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Jonald P. Fenecios; Emmanuel A. Cabral; Abraham P. Racca
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BAIRE ONE FUNCTIONS AND THEIR SETS OF DISCONTINUITY

Jonald P. Fenecios, Davao City, Emmanuel A. Cabral, Quezon City,
Abraham P. Racca, Silang

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Abstract. A characterization of functions in the first Baire class in terms of their sets of discontinuity is given. More precisely, a function $f\colon \mathbb{R}\to\mathbb{R}$ is of the first Baire class if and only if for each $\varepsilon>0$ there is a sequence of closed sets $\{C_n\}_{n=1}^{\infty}$ such that $D_f=\bigcup\limits_{n=1}^{\infty} C_n$ and $\omega_f(C_n)<\varepsilon$ for each n where

$$\omega_f(C_n)=\sup\{|f(x)-f(y)|\colon\ x,y\in C_n\}$$

and D_f denotes the set of points of discontinuity of f. The proof of the main theorem is based on a recent ε-δ characterization of Baire class one functions as well as on a well-known theorem due to Lebesgue. Some direct applications of the theorem are discussed in the paper.

Keywords: Baire class one function; set of points of discontinuity; oscillation of a function

MSC 2010: 26A21

1. Introduction

A function $f\colon \mathbb{R}\to\mathbb{R}$ is Baire class one or of first Baire class or simply Baire one if it is a pointwise limit of a sequence of continuous function on $\mathbb{R}$. Henri Lebesgue showed in 1904 that a function is of the first Baire class if and only if for each $k\in\mathbb{N}$, the domain can be represented as a countable union of closed sets so that the oscillation of f on each set is strictly less than $1/k$, see [2], page 116. For easy reference, we shall call this theorem Lebesgue's theorem. In the process it was proved that the set of points of discontinuity of f is a set of the first category. From this, a question emerges: Does a function whose set of points of discontinuity is of the first category have to be a Baire class one function? It turns out the answer is no.

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There is a function $f\colon \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is of the first category and at the same time of Lebesgue measure 0 but which is not Baire class one. Hence, from this perspective it is hard to obtain a characterization of Baire class one functions both in terms of the category and measure of its set of points of discontinuity.

However, a natural problem arises: can one still obtain a characterization of Baire class one functions in terms of their set of points of discontinuity? We answer this question in the affirmative.

2. A new characterization

Throughout the paper, we let C_f and D_f denote the set of points of continuity and the set of points of discontinuity of f, respectively. Before presenting the main result, we need the following useful propositions.

Proposition 2.1 ([8]). *If $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} E_n$ with each E_n an F_σ in $\mathbb{R}$ then there are disjoint F_σ sets F_n, $n = 1, 2, \ldots$ in $\mathbb{R}$ such that $F_n \subseteq E_n$ and $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} F_n$.*

Proposition 2.2 ([8]). *Let $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} F_n$ where F_n's are disjoint F_σ sets. Then there is a positive function $\delta(\cdot)$ on $\mathbb{R}$ such that $x \in F_n$, $y \in F_m$ and $n \neq m$ imply*

$$|x - y| \geqslant \min\{\delta(x), \delta(y)\}.$$

We shall now prove our main result.

Theorem 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$. The following statements are equivalent:*

(1) *f is Baire class one.*

(2) *For each $\varepsilon > 0$ there is a sequence of closed sets $\{C_n\}$ such that $D_f = \bigcup\limits_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \varepsilon$ for each n where*

$$\omega_f(C_n) = \sup\{|f(x) - f(y)|\colon\ x, y \in C_n\}.$$

Proof. (1) $\Rightarrow$ (2). Let $\varepsilon > 0$ be given. By Lebesgue's theorem, there exists a sequence of closed sets $\{E_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} E_n$ and $\omega_f(E_n) < \varepsilon$ for each n.

Since D_f is known to be an F_σ then there is a sequence of closed sets $\{F_n\}_{n=1}^{\infty}$ such that $D_f = \bigcup_{n=1}^{\infty} F_n$. It follows that we can express D_f as

$$D_f = \bigcup_{i,j \in \mathbb{N}} (E_i \cap F_j).$$

Clearly, $\omega_f(E_i \cap F_j) < \varepsilon$ for any pair (i, j).

(2) $\Rightarrow$ (1). We will use the characterization of Baire class one function due to Lee, Tang and Zhao [6] to establish (1). Let $\varepsilon > 0$. By assumption, there is a sequence of closed sets $\{C_n\}$ such that $D_f = \bigcup_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \varepsilon$ for each n. For each $x \in C_f$ there is a corresponding positive number $\delta_x > 0$ such that

$$y \in (x - \delta_x, x + \delta_x) \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Let $\mathcal{C} = \{G\colon\ G = (x - \delta_x, x + \delta_x) \text{ and } x \in C_f\}$. Then $C_f \subseteq \bigcup_{G \in \mathcal{C}} G$. We can find a countable subcollection $\{G_n\}$ of $\mathcal{C}$ such that $C_f \subseteq \bigcup_{n=1}^{\infty} G_n$. It follows that

$$\mathbb{R} = \left(\bigcup_{i=1}^{\infty} G_i\right) \cup \left(\bigcup_{j=1}^{\infty} C_j\right).$$

By reindexing, we can write $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ where $E_n = G_i$ for some i or $E_n = C_j$ for some j. By Proposition 2.1 we can find a disjoint sequence of F_σ sets $\{F_n\}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ and $F_n \subseteq E_n$ for each n. By Proposition 2.2 there is a positive function $\delta\colon\ \mathbb{R} \to \mathbb{R}^+$ such that $x \in F_m$, $y \in F_n$ with $m \neq n$ implies

$$|x - y| \geqslant \min\{\delta(x), \delta(y)\}.$$

Let $x, y \in \mathbb{R}$ and $|x - y| < \min\{\delta(x), \delta(y)\}$. By the property of the positive function $\delta(\cdot)$ there is a unique n such that $x, y \in F_n$. Since $F_n \subseteq E_n$ implies $|f(x) - f(y)| < \varepsilon$, all these show that f is Baire class one. $\square$

Remark. The theorem is saying that to decide whether a function belongs to the first Baire class, one no longer needs to examine the whole domain of the function, as Lebesgue's theorem is suggesting, but one examines instead the set of points of discontinuity of the function. In this sense, Theorem 2.1 may be viewed as an improvement of Lebesgue's theorem.

It may be observed that Theorem 2.1 can be expressed in a slightly different manner that may prove useful in some cases. We will state it as a corollary.

Corollary 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$. The following statements are equivalent:*

(1) *f is Baire class one.*

(2) *For each $\varepsilon > 0$ there is a sequence of closed sets $\{C_n\}$ such that $D_f \subseteq \bigcup_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \varepsilon$ for each n where*

$$\omega_f(C_n) = \sup\{|f(x) - f(y)|\colon\ x, y \in C_n\}.$$

3. Some applications

In this section, we shall try to give some applications of Theorem 2.1. A short and quick proof that a function with countable set of discontinuity is Baire class one is perhaps through a theorem due to René Baire: A function $f\colon \mathbb{R} \to \mathbb{R}$ is Baire class one if and only if for every closed set K, the restriction $f|_K$ has a point of continuity in K. However, the statement above also admits a straighforward justification using Theorem 2.1. For other proofs one may see [3], [4], [5], [7].

Theorem 3.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$. If D_f is countable then f is Baire class one.*

Proof. Note that a countable set is a countable union of singletons. Clearly the oscillation of the function on each singleton is zero. □

Theorem 3.2. *Let $f\colon \mathbb{R} \to \mathbb{R}$. If $f(D_f)$ is countable and for each $y \in f(D_f)$ the set $\{x \in D_f\colon\ f(x) = y\}$ is F_σ then f is Baire class one.*

Proof. Let $f(D_f) = \{r_1, r_2, \ldots, r_n, \ldots\}$ and let $F_i = \{x \in D_f\colon\ f(x) = r_i\}$ for each i. By assumption, each F_i is F_σ. Note that $D_f \subseteq \bigcup_{i=1}^{\infty} F_i$ and $\omega_f(F_i) = 0$ for each i. By Corollary 2.1, f is Baire class one. □

Theorem 3.3. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ such that $D_f \subseteq D_g$. If g is Baire class one and $\omega_f(A) \leqslant \omega_g(A)$ for every $A \subseteq D_f$ then f is Baire class one.*

Proof. Let $\varepsilon > 0$ be given. Since g is Baire class one then there exists a sequence of closed sets $\{D_n\}$ such that $D_g = \bigcup_{n=1}^{\infty} D_n$ and $\omega_g(D_n) < \varepsilon$ for all n. Since $D_f \subseteq D_g$ and D_f is F_σ then we can find a countable collection $\{E_n\}$ of closed sets such that $D_f = \bigcup_{n=1}^{\infty} E_n$ and $\omega_g(E_n) < \varepsilon$ for each n. From the hypothesis, it immediately follows that $\omega_f(E_n) \leqslant \omega_g(E_n) < \varepsilon$ for each n. Thus, f is Baire class one. □

Lastly, we will show that if $f|_{D_f}$ is continuous on D_f then f is Baire class one. This class of functions is called B_1^{**}, see [1]. Moreover, we use the new characterization to show that if there is a sequence of closed sets $\{E_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} E_n$ and the restriction $f|_{E_n}$is continuous on E_n for each n then f is Baire class one. Such functions are known as piecewise continuous functions. We need first the following lemma.

Lemma 3.1. *Let K be a closed subset of $\mathbb{R}$. If $f|_K$ is continuous on K then for each $\varepsilon > 0$ there exists a sequence $\{K_n\}$ of closed sets covering K satisfying $\omega_f(K_n) < \varepsilon$ for each n.*

Proof. Let $\varepsilon > 0$. For each $x \in K$ there is an open interval I_x containing x such that

$$y \in I_x \cap K \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Note that the collection $\mathcal{C} = \{I_x \cap K \colon\ x \in K\}$ forms an open cover of K where K is viewed as a subspace of $\mathbb{R}$. Since K is a Lindelöf subspace of $\mathbb{R}$, we can find a countable subcollection of $\mathcal{C}$ such that

$$K = \bigcup_{i=1}^{\infty} (I_{\xi_i} \cap K).$$

Note that $\omega_f(I_{\xi_i} \cap K) < \varepsilon$ and each $I_{\xi_i} \cap K$ is F_σ. The lemma follows. □

Theorem 3.4. *Let $f\colon\ \mathbb{R} \to \mathbb{R}$. If $f|_{D_f}$ is continuous on D_f then f is Baire class one.*

Proof. Since D_f is F_σ then there exists a sequence of closed sets $\{K_n\}$ in $\mathbb{R}$ such that $D_f = \bigcup\limits_{n=1}^{\infty} K_n$. Since $f|_{D_f}$ is continuous, $f|_{K_n}$ is continuous for each n. We apply Lemma 3.1 and Theorem 2.1 to conclude that f is Baire class one. □

Theorem 3.5. *Let $f\colon\ \mathbb{R} \to \mathbb{R}$. If there is a sequence of closed sets $\{E_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} E_n$ and the restriction $f|_{E_n}$is continuous on E_n for each n then f is Baire class one.*

Proof. Let $\varepsilon > 0$ be given. Since $\{E_n\}$ covers $\mathbb{R}$ it certainly covers D_f. Since $f|_{E_n}$ is continuous for each n by Lemma 3.1 there exists a sequence of closed sets $\{F_n\}$ such that $D_f \subset \bigcup\limits_{n=1}^{\infty} F_n$ and $\omega_f(F_n) < \varepsilon$ for each n. Thus by Corollary 2.1, f is Baire class one. □

It is also interesting to note that if there is a sequence $\{D_n\}$ of closed sets such that $D_f = \bigcup_{n=1}^{\infty} D_n$ and $f|_{D_n}$ is continuous for each n then f is Baire one on the whole of $\mathbb{R}$. Clearly, this class of functions lies between the class of all piecewise continuous functions and the class of Baire one functions. It is not clear whether this class of functions is the same as the class of all Baire one functions.

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References

[1] *A. Bąkowska, R. J. Pawlak*: On some characterizations of Baire class one functions and Baire class one like functions. Tatra Mt. Math. Publ. *46* (2010), 91–106.
[2] *D. M. Bressoud*: A Radical Approach to Lebesgue's Theory of Integration. MAA Textbooks, Cambridge University Press, Cambridge, 2008.
[3] *A. M. Bruckner, J. B. Bruckner, B. S. Thomson*: Real Analysis. Prentice-Hall International, Upper Saddle River, 1997.
[4] *R. A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. Graduate Studies in Mathematics 4, American Mathematical Society, Providence, 1994.
[5] *K. Kuratowski*: Topology. I. Academic Press, New York; Państwowe Wydawnictwo Naukowe, Warszawa; Mir, Moskva, 1966. (In Russian.)
[6] *P.-Y. Lee, W.-K. Tang, D. Zhao*: An equivalent definition of functions of the first Baire class. Proc. Am. Math. Soc. *129* (2001), 2273–2275.
[7] *I. P. Natanson*: Theory of Functions of a Real Variable. II. Frederick Ungar Publishing, New York, 1961. (In German.)
[8] *D. Zhao*: Functions whose composition with Baire class one functions are Baire class one. Soochow J. Math. *33* (2007), 543–551.

Authors' addresses: *Jonald P. Fenecios*, Department of Mathematics, Ateneo de Davao University, E. Jacinto Street, 8016 Davao, Philippines, e-mail: `jpfenecios@addu.edu.ph`; *Emmanuel A. Cabral*, Department of Mathematics, Ateneo de Manila University, Loyola Heights Campus, Katipunan Avenue, 1108 Quezon, Philippines, e-mail: `ecabral@ateneo.edu`; *Abraham P. Racca*, Department of Mathematics and Physics, Adventist University of the Philippines, Puting Kahoy, Silang, 4118 Cavite, Philippines, e-mail: `abraham.racca@yahoo.com`.