Mathematica Bohemica

Abdellatif Chahbi; Samir Kabbaj; Ahmed Charifi

Linear maps preserving A-unitary operators

Mathematica Bohemica, Vol. 141 (2016), No. 1, 59–70

Persistent URL: `http://dml.cz/dmlcz/144851`

LINEAR MAPS PRESERVING A-UNITARY OPERATORS

Abdellatif Chahbi, Samir Kabbaj, Ahmed Charifi, Kenitra

Received November 12, 2013
Communicated by Radomír Halaš

Abstract. Let $\mathcal{H}$ be a complex Hilbert space, A a positive operator with closed range in $\mathscr{B}(\mathcal{H})$ and $\mathscr{B}_A(\mathcal{H})$ the sub-algebra of $\mathscr{B}(\mathcal{H})$ of all A-self-adjoint operators. Assume $\varphi\colon \mathscr{B}_A(\mathcal{H})$ onto itself is a linear continuous map. This paper shows that if φ preserves A-unitary operators such that $\varphi(I) = P$ then ψ defined by $\psi(T) = P\varphi(PT)$ is a homomorphism or an anti-homomorphism and $\psi(T^\sharp) = \psi(T)^\sharp$ for all $T \in \mathscr{B}_A(\mathcal{H})$, where $P = A^+A$ and A^+ is the Moore-Penrose inverse of A. A similar result is also true if φ preserves A-quasi-unitary operators in both directions such that there exists an operator T satisfying $P\varphi(T) = P$.

Keywords: linear preserver problem; semi-inner product

MSC 2010: 15A86, 46C50

1. Introduction

Linear preserver problems are an active research area in matrix and operator theory and Banach algebras. It has attracted the attention of many mathematicians in the last few decades ([3], [4], [9]–[11], [17]–[19]). By a linear preserver we mean a linear map of an algebra $\mathscr{A}$ into itself which, roughly speaking, preserves certain properties of some elements in $\mathscr{A}$. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. In this paper, we concentrate on the case when $\mathscr{A} = \mathscr{B}(\mathcal{H})$, the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We point out that a great deal of work has been devoted to the case when $\mathcal{H}$ is finite dimensional, that is, the case when $\mathscr{A}$ is a matrix algebra (see survey articles [9], [11], [15]), and that the first papers concerning this case date back to the 19th century [7].

DOI: 10.21136/MB.2016.4

The aim of this paper is to prove some results for A-self-adjoint operators, A-projections, A-unitary operators and A-quasi-unitary operators, which are useful to give the concrete form of continuous linear maps

$$\varphi\colon\ \mathscr{B}_A(\mathcal{H}) \to \mathscr{B}_A(\mathcal{H})$$

that preserve A-unitary-operators. As a consequence, we describe surjective continuous linear maps from $\mathscr{B}_A(\mathcal{H})$ onto itself that preserve A-quasi-unitary operators.

2. Preliminaries and results

Throughout, $\mathcal{H}$ denotes a Hilbert space with an inner product $\langle , \rangle$. By $\mathscr{B}(\mathcal{H})$ we denote the algebra of all linear bounded operators on $\mathcal{H}$. Also, for $T \in \mathscr{B}(\mathcal{H})$, $R(T)$ denotes the range of T and $N(T)$ the kernel of T.

Any $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form as follows:

$$\langle , \rangle_A\colon\ \mathcal{H} \times \mathcal{H} \to \mathbb{C};\quad \langle x, y\rangle_A = \langle Ax, y\rangle.$$

By $\|\cdot\|_A$ we denote the semi-norm induced by $\langle x, y\rangle_A$, i.e., $\|x\|_A = \langle x, x\rangle_A^{1/2}$. Observe that $\|x\|_A = 0$ if and only if $x \in N(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is an injective operator. Moreover, $\|\cdot\|_A$ induces a semi-norm on a certain subset of $\mathcal{B}(\mathcal{H})$, namely, on the subset of all $T \in \mathcal{B}(\mathcal{H})$ for which there exists a constant $c > 0$ such that $\|Tx\|_A \leqslant c\|x\|_A$ for all $x \in H$. These operators satisfy

$$\|T\|_A = \sup_{x\in\overline{R(A)}, x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

Definition 2.1. Given an operator $T \in \mathcal{B}(\mathcal{H})$, an operator $W \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if

$$\langle Tx, y\rangle_A = \langle x, Wy\rangle_A$$

for every $x, y \in \mathcal{H}$. So, T is called A-self-adjoint if $AT = T^*A$, and T is called A-positive if AT is positive.

The following theorem due to Douglas will be used frequently (see [5], [6]).

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(1) $R(B) \subset R(A)$.

(2) *There exists a positive number λ such that $BB^* \leqslant \lambda AA^*$.*

(3) *There exists $D \in \mathcal{B}(\mathcal{H})$ such that $AD = B$.*

If one of these conditions holds then there exists a unique operator $D \in \mathcal{B}(\mathcal{H})$ such that $AD = B$, $R(D) \subset R(A^*)$ and $N(D) = N(B)$. Moreover, $\|D\|^2 = \inf\{\lambda > 0\colon BB^* \leqslant \lambda AA^*\}$. We shall call D the reduced solution of $AX = B$.

The reduced solution of the equation $AX = B$ can be obtained by means of the Moore-Penrose inverse of A. Recall that given $A \in \mathcal{B}(\mathcal{H})$ the Moore-Penrose inverse of A, denoted by A^+, is defined as the unique linear extension of $\widetilde{A}^{-1}$ to $D(A^+) := R(A) + R(A)^\perp$ with $N(A^+) = R(A)^\perp$, where $\widetilde{A}$ is the isomorphism

$$A_{|N(A)^\perp}\colon\ N(A)^\perp \to R(A).$$

Moreover, A^+ is the unique solution of the four Moore-Penrose equations

$$AXA = A; \quad XAX = X; \quad XA = P_{|N(A)^\perp}; \quad AX = (P_{\overline{R(A)}})_{|D(A^+)}.$$

It is easy to prove that A^+ has closed graph and is bounded if and only if $R(A)$ is closed. As a consequence, given $A, B \in \mathcal{B}(\mathcal{H})$ such that $R(B) \subset R(A)$ then $A^+B \in \mathcal{B}(\mathcal{H})$ even if A^+ is not bounded. Moreover, A^+B is the reduced solution of the equation $AX = B$. In fact, $AA^+B = (P_{\overline{R(A)}})_{|D(A^+)}B = B$. Furthermore, as $A^+B \in \mathcal{B}(\mathcal{H})$ and $R(A^+B) \subset R(A)$, A^+B is the reduced solution of the equation $AX = B$.

Note every $T \in \mathscr{B}(\mathcal{H})$ admits an A-adjoint operator. In fact, $T \in \mathscr{B}(\mathcal{H})$ has an A-adjoint operator if and only if there exists $W \in \mathscr{B}(\mathcal{H})$ such that $AW = T^*A$, if and only if the equation $AX = T^*A$ has a solution; then, by the Douglas theorem, T admits an A-adjoint operator if and only if $R(T^*A) \subset R(A)$.

From now on, $\mathscr{B}_A(\mathcal{H})$ denotes the set of all $T \in \mathscr{B}(\mathcal{H})$ which admit an A-adjoint, that is

$$\mathscr{B}_A(\mathcal{H}) = \{T \in \mathscr{B}(\mathcal{H})\colon\ R(T^*A) \subset R(A)\}.$$

$\mathscr{B}_A(\mathcal{H})$ is a sub-algebra of $\mathscr{B}(\mathcal{H})$ which is neither closed nor dense in $\mathscr{B}(\mathcal{H})$. In fact, if A has closed range, we have that $\mathscr{B}_A(\mathcal{H})$ is closed and therefore complete.

If $T \in \mathscr{B}(\mathcal{H})$ admits an A-adjoint operator, i.e., if $R(T^*A) \subset R(A)$, then there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e., A^+T^*A. We denote this operator by $T^\sharp$. Therefore, $T^\sharp = A^+T^*A$ and

$$AT^\sharp = T^*A, \quad R(T^\sharp) \subset \overline{R(A)} \quad \text{and} \quad N(T^\sharp) = N(T^*A).$$

In the sequel we give some important properties of $T^\sharp$ without proof (cf. [1], [2]).

Theorem 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then:*

(1) $(A^t)^\sharp = A^t$ *for every* $t > 0$.
(2) *If* $AT = TA$ *then* $T^\sharp = PT^*$.
(3) *If* $AT = T^*A$ *then* $(A^{1/2})^+ T^* A^{1/2}$ *is self-adjoint.*
(4) *If* $W \in \mathcal{B}_A(\mathcal{H})$ *then* $TW \in \mathcal{B}_A(\mathcal{H})$ *and* $(TW)^\sharp = W^\sharp T^\sharp$.
(5) $T^\sharp \in \mathcal{B}_A(\mathcal{H})$, $(T^\sharp)^\sharp = PTP$ *and* $((T^\sharp)^\sharp)^\sharp = T^\sharp$.
(6) $T^\sharp T$ *and* $TT^\sharp$ *are* A*-self-adjoint and* A*-positive.*

For more details cf. [1], [2].

Throughout this paper A always denotes a positive operator with closed range in $\mathscr{B}(\mathcal{H})$. So $P_{|N(A)^\perp} = (P_{\overline{R(A)}})_{|D(A^+)}$ will be denoted by $P = AA^+ = A^+A$.

Now we shall prove some natural properties of A-self-adjoint operators, A-projections and A-unitary operators, which will be useful in our main results.

Proposition 2.4. *Let T be an operator in $\mathscr{B}_A(\mathcal{H})$. Then the following assertions are equivalent:*

(1) T *is* A*-self-adjoint.*
(2) $PT = T^\sharp$.
(3) $(T^\sharp)^\sharp = T^\sharp$.

P r o o f. $1 \Rightarrow 2$. From Definition 2.1, if T is A-self-adjoint, then $AT = T^*A$, and by left multiplication of this equality by A^+ we get $PT = T^\sharp$.

$2 \Rightarrow 3$. If $PT = T^\sharp$, since $T^\sharp P = PT^\sharp = T^\sharp$ for all $T \in \mathscr{B}_A(\mathcal{H})$, so $(T^\sharp)^\sharp = T^\sharp$.

$3 \Rightarrow 1$. If $(T^\sharp)^\sharp = T^\sharp$, so $PTP = A^+T^*A$, and by left multiplication by A we get $APTP = AA^+T^*A$; since $AP = A$, $ATP = AT$, and $PT^*A = T^*A$, we have $AT = T^*A$, therefore T is A-self-adjoint. □

Proposition 2.5. *Let p and q be two A-projections, then $p+q$ is an A-projection if and only if $Apq = Aqp = 0$.*

P r o o f. If $p + q$ is an A-projection so

$$A(p+q)^2 = A(p+q) = (p+q)^*A,$$

and so

$$Apq + Aqp = 0. \tag{2.1}$$

Now, right multiplication of (2.1) by p gives

$$Apqp + Aqp = 0.$$

Afterward, by left multiplication of the equality (2.1) by p^*, we obtain

$$Apq + Apqp = 0,$$

and these three equalities yield $Apq = Aqp = 0$.

Conversely, if p and q are A-projections such that $Apq = Aqp = 0$, then

$$A(p+q)^2 = Ap^2 + Apq + Aqp + Aq^2 = Ap^2 + Aq^2 = (p+q)^*A = A(p+q).$$

□

Proposition 2.6. *Let p and q be two A-projections. Then $p-q$ is an A-projection if and only if $Apq = Aqp = Aq$.*

Proof. From Proposition 2.5, $p-q$ and q being A-projections is equivalent to

$$A(p-q)q = Aq(p-q) = 0$$

and this is equivalent to $Apq = Aqp = Aq$. □

Proposition 2.7. *Let B in $\mathcal{B}_A(\mathcal{H})$. Then the following statements are equivalent:*

(1) *B is an A-projection.*
(2) $PB^2 = PB = B^\sharp$.
(3) $PB = B^\sharp = (B^2)^\sharp$.

Proof. $1 \Rightarrow 2$. If B is an A-projection, so $AB^2 = AB = B^*A$, and by left multiplication of this equality by A^+, we get $PB^2 = PB = B^\sharp$.

$2 \Rightarrow 3$. If we have $PB^2 = PB = B^\sharp$, right multiplication this equality by B yields $PB^2 = PB = B^\sharp B$, since $B^\sharp P = B^\sharp$, this implies that $PB = B^\sharp = (B^2)^\sharp$.

$3 \Rightarrow 1$. If $PB = B^\sharp = (B^2)^\sharp$, right multiplication of this equality by B yields $PB^2 = PB = B^\sharp B$; now left multiplication of this equality by A yields $AB^2 = AB = PB^*A$; since $PB^*A = B^*A$, it follows that B is an A-projection. □

Proposition 2.8. *Let B and C be A-projections. Then BC is an A-projection if and only if $ABC = ACB$.*

Proof. If BC is A-projection, then

$$ABCBC = ABC = C^*B^*A;$$

since B and C are A-projections, we have

$$ABC = C^*B^*A = ACB.$$

For the converse implication, if we have $ABC = ACB$ and B and C are A-projections, then

$$\begin{aligned} ABCBC &= C^*B^*ABC = C^*B^*ACB = C^*AB^2C \\ &= C^*ABC = C^*ACB = ACB = ABC = C^*B^*A. \end{aligned}$$

This completes the proof. □

Proposition 2.9. *Let $U \in \mathscr{B}_A(\mathcal{H})$. The following assertions are equivalent:*

(1) *U is an A-unitary operator.*
(2) *PU is an A-unitary operator.*
(3) *$U^\sharp$ is an A-unitary operator.*

Proof. (1) ⇔ (2). We know that

$$PB\sharp = B^\sharp P = B^\sharp$$

for all $B \in \mathscr{B}_A(\mathcal{H})$. So

$$U^\sharp U = (U^\sharp)^\sharp U^\sharp = P$$

is equivalent to

$$(PU)^\sharp PU = ((PU)^\sharp)^\sharp (PU)^\sharp = P,$$

which completes the proof.

(1) ⇔ (3). U being an A-unitary operator is equivalent to

$$U^\sharp U = (U^\sharp)^\sharp U^\sharp = P,$$

this is equivalent to

$$(U^\sharp U)^\sharp = ((U^\sharp)^\sharp U^\sharp)^\sharp = P,$$

this is equivalent to

$$((U^\sharp)^\sharp)^\sharp (U^\sharp)^\sharp = (U^\sharp)^\sharp U^\sharp = P,$$

which is equivalent to $U^\sharp$ being an A-unitary operator. □

Proposition 2.10. *For all A-self-adjoint $S \in \mathscr{B}_A(\mathcal{H})$, $\exp(\mathrm{i}tS)$ is an A-unitary operator for every $t \in \mathbb{R}$.*

P r o o f. We have $(S^\sharp)^k = A^+(S^*)^k A$; since S is A-self-adjoint so $S^*A = AS$ and $(S^*)^k A = AS^k$, consequently $(S^\sharp)^k = PS^k$, then $(\exp(\mathrm{i}tS))^\sharp = P\exp(-\mathrm{i}tS)$. As $\mathscr{B}_A(\mathcal{H})$ is complete, it follows that

$$(\exp(\mathrm{i}tS))^\sharp = \sum_{k=0}^{\infty} \frac{(-\mathrm{i}t)^k}{k!}(S^\sharp)^k = P\sum_{k=0}^{\infty} \frac{(-\mathrm{i}t)^k}{k!}S^k = P\exp(-\mathrm{i}tS),$$

then $(\exp(\mathrm{i}tS))^\sharp = P\exp(-\mathrm{i}tS)$, so $(\exp(\mathrm{i}tS))^\sharp(\exp(\mathrm{i}tS)) = P$.

Conversely, we have $((S^\sharp)^\sharp)^k = (S^\sharp)^k$, then

$$\begin{aligned}((\exp(\mathrm{i}tS))^\sharp)^\sharp(\exp(\mathrm{i}tS))^\sharp &= (\exp(-\mathrm{i}tS))^\sharp P(\exp(-\mathrm{i}tS)) \\ &= P(\exp(\mathrm{i}tS))(\exp(-\mathrm{i}tS)) = P.\end{aligned}$$

Consequently, $\exp(\mathrm{i}tS)$ is an A-unitary operator. □

In 1977 Phadke et al. in [14] introduced the notion of a quasi-unitary operator on a Hilbert space as follows.

Definition 2.11. An operator T on a Hilbert space $\mathcal{H}$ is called quasi-unitary if

$$TT^* = T^*T = T + T^*.$$

It is easy to see that the following proposition holds true.

Proposition 2.12. *An operator T is quasi-unitary in a Hilbert space if and only if $I - T$ is a unitary operator.*

For more information on quasi-unitary operators the reader can see [12]–[14], [16].

By combining definitions of A-unitary and quasi-unitary operators, we define an A-quasi-unitary operator as follows.

Definition 2.13. An operator T on a Hilbert space $\mathcal{H}$ is called A-quasi-unitary if

$$T^\sharp T = (T^\sharp)^\sharp T^\sharp = T^\sharp + (T^\sharp)^\sharp.$$

E x a m p l e 2.14. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2^+(\mathbb{C})$$

then

$$T = \begin{pmatrix} 0 & n-1 \\ 0 & 1-n \end{pmatrix}$$

is A-quasi-unitary for all $n \in \mathbb{N}$.

Proposition 2.15. *An operator U is an A-quasi-unitary operator on a Hilbert space $\mathcal{H}$ if and only if $I-U$ is an A-unitary operator.*

P r o o f. For the first implication, we have $(I-U)^\sharp(I-U) = P - U^\sharp - PU + U^\sharp U$ and

$$((I-U)^\sharp)^\sharp(I-U)^\sharp = P - (U^\sharp)^\sharp - U^\sharp + (U^\sharp)^\sharp U^\sharp;$$

since U is an A-quasi-unitary operator and $PU + U^\sharp = (U^\sharp)^\sharp + U^\sharp$, so

$$(I-U)^\sharp(I-U) = ((I-U)^\sharp)^\sharp(I-U)^\sharp = P.$$

For the converse implication, we have

$$(I-U)^\sharp(I-U) = ((I-U)^\sharp)^\sharp(I-U)^\sharp = P,$$

then by $PU + U^\sharp = (U^\sharp)^\sharp + U^\sharp$ we have $U^\sharp U = (U^\sharp)^\sharp U^\sharp = (U^\sharp)^\sharp + U^\sharp$, consequently U is A-quasi-unitary. □

Proposition 2.16. *Let $U \in \mathscr{B}_A(\mathcal{H})$. The following assertions are equivalent:*

(1) *U is an A-quasi-unitary operator.*
(2) *PU is an A-quasi-unitary operator.*
(3) *$U^\sharp$ is an A-quasi-unitary operator.*

P r o o f. This proof is similar to the proof of Proposition 2.9. □

A linear map φ from an algebra $\mathscr{A}$ into an algebra $\mathscr{B}$ is called a Jordan homomorphism if $\varphi(x^2) = \varphi(x)^2$ for every $x \in \mathscr{A}$. A well known result of Herstein ([8], Theorem 3.1) shows that a Jordan homomorphism on a prime algebra is either a homomorphism or an anti-homomorphism.

3. Linear maps preserving A-unitary operators

Theorem 3.1. *Let $\mathcal{H}$ be a complex Hilbert space and let $\varphi\colon \mathscr{B}_A(\mathcal{H}) \to \mathscr{B}_A(\mathcal{H})$ be a linear continuous map such that $\varphi(I) = P$. If φ preserves A-unitary operators then ψ is a homomorphism or an anti-homomorphism and $\psi(T^\sharp) = \psi(T)^\sharp$ for all $T \in \mathscr{B}_A(\mathcal{H})$, where ψ is defined by $\psi(T) = P\varphi(PT)$ for all $T \in \mathscr{B}_A(\mathcal{H})$.*

P r o o f. Pick an A-self-adjoint $S \in \mathscr{B}_A(\mathcal{H})$. According to Proposition 2.10, $\exp(\mathrm{i}tS)$ is an A-unitary operator for every $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
P &= \varphi(\exp(\mathrm{i}tS))^\sharp\varphi(\exp(\mathrm{i}tS)) \\
&= \varphi\Big(I + \mathrm{i}tS + \frac{(\mathrm{i}t)^2}{2!}S^2 + \dots\Big)^\sharp\varphi\Big(I + \mathrm{i}tS + \frac{(\mathrm{i}t)^2}{2!}S^2 + \dots\Big) \\
&= P + \mathrm{i}tP(\varphi(S) - \varphi(S)^\sharp) - t^2\frac{P\varphi(S^2)^\sharp + P\varphi(S^2)}{2} + t^2\varphi(S)^\sharp\varphi(S) + \dots
\end{aligned}$$

Hence

$$P\varphi(S) = P\varphi(S)^\sharp \tag{3.1}$$

and

$$-\frac{P\varphi(S^2)^\sharp + P\varphi(S^2)}{2} + P\varphi(S)^\sharp P\varphi(S) = 0 \tag{3.2}$$

for all A-self-adjoint S. If S is A-self-adjoint so $S^\sharp$ is A-self-adjoint, and so

$$\psi(S^\sharp) = \psi(S^\sharp)^\sharp \tag{3.3}$$

for any A-self-adjoint operator S. Using the fact that every $B \in \mathscr{B}_A(\mathcal{H})$, can be written as $B = S + \mathrm{i}T$ with A-self-adjoint operators S, T, we conclude that $B^\sharp = S^\sharp - \mathrm{i}T^\sharp$ and $(B^\sharp)^\sharp = S^\sharp + \mathrm{i}T^\sharp$. Note that if S is A-self-adjoint, then $S^\sharp$ is A-self-adjoint, hence $S^\sharp = PS$. By (3.1) and linearity of φ we get that $\psi(B^\sharp) = \psi(B)^\sharp$.

Now we will prove that ψ is a Jordan homomorphism.

Every operator $B \in \mathscr{B}_A(\mathcal{H})$ can be written as $B = S + \mathrm{i}T$ with A-self-adjoint operators S, T. Hence

$$B^\sharp = S^\sharp - \mathrm{i}T^\sharp$$

and

$$B^\sharp = PS - \mathrm{i}PT.$$

By (3.2),

$$\psi(S^2) = \psi(S)^2 \tag{3.4}$$

for every A-self-adjoint operator S. As $S + T$ is an A-self-adjoint operator, hence

$$\psi((S+T)^2) = (\psi(S) + \psi(T))^2,$$

so

$$\psi(ST + TS) = \psi(S)\psi(T) + \psi(T)\psi(S),$$

and consequently,

$$\begin{aligned}
\psi(B^2) &= \psi((S+\mathrm{i}T)^2) = \psi(S^2 - T^2 + \mathrm{i}(ST+TS)) \\
&= \psi(S^2) - \psi(T^2) + \mathrm{i}\psi(ST+TS) \\
&= \psi(S)^2 - \psi(T)^2 + \mathrm{i}\psi(S)\psi(T) + \psi(T)\psi(S) = \psi(S+\mathrm{i}T)^2 = \psi(B)^2.
\end{aligned}$$

Then we get $\psi(B^2) = \psi(B)^2$ for every operator B in $\mathcal{B}_A(\mathcal{H})$. It follows that ψ is a Jordan homomorphism. Note that $\mathcal{H} = R(A) \oplus N(A)$. Under this decomposition,

$$P = A^+A = AA^+ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\psi(T) = \begin{pmatrix} (\psi(T))_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

where $(\psi(T))_{11} \in \mathcal{B}_A(R(A))$, $T \in \mathcal{B}_A(\mathcal{H})$. Since $R(A)$ with $\langle , \rangle_A$ is a Hilbert space, this implies that $\mathcal{B}_A(R(A))$ is a prime algebra. As ψ is a Jordan homomorphism, so $(\psi)_{11}$ is a Jordan homomorphism in the prime algebra $\mathcal{B}_A(R(A))$. Since it is known that a Jordan homomorphism in a prime algebra is a homomorphism or an anti-homomorphism, ψ is a homomorphism or an anti-homomorphism. $\square$

Corollary 3.2. *Let $\mathcal{H}$ be a complex Hilbert space and let $\varphi\colon \mathcal{B}_A(\mathcal{H}) \to \mathcal{B}_A(\mathcal{H})$ be a linear continuous subjective map. If φ preserves A-unitary operators, then $\varphi(I)^\sharp\psi$ is a homomorphism or an anti-homomorphism.*

Proof. If I is an A-unitary operator, then $\varphi(I)$ is an A-unitary operator. Now we consider φ_1 such that $\varphi_1(B) = (\varphi(I)^\sharp)\varphi(B)$. We can show that φ_1 is a linear continuous map that preserves A-unitary operators and $\varphi_1(I) = P$. From Theorem 3.1 we get that $\psi_1 = (\varphi(I)^\sharp)\psi$ is a homomorphism or an anti-homomorphism. $\square$

Corollary 3.3. *Let $\varphi\colon \mathscr{B}_A(\mathcal{H}) \to \mathscr{B}_A(\mathcal{H})$ be a linear continuous map. If $\varphi(U^\sharp)$ is an A-unitary operator if U is an A-unitary operator, then ψ is a homomorphism or an anti-homomorphism and $\psi(T^\sharp) = \psi(T)^\sharp$ for all $T \in \mathscr{B}_A(\mathcal{H})$.*

Proof. If U is an A-unitary operator in $\mathscr{B}_A(\mathcal{H})$, so $U^\sharp$ is an A-unitary operator, and so $\varphi((U^\sharp)^\sharp) = \varphi(PUP) = \varphi(PU)$ is an A-unitary operator. Consequently $\psi(U)$ is an A-unitary operator provided U is an A-unitary operator in $\mathscr{B}_A(\mathcal{H})$. Now from Theorem 3.1 we get that ψ is a homomorphism or an anti-homomorphism and $\psi(T^\sharp) = \psi(T)^\sharp$ for all $T \in \mathscr{B}_A(\mathcal{H})$. $\square$

4. Linear maps preserving A-quasi-unitary operators

Theorem 4.1. *Let $\varphi\colon \mathscr{B}_A(\mathcal{H}) \to \mathscr{B}_A(\mathcal{H})$ be a linear continuous map such that there exists an operator T satisfying $P\varphi(T) = P$. Suppose that φ preserves A-quasi-unitary operators in both directions. Then ψ is a homomorphism or an anti-homomorphism where ψ is defined by $\psi(T) = P\varphi(PT)$ for all $T \in \mathscr{B}_A(\mathcal{H})$.*

We prove some elementary results which are useful in the proofs of this theorem.

Lemma 4.2. *$P\varphi$ preserve A-projection in both directions.*

Proof. Let q be an A-projection. We consider a scalar $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \overline{\lambda} + \lambda$. It is easy to see that λq is an A-quasi-unitary operator, since φ preserves A-quasi-unitary operators in both directions. Hence $\lambda\varphi(q)$ is also an A-quasi-unitary operator and we get

$$|\lambda|^2\varphi(q)^\sharp\varphi(q) = |\lambda|^2(\varphi(q)^\sharp)^\sharp\varphi(q)^\sharp = \lambda(\varphi(q)^\sharp)^\sharp + \overline{\lambda}\varphi(q)^\sharp = \lambda P\varphi(q) + \overline{\lambda}\varphi(q)^\sharp. \tag{4.1}$$

If we replace λ by 2 in (4.1) we obtain

$$2\varphi(q)^\sharp P\varphi(q) = 2(\varphi(q)^\sharp)^\sharp\varphi(q)^\sharp = P\varphi(q) + \varphi(q)^\sharp. \tag{4.2}$$

In view of (4.1) and (4.2) we get

$$\frac{\lambda + \overline{\lambda}}{2}(P\varphi(q) + \varphi(q)^\sharp) = \overline{\lambda}\varphi(q)^\sharp + \lambda P\varphi(q), \tag{4.3}$$

which gives after an easy computation that

$$\frac{\lambda - \overline{\lambda}}{2}P\varphi(q) = \frac{\lambda - \overline{\lambda}}{2}\varphi(q)^\sharp.$$

Now if we take $\lambda \in \mathbb{C} - \mathbb{R}$ we have $P\varphi(q) = \varphi(q)^\sharp$. Recall the result obtained in (4.2). We get $\varphi(q)^\sharp = (\varphi(q)^\sharp)^2 = P\varphi(q)$ and consequently $P\varphi(q)$ is an A-projection. Now, if $P\varphi(q)$ is an A-projection, then $\varphi(q)$ is an A-projection. Repeating the same argument with $\varphi(q)$ being an A-projection we obtain that q is an A-projection, hence the proof is complete. □

Lemma 4.3. *$P\varphi(I) = P$.*

Proof. Let $B \in \mathcal{H}$ be such that $\varphi(I) = B$. We have $P\varphi(T) = P$. From Lemma 4.2 and since P is an A-projection, we get that T is an A-projection. It follows that $I - T$ is an A-projection, since $P\varphi$ preserves A-projections in both directions, thus $PB - P$ is an A-projection, consequently $ABP = APB = AP$, hence $AB = A$ and multiplying this equality by A^+ we get $PB = P$, which completes the proof. □

We will prove now Theorem 4.1.

Proof. By Lemmas 4.2 and 4.3 we have $P\varphi(I) = P$, so U is an A-unitary operator; this implies that $I - U$ is A-quasi-unitary. Since φ preserves A-quasi-unitary operators, $P\varphi(I - U) = P(I - \varphi(U))$ is an A-quasi-unitary operator, so $I - \varphi(U)$ is an A-quasi-unitary operator, and so $\varphi(U)$ is an A-unitary operator. It follows that $P\varphi(U)$ is an A-unitary operator. Consequently $P\varphi$ preserves A-unitary operators and $P\varphi(I) = P$. From Theorem 3.1 the result follows. □

A c k n o w l e d g e m e n t. We are greatly indebted to the referee for valuable suggestions that led to an overall improvement of the paper.

References

[1] *M. L. Arias, G. Corach, M. C. Gonzalez*: Metric properties of projections in semi-Hilbertian spaces. Integral Equations Oper. Theory *62* (2008), 11–28.

[2] *M. L. Arias, G. Corach, M. C. Gonzalez*: Partial isometries in semi-Hilbertian spaces. Linear Algebra Appl. *428* (2008), 1460–1475.

[3] *M. Brešar, P. Šemrl*: Linear maps preserving the spectral radius. J. Funct. Anal. *142* (1996), Article No. 0153, 360–368.

[4] *M. Brešar, P. Šemrl*: Mappings which preserve idempotents, local automorphisms, and local derivations. Can. J. Math. *45* (1993), 483–496.

[5] *R. G. Douglas*: On the operator equation $S^*XT = X$ and related topics. Acta Sci. Math. *30* (1969), 19–32.

[6] *R. G. Douglas*: On majorization, factorization, and range inclusion of operators on Hilbert space. Proc. Am. Math. Soc. *17* (1966), 413–415.

[7] *G. Frobenius*: Über die Darstellung der endlichen Gruppen durch lineare Substitutionen. Berl. Ber. *1897* (1897), 994–1015. (In German.)

[8] *I. N. Herstein*: Topics in Ring Theory. Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1969.

[9] *C.-K. Li, N.-K. Tsing*: Linear preserver problems: A brief introduction and some special techniques. Linear Algebra Appl. *162–164* (1992), 217–235.

[10] *M. Mbekhta*: Linear maps preserving the generalized spectrum. Extr. Math. *22* (2007), 45–54.

[11] *M. Omladič, P. Šemrl*: Linear mappings that preserve potent operators. Proc. Am. Math. Soc. *123* (1995), 1069–1074.

[12] *T. W. Palmer*: Banach Algebras and the General Theory of $*$-algebras, II. Encyclopedia of Mathematics and Its Applications 79, Cambridge University Press, Cambridge, 2001.

[13] *T. W. Palmer*: *-representations of U^*-algebras. Proc. Int. Symp. on Operator Theory (Indiana Univ., Bloomington, Ind., 1970) 20. Cambridge University Press, Cambridge, 2001, pp. 929–933.

[14] *S. V. Phadke, S. K. Khasbardar, N. K. Thakare*: On QU-operators. Indian J. Pure Appl. Math. *8* (1977), 335–343.

[15] *S. Pierce*: General introduction: A survey of linear preserver problems. Linear and Multilinear Algebra *33* (1992), 3–5.

[16] *V. Pták*: Banach algebras with involution. Manuscr. Math. *6* (1972), 245–290.

[17] *B. Russo, H. A. Dye*: A note on unitary operators in C^*-algebras. Duke Math. J. *33* (1966), 413–416.

[18] *P. Šemrl*: Two characterizations of automorphisms on $B(X)$. Stud. Math. *105* (1993), 143–149.

[19] *W. Watkins*: Linear maps that preserve commuting pairs of matrices. Linear Algebra Appl. *14* (1976), 29–35.

Authors' address: *Abdellatif Chahbi* (corresponding author), *Samir Kabbaj*, *Ahmed Charifi*, Department of Mathematics, Faculty of Sciences, Ibn Tofaïl University, B.P. 242, Kenitra, Morocco, e-mail: `ab-1980@live.fr`, `samkabbaj@yahoo.fr`, `charifi2000@yahoo.fr`.