the counting function (reduced counting function) of those \( a \)-points of \( f \) whose multiplicities are not greater than \( k \);  
(iii) \( N_k(r,a;f) \) denotes the sum \( \overline{N}(r,a;f) + \sum_{j=2}^{k} \overline{N}_j(r,a;f) \).

Clearly \( N_1(r,a;f) = \overline{N}(r,a;f) \) and \( N_k(r,a;f) \leq k \overline{N}(r,a;f) \).

Rubel-Yang [10], Mues-Steinmetz [9], Gundersen [5], Yang [12] and others considered the uniqueness problem of entire functions when their first and \( k \)th derivatives share two values CM or IM.

Brück [4] considered the uniqueness problem of an entire function when it shares a single value CM with its first derivative and proved the following theorem.

**Theorem A** ([4]). Let \( f \) be a nonconstant entire function. If \( f \) and \( f' \) share the value 1 CM and \( N(r,0;f') = S(r,f) \), then \( f - 1 = c(f' - 1) \), where \( c \) is a nonzero constant.

Yang [11] considered an entire function of finite order and proved the following result.

**Theorem B** ([11]). Let \( f \) be a nonconstant entire function of finite order and let \( a \neq 0 \) be a finite constant. If \( f, f^{(k)} \) share the value \( a \) CM, then \( f - a = c(f^{(k)} - a) \), where \( c \) is a nonzero constant and \( k \geq 1 \) is an integer.

Zhang [14] extended Theorem A to meromorphic functions and proved the following results.

**Theorem C** ([14]). Let \( f \) be a nonconstant meromorphic function. If \( f \) and \( f' \) share 1 CM and if

\[
2\overline{N}(r,\infty;f) + 2N(r,0;f') < \lambda T(r,f') + S(r,f')
\]

for some constant \( \lambda \in (0,1) \), then \( f - 1 = c(f' - 1) \), where \( c \) is a nonzero constant.
**Theorem D** ([14]). Let $f$ be a nonconstant meromorphic function. If $f$ and $f^{(k)}$ share 1 CM and if

\[
2\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N(r, 0; f^{(k)}) < \lambda T(r, f^{(k)}) + S(r, f^{(k)})
\]

for some constant $\lambda \in (0, 1)$, then $f - 1 = c(f^{(k)} - 1)$, where $c$ is a nonzero constant.

Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$. A meromorphic function $a = a(z)$, defined in $\mathbb{C}$, is called a small function of $f$ if $T(r, a) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f)/T(r, f) \to 0$ as $r \to \infty$, possibly outside a set of finite linear measure.

Yu [13] considered the uniqueness problem of an entire function or a meromorphic function when it shares one small function with its derivative. The next two theorems are the results of Yu [13].

**Theorem E** ([13]). Let $f$ be a nonconstant entire function and let $a = a(z)$ ($\not\equiv 0, \infty$) be a small function of $f$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0; f) > 3/4$, then $f \equiv f^{(k)}$, where $k$ is a positive integer.

**Theorem F** ([13]). Let $f$ be a nonentire meromorphic function and $a = a(z)$ ($\not\equiv 0, \infty$) a small function of $f$. If

(i) $f$ and $a$ have no common pole,
(ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM,
(iii) $4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k$,

then $f \equiv f^{(k)}$, where $k$ is a positive integer.


**Theorem G** ([8]). Let $f$ be a nonconstant meromorphic function and $a = a(z)$ ($\not\equiv 0, \infty$) a small function of $f$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and $a = a(z)$ do not have any common pole of the same multiplicity and $2\delta(0; f) + 4\Theta(\infty; f) > 5$, then $f \equiv f^{(k)}$, where $k$ is a positive integer.

Al-Khaladi [3] observed by considering $f(z) = 1 + \exp(e^z)$ and $a(z) = e^z/(e^z - 1)$ that in Theorem A it is not possible to replace the value 1 by a small function. Instead, he proved the following result.
Theorem H ([3]). Let \( f \) be a nonconstant entire function satisfying \( N(r, 0; f') = S(r, f) \) and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f' - a \) share 0 CM, then \( f - a = (1 + c/a)(f' - a) \), where \( 1 + c/a = e^\beta \), \( c \) is a constant and \( \beta \) is an entire function.

In 2005 Al-Khaladi [2] considered the general order derivative of an entire function and proved the following result.

Theorem I ([2]). Let \( f \) be a nonconstant entire function satisfying \( N(r, 0; f^{(k)}) = S(r, f) \) and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f^{(k)} - a \) share 0 CM, then \( f - a = (1 + P_{k-1}/a)(f^{(k)} - a) \), where \( 1 + P_{k-1}/a = e^\beta \), \( P_{k-1} \) is a polynomial of degree at most \( k - 1 \) and \( \beta \) is an entire function.

Recently Al-Khaladi [1] extended Theorem I to meromorphic functions and proved the following theorem.

Theorem J ([1]). Let \( f \) be a nonconstant meromorphic function and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f^{(k)} - a \) share 0 CM and

\[
(k + 1)N(r, \infty; f) + (k + 1)\overline{N}(r, 0; f^{(k)}) < \lambda T(r, f^{(k)}) + S(r, f^{(k)})
\]

for some constant \( \lambda \in (0, 1) \), then \( f - a = (1 + P_{k-1}/a)(f^{(k)} - a) \), where \( P_{k-1} \) is a polynomial of degree at most \( k - 1 \) and \( 1 + P_{k-1}/a \neq 0 \).

For a nonconstant meromorphic function \( f \) we denote by \( L = L(f) \) a linear differential polynomial of the form

\[ L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)}, \]

where \( a_1, a_2, \ldots, a_k (\neq 0) \) are constants.

In the paper we prove the following theorem, which involves the sharing of a small function by \( f \) and \( L \).

Theorem 1.1. Let \( f \) be a nonconstant meromorphic function such that \( L \) is nonconstant. Suppose that \( a = a(z) (\neq 0, \infty) \) is a small function of \( f \). If \( f - a \) and \( L - a \) share 0 CM and

\[
(k + 1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_k(r, 0; f') < \lambda T(r, f') + S(r, f')
\]

for some real constant \( \lambda \in (0, 1) \), then \( f - a = (1 + c/a)(L - a) \), where \( c \) is a constant and \( 1 + c/a \neq 0 \).
2. Lemmas

In this section we present some necessary lemmas.

**Lemma 2.1** ([6], page 55, Theorem 3.1). Let \( f \) be a nonconstant meromorphic function. Then

\[
T(r, f) \leq (k + 1)T(r, f') + S(r, f).
\]

**Lemma 2.2.** Let \( f \) be a nonconstant meromorphic function such that \( L \) is nonconstant. Suppose that \( a = a(z) \) \((\neq 0, \infty)\) is a small function of \( f \). If \( f - a \) and \( L - a \) share 0 IM, then

\[
T(r, f) \leq \left( \frac{1}{k + 1} + k + 2 \right) T(r, L) + S(r, f) \leq \{(k + 1)(k + 2) + 1\}T(r, f') + S(r, f).
\]

**Proof.** By Milloux’s basic result [6], page 57, Theorem 3.2, we get

\[
T(r, f) \leq N(r, \infty; f) + N(r, 0; f) + \overline{N}(r, 1; L) - N_0(r, 0; L') + S(r, f),
\]

where \( N_0(r, 0; L') \) is the counting function of those zeros of \( L' \) which are not the 1-points of \( L \).

Now \( N(r, 0; f) - N_0(r, 0; L') \leq (k + 1)\overline{N}(r, 0; f) \) and \((k + 1)\overline{N}(r, \infty; f) \leq N(r, \infty; L) \leq T(r, L). \) Therefore

\[
(2.1) \quad T(r, f) \leq T(r, L) + \overline{N}(r, 1; L) + (k + 1)\overline{N}(r, 0; f) + S(r, f)
\]

\[
\leq \left( \frac{1}{k + 1} + 1 \right) T(r, L) + (k + 1)\overline{N}(r, 0; f) + S(r, f).
\]

Since \( L(f - a) = L(f) - \sum_{j=1}^{k} a_j a^{(j)} \), we have \( T(r, L(f - a)) = T(r, L) + S(r, f) \).

Now replacing \( f \) by \( f - a \) in (2.1) and noting that \( f - a \) and \( L - a \) share 0 IM we get

\[
T(r, f - a) \leq \left( \frac{1}{k + 1} + 1 \right) T(r, L) + (k + 1)\overline{N}(r, 0; f - a) + S(r, f)
\]

and so

\[
(2.2) \quad T(r, f) \leq \left( \frac{1}{k + 1} + k + 2 \right) T(r, L) + S(r, f).
\]

By Lemma 2.1 we get

\[
(2.3) \quad T(r, L) \leq (k + 1)T(r, f') + S(r, f).
\]

Now the lemma follows from (2.2) and (2.3). \( \square \)
Lemma 2.3 ([6], page 47, Theorem 2.5). Let $f$ be a nonconstant meromorphic function and $a_1, a_2, a_3$ three distinct small functions of $f$. Then

$$T(r, f) \leq N(r, 0; f - a_1) + N(r, 0; f - a_2) + N(r, 0; f - a_3) + S(r, f).$$

Lemma 2.4 ([7]). Let $f$ be a nonconstant meromorphic function and $k$ a positive integer. If $f$ and $f^{(k)}$ share $1$ IM and $f^{(k)} = (Af + B)/(Cf + D)$, where $A, B, C, D$ are constants, then $(f^{(k)} - 1)/(f - 1)$ is a nonzero constant.

3. Proof of Theorem 1.1

Proof. Let $h = (f - a)/(L - a)$. Then $f - a = h(L - a)$ and differentiating we get

$$f' - a' = (hL)' - (ha)'.$$  \hfill (3.1)

We now consider the following cases.

Case I: Let $a' \neq 0$. We put

$$W = \frac{(hL)'}{hf'} - \frac{(ha)'}{ha'}.$$  \hfill (3.2)

If $z_0$ is a zero of $f' - a'$ with $a'(z_0) \neq 0, \infty$, then we get from (3.1) that $W(z_0) = 0$. Let $W \neq 0$. Then

$$\overline{N}(r, 0; f' - a') \leq N(r, 0; W) + S(r, f) \leq T(r, W) + S(r, f)$$

$$= N(r, W) + m(r, W) + S(r, f) = N(r, W) + S(r, f).$$  \hfill (3.3)

From (3.2) we get

$$W = \frac{(hL)'}{hL} \cdot \frac{L}{f'} + \frac{(ha)'}{ha} \cdot \frac{a}{a'}.$$  \hfill (3.4)

Let $z_1$ be a pole of $f$ with multiplicity $p$ such that $a(z_1) \neq 0, \infty$ and $a'(z_1) \neq 0$. Then $z_1$ is a pole of $hL$ with multiplicity $p$ and a pole of $L/f'$ with multiplicity $k - 1$. Hence $z_1$ is a pole of $W$ with multiplicity at most $k$.

Let $z_2$ be a zero of $f'$ with multiplicity $q$ such that $a(z_2) \neq 0, \infty$ and $a'(z_2) \neq 0$. If $q \leq k - 1$ and $L(z_2) \neq 0$, then $z_2$ is a pole of $(hL)'/hL \cdot L/f'$ with multiplicity $q \leq k - 1$. Also, if $q \leq k - 1$ and $z_2$ is a zero of $L$ with multiplicity $t \geq 1$, then $z_2$ is a pole of $(hL)'/hL \cdot L/f'$ with multiplicity $q - (t - 1) \leq q \leq k - 1$. 

6
If \( q \geq k \), then \( z_2 \) is a pole of \( L/f' \) with multiplicity \( k - 1 \) and a pole of \((hL)'/(hL)\) with multiplicity 1. Hence \( z_2 \) is a pole of \((hL)'/(hL) \cdot L/f' \) with multiplicity \( k \).

Therefore from (3.4) we get

\[
N(r, W) \leq k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f).
\]

From (3.3) and (3.5) we obtain

\[
\overline{N}(r, 0; f' - a') \leq k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f).
\]

Since by Lemma 2.1 and Lemma 2.2, \( a' = a'(z) \) is a small function of \( f' \) and \( S(r, f) \) is interchangeable with \( S(r, f') \), we get by Lemma 2.3 and (3.6)

\[
T(r, f') \leq \overline{N}(r, 0; f' - a') + \overline{N}(r, 0; f') + \overline{N}(r, \infty; f') + S(r, f')
\]

\[
\leq (k + 1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_k(r, 0; f') + S(r, f'),
\]

which contradicts the hypothesis.

Therefore \( W \equiv 0 \) and so by (3.1) and (3.2) we get \((f' - a')(ha)' = (f' - a')a'\).

Since \( f' \not\equiv a' \), we have \((ha)' = a' \) and so \( ha = a + c \), where \( c \) is a constant. Hence

\[
f - a = h(L - a) = \left(1 + \frac{c}{a}\right)(L - a),
\]

where \( 1 + c/a \not\equiv 0 \).

**Case II:** Let \( a' \equiv 0 \) so that \( a \) is a constant. We now consider the following subcases.

**Subcase (i):** Let \( k \geq 2 \). From (3.1) we get

\[
f' = (hL)' - ah' = h\left\{\frac{(hL)'}{h} - a\frac{h'}{h}\right\}
\]

and so

\[
\frac{1}{h} = \frac{(hL)'}{hf'} - a\frac{h'}{h} \cdot \frac{1}{f'}.
\]

We put \( F = f' \), \( G = (hL)'/(hf') \) and \( b = ah'/h \). Then

\[
\frac{1}{h} = G - \frac{b}{F}.
\]

Differentiating (3.7) we obtain

\[
\frac{1}{h} \cdot \frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F} \cdot G' + \frac{b F'}{F^2}.
\]
Eliminating $1/h$ from (3.7) and (3.8) we get

\begin{equation}
\frac{A}{F} = G' + \frac{Gh'}{h},
\end{equation}

where $A = b \cdot h'/h + b' - b \cdot F'/F$.

First we suppose that $G \equiv 0$. Then $hL = d$, a nonzero constant. Putting $h = (f - a)/(L - a)$ we have $L(f - a) = d(L - a)$. This implies that $f$ is an entire function. Therefore, $h$ is an entire function having no zero. We now put $h = e^\alpha$, where $\alpha$ is an entire function.

Now $f = a + h(L - a) = a + d - ae^\alpha$ and $L = de^{-\alpha}$. Also we see that $L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = P(\alpha')e^\alpha$, where $P(\alpha')$ is a differential polynomial in $\alpha'$. Therefore $P(\alpha')e^\alpha = de^{-\alpha}$ and so $P(\alpha')e^{2\alpha} = d$. This implies $2T(r, e^\alpha) = T(r, P(\alpha')) = S(r, e^\alpha)$, a contradiction. Hence $G \not\equiv 0$.

Next we suppose that $A \equiv 0$. Then from (3.9) we get $G'/G + h'/h = 0$. Integrating we obtain $Gh = K$, where $K$ is a nonzero constant. Hence $(hL)' = kf'$ and again integration yields $hL = Kf + M$, where $M$ is a constant. Since $f - a = hL - ah$, we get

\begin{equation}
(1 - K)f = a(1 - h) + M.
\end{equation}

If $K = 1$, from (3.10) we see that $h$ is a constant. Hence $f - a = (1 + c/a)(L - a)$, where we put $h = 1 + c/a$ for some constant $c$ such that $1 + c/a \neq 0$.

Let $K \neq 1$. Then from (3.10) we see that $h$ is nonconstant. Since $h$ is entire, (3.10) implies that $f$ is also entire. Therefore $h = (f - a)/(L - a)$ has no zero. So we can put $h = e^\beta$, where $\beta$ is an entire function. Hence from (3.10) we get

$$f = \frac{a + M}{1 - K} - \frac{ae^\beta}{1 - K}$$

and so

\begin{equation}
L = K\frac{f}{h} + \frac{M}{h} = \frac{Ka + M}{1 - K}e^{-\beta} - \frac{a}{1 - K}.
\end{equation}

Also

\begin{equation}
L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = Q(\beta')e^\beta,
\end{equation}

where $Q(\beta')$ is a differential polynomial in $\beta'$.

Since $L$ is nonconstant, we see that $Ka + M \neq 0$. Hence from (3.11) and (3.12) we get

$$Q(\beta')e^{2\beta} = \frac{Ka + M}{1 - K} - \frac{a}{1 - K}e^\beta.$$
This implies by the first fundamental theorem

\[ 2T(r, e^\beta) \leq T(r, e^\beta) + T(r, Q(\beta')) + O(1) = T(r, e^\beta) + S(r, e^\beta), \]

a contradiction.

Finally we suppose that \( A \neq 0 \). Now \( m(r, A) \leq 2m(r, b) + m(r, b') + m(r, h'/h) + m(r, F'/F) = S(r, f) \). Since \( A = a(h'/h)^2 + a(h'/h)' - h'/h \cdot F'/F \), we see that \( N(r, \infty; A) \leq 2N(r, \infty; f) + N(r, 0; f') \). Hence

(3.13) \[ T(r, A) \leq 2N(r, \infty; f) + N(r, 0; f') + S(r, f). \]

Now from (3.9) and (3.13) we get

(3.14) \[ m\left(\frac{r}{F}\right) \leq m\left(\frac{r}{A}\right) + m\left(\frac{r}{G'} + G\frac{h'}{h}\right) \]
\[ \leq T(r, A) + S(r, f) \]
\[ \leq 2N(r, \infty; f) + N(r, 0; f') + S(r, f). \]

Since \( A \neq 0 \), it is clear that \( b \neq 0 \). Let \( z_3 \) be a zero of \( F \) with multiplicity \( q \geq k + 1 \). Then \( z_3 \) is a zero of \( b = af'/f - aL'/L - a \) with multiplicity at least \( q - k \). Hence

\[ N(k+1)\left(\frac{r}{F}\right) - kN(k+1)\left(\frac{r}{F}\right) \leq N(r, 0; b) \]

and so

\[ N(k+1)\left(\frac{r}{F}\right) \leq kN(k+1)\left(\frac{r}{F}\right) + N(r, 0; b) \]
\[ \leq kN(k+1)\left(\frac{r}{F}\right) + T(r, b) + O(1) \]
\[ = kN(k+1)\left(\frac{r}{F}\right) + N(r, b) + S(r, f) \]
\[ \leq kN(k+1)\left(\frac{r}{F}\right) + N(r, \infty; f) + S(r, f). \]

So

(3.15) \[ N\left(\frac{r}{F}\right) = N_k\left(\frac{r}{F}\right) + N_{k+1}\left(\frac{r}{F}\right) \]
\[ \leq N_k(r, 0; f') + N(r, 0; f') + S(r, f). \]

Adding (3.14) and (3.15) and using the first fundamental theorem we get

\[ T(r, f') \leq 3N(r, \infty; f) + N_k(r, 0; f') + N(r, 0; f') + S(r, f), \]

which is a contradiction with the hypothesis for \( k \geq 2 \).
Subcase (ii): Let $k = 1$. We put $g = f/a$ and $R = L/a$. Then $g$ and $R$ share the value $1$ CM. Let 

$$H = \left( \frac{g''}{g'} - \frac{2g'}{g - 1} \right) - \left( \frac{R''}{R'} - \frac{2R'}{R - 1} \right).$$

We first suppose that $H \not\equiv 0$. Since $g$ and $R$ share $1$ CM, we get 

$$N(r, H) = N(r, H) \leq N(r, \infty; f) + \overline{N}(r, 0; f') - \overline{N}(2(r, a; f) + N_*(r, 0; f^{(2)}),$$

where $N_*(r, 0; f^{(2)})$ denotes the reduced counting function of those zeros of $f^{(2)}$ which are not the zeros of $(f - a)f'$.

Since $g$ and $R$ share the value $1$ CM, it is easy to see that 

$$N_1(r, a; f) = N_1(r, 1; g) \leq N(r, 0; H) \leq T(r, H) + O(1) = N(r, H) + S(r, f)$$

\begin{align*}
&\leq N(r, \infty; f) + \overline{N}(r, 0; f') - \overline{N}(2(r, a; f) + N_*(r, 0; f^{(2)}) + S(r, f)
\end{align*}

and so

$$(3.16) \quad N(r, a; f) = N_1(r, a; f) + N_2(r, a; f)$$

\begin{align*}
&\leq N(r, \infty; f) + \overline{N}(r, 0; f') + N_*(r, 0; f^{(2)}) + S(r, f).
\end{align*}

Now by the second fundamental theorem and (3.16) we get in view of the fact that $L - a$ and $f - a$ share $0$ CM:

$$T(r, f') = T(r, L) + O(1)$$

\begin{align*}
&\leq \overline{N}(r, \infty; L) + \overline{N}(r, 0; L) + \overline{N}(r, a; L) - \overline{N}_*(r, 0; f^{(2)}) + S(r, L)
\end{align*}

$$= \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N(r, a; f) - \overline{N}_*(r, 0; f^{(2)}) + S(r, f')$$

\begin{align*}
&\leq 2\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_1(r, 0; f') + S(r, f'),
\end{align*}

a contradiction with the hypothesis.

Therefore $H \equiv 0$ and so integration yields $R = (Ag + B)/(Cg + D)$, where $A$, $B$, $C$, $D$ are constants. Hence by Lemma 2.4 we get $(g - 1)/(R - 1)$ is a nonzero constant. So we can put $f - a = (1+c/a)(L - a)$, where $c$ is a constant and $1+c/a \neq 0$. This proves the theorem. \[\square\]
References


Authors’ addresses: Indrajit Lahiri, Department of Mathematics, University of Kalyani, Block-C, University Area, Kalyani-741235, Nadia, West Bengal, India, e-mail: ilahiri@hotmail.com; Amit Sarkar, Ramnagar High School, P.O.-Kumari Ramnagar, Ramnagar-741502, Nadia, West Bengal, India, e-mail: amit83sarkar@gmail.com.