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BAIRE ONE FUNCTIONS AND THEIR SETS OF DISCONTINUITY

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Abstract. A characterization of functions in the first Baire class in terms of their sets of discontinuity is given. More precisely, a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is of the first Baire class if and only if for each \( \varepsilon > 0 \) there is a sequence of closed sets \( \{C_n\}_{n=1}^{\infty} \) such that \( D_f = \bigcup_{n=1}^{\infty} C_n \) and \( \omega_f(C_n) < \varepsilon \) for each \( n \) where

\[
\omega_f(C_n) = \sup\{|f(x) - f(y)| : x, y \in C_n\}
\]

and \( D_f \) denotes the set of points of discontinuity of \( f \). The proof of the main theorem is based on a recent \( \varepsilon-\delta \) characterization of Baire class one functions as well as on a well-known theorem due to Lebesgue. Some direct applications of the theorem are discussed in the paper.

Keywords: Baire class one function; set of points of discontinuity; oscillation of a function
MSC 2010: 26A21

1. Introduction

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is Baire class one or of first Baire class or simply Baire one if it is a pointwise limit of a sequence of continuous function on \( \mathbb{R} \). Henri Lebesgue showed in 1904 that a function is of the first Baire class if and only if for each \( k \in \mathbb{N} \), the domain can be represented as a countable union of closed sets so that the oscillation of \( f \) on each set is strictly less than \( 1/k \), see [2], page 116. For easy reference, we shall call this theorem Lebesgue’s theorem. In the process it was proved that the set of points of discontinuity of \( f \) is a set of the first category. From this, a question emerges: Does a function whose set of points of discontinuity is of the first category have to be a Baire class one function? It turns out the answer is no.
There is a function $f: \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is of the first category and at the same time of Lebesgue measure 0 but which is not Baire class one. Hence, from this perspective it is hard to obtain a characterization of Baire class one functions both in terms of the category and measure of its set of points of discontinuity.

However, a natural problem arises: can one still obtain a characterization of Baire class one functions in terms of their set of points of discontinuity? We answer this question in the affirmative.

2. A new characterization

Throughout the paper, we let $C_f$ and $D_f$ denote the set of points of continuity and the set of points of discontinuity of $f$, respectively. Before presenting the main result, we need the following useful propositions.

**Proposition 2.1** ([8]). If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ with each $E_n$ an $F_\sigma$ in $\mathbb{R}$ then there are disjoint $F_\sigma$ sets $F_n$, $n = 1, 2, \ldots$ in $\mathbb{R}$ such that $F_n \subseteq E_n$ and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$.

**Proposition 2.2** ([8]). Let $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ where $F_n$'s are disjoint $F_\sigma$ sets. Then there is a positive function $\delta(\cdot)$ on $\mathbb{R}$ such that $x \in F_n$, $y \in F_m$ and $n \neq m$ imply

$$|x - y| \geq \min\{\delta(x), \delta(y)\}.$$

We shall now prove our main result.

**Theorem 2.1.** Let $f: \mathbb{R} \to \mathbb{R}$. The following statements are equivalent:

1. $f$ is Baire class one.
2. For each $\varepsilon > 0$ there is a sequence of closed sets $\{C_n\}$ such that $D_f = \bigcup_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \varepsilon$ for each $n$ where

$$\omega_f(C_n) = \sup\{|f(x) - f(y)|: x, y \in C_n\}.$$

**Proof.** (1) ⇒ (2). Let $\varepsilon > 0$ be given. By Lebesgue’s theorem, there exists a sequence of closed sets $\{E_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ and $\omega_f(E_n) < \varepsilon$ for each $n$. 

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Since $D_f$ is known to be an $F_\sigma$ then there is a sequence of closed sets $\{F_n\}_{n=1}^\infty$ such that $D_f = \bigcup_{n=1}^\infty F_n$. It follows that we can express $D_f$ as

$$D_f = \bigcup_{i,j \in \mathbb{N}} (E_i \cap F_j).$$

Clearly, $\omega_f(E_i \cap F_j) < \varepsilon$ for any pair $(i, j)$.

(2) $\Rightarrow$ (1). We will use the characterization of Baire class one function due to Lee, Tang and Zhao [6] to establish (1). Let $\varepsilon > 0$. By assumption, there is a sequence of closed sets $\{C_n\}$ such that $D_f = \bigcup_{n=1}^\infty C_n$ and $\omega_f(C_n) < \varepsilon$ for each $n$. For each $x \in C_f$ there is a corresponding positive number $\delta_x > 0$ such that

$$y \in (x - \delta_x, x + \delta_x) \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$ 

Let $\mathcal{C} = \{G: G = (x - \delta_x, x + \delta_x) \text{ and } x \in C_f\}$. Then $C_f \subseteq \bigcup_{G \in \mathcal{C}} G$. We can find a countable subcollection $\{G_n\}$ of $\mathcal{C}$ such that $C_f \subseteq \bigcup_{n=1}^\infty G_n$. It follows that

$$\mathbb{R} = \left(\bigcup_{i=1}^\infty G_i\right) \cup \left(\bigcup_{j=1}^\infty C_j\right).$$

By reindexing, we can write $\mathbb{R} = \bigcup_{n=1}^\infty E_n$ where $E_n = G_i$ for some $i$ or $E_n = C_j$ for some $j$. By Proposition 2.1 we can find a disjoint sequence of $F_\sigma$ sets $\{F_n\}$ such that $\mathbb{R} = \bigcup_{n=1}^\infty F_n$ and $F_n \subseteq E_n$ for each $n$. By Proposition 2.2 there is a positive function $\delta: \mathbb{R} \to \mathbb{R}^+$ such that $x \in F_m$, $y \in F_n$ with $m \neq n$ implies

$$|x - y| \geq \min\{\delta(x), \delta(y)\}.$$ 

Let $x, y \in \mathbb{R}$ and $|x - y| < \min\{\delta(x), \delta(y)\}$. By the property of the positive function $\delta(\cdot)$ there is a unique $n$ such that $x, y \in F_n$. Since $F_n \subseteq E_n$ implies $|f(x) - f(y)| < \varepsilon$, all these show that $f$ is Baire class one. \qed

Remark. The theorem is saying that to decide whether a function belongs to the first Baire class, one no longer needs to examine the whole domain of the function, as Lebesgue’s theorem is suggesting, but one examines instead the set of points of discontinuity of the function. In this sense, Theorem 2.1 may be viewed as an improvement of Lebesgue’s theorem.

It may be observed that Theorem 2.1 can be expressed in a slightly different manner that may prove useful in some cases. We will state it as a corollary.
Corollary 2.1. Let $f: \mathbb{R} \to \mathbb{R}$. The following statements are equivalent:

1. $f$ is Baire class one.
2. For each $\varepsilon > 0$ there is a sequence of closed sets $\{C_n\}$ such that $D_f \subseteq \bigcup_{n=1}^{\infty} C_n$ and $\omega_f(C_n) < \varepsilon$ for each $n$ where
   \[ \omega_f(C_n) = \sup\{|f(x) - f(y)| : x, y \in C_n\}. \]

3. Some applications

In this section, we shall try to give some applications of Theorem 2.1. A short and quick proof that a function with countable set of discontinuity is Baire class one is perhaps through a theorem due to René Baire: A function $f: \mathbb{R} \to \mathbb{R}$ is Baire class one if and only if for every closed set $K$, the restriction $f|_K$ has a point of continuity in $K$. However, the statement above also admits a straightforward justification using Theorem 2.1. For other proofs one may see [3], [4], [5], [7].

Theorem 3.1. Let $f: \mathbb{R} \to \mathbb{R}$. If $D_f$ is countable then $f$ is Baire class one.

Proof. Note that a countable set is a countable union of singletons. Clearly the oscillation of the function on each singleton is zero. □

Theorem 3.2. Let $f: \mathbb{R} \to \mathbb{R}$. If $f(D_f)$ is countable and for each $y \in f(D_f)$ the set $\{x \in D_f : f(x) = y\}$ is $F_\sigma$ then $f$ is Baire class one.

Proof. Let $f(D_f) = \{r_1, r_2, \ldots, r_n, \ldots\}$ and let $F_i = \{x \in D_f : f(x) = r_i\}$ for each $i$. By assumption, each $F_i$ is $F_\sigma$. Note that $D_f \subseteq \bigcup_{i=1}^{\infty} F_i$ and $\omega_f(F_i) = 0$ for each $i$. By Corollary 2.1, $f$ is Baire class one. □

Theorem 3.3. Let $f, g: \mathbb{R} \to \mathbb{R}$ such that $D_f \subseteq D_g$. If $g$ is Baire class one and $\omega_f(A) \leq \omega_g(A)$ for every $A \subseteq D_f$ then $f$ is Baire class one.

Proof. Let $\varepsilon > 0$ be given. Since $g$ is Baire class one then there exists a sequence of closed sets $\{D_n\}$ such that $D_g = \bigcup_{n=1}^{\infty} D_n$ and $\omega_g(D_n) < \varepsilon$ for all $n$. Since $D_f \subseteq D_g$ and $D_f$ is $F_\sigma$ then we can find a countable collection $\{E_n\}$ of closed sets such that $D_f = \bigcup_{n=1}^{\infty} E_n$ and $\omega_g(E_n) < \varepsilon$ for each $n$. From the hypothesis, it immediately follows that $\omega_f(E_n) \leq \omega_g(E_n) < \varepsilon$ for each $n$. Thus, $f$ is Baire class one. □
Lastly, we will show that if \( f|_{D_f} \) is continuous on \( D_f \) then \( f \) is Baire class one. This class of functions is called \( B_1^* \), see [1]. Moreover, we use the new characterization to show that if there is a sequence of closed sets \( \{E_n\}_{n=1}^\infty \) such that \( \mathbb{R} = \bigcup_{n=1}^\infty E_n \) and the restriction \( f|_{E_n} \) is continuous on \( E_n \) for each \( n \) then \( f \) is Baire class one. Such functions are known as piecewise continuous functions. We need first the following lemma.

**Lemma 3.1.** Let \( K \) be a closed subset of \( \mathbb{R} \). If \( f|_K \) is continuous on \( K \) then for each \( \varepsilon > 0 \) there exists a sequence \( \{K_n\} \) of closed sets covering \( K \) satisfying \( \omega_f(K_n) < \varepsilon \) for each \( n \).

**Proof.** Let \( \varepsilon > 0 \). For each \( x \in K \) there is an open interval \( I_x \) containing \( x \) such that
\[
y \in I_x \cap K \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.
\]
Note that the collection \( C = \{I_x \cap K : x \in K\} \) forms an open cover of \( K \) where \( K \) is viewed as a subspace of \( \mathbb{R} \). Since \( K \) is a Lindelöf subspace of \( \mathbb{R} \), we can find a countable subcollection of \( C \) such that
\[
K = \bigcup_{i=1}^\infty (I_{\xi_i} \cap K).
\]
Note that \( \omega_f(I_{\xi_i} \cap K) < \varepsilon \) and each \( I_{\xi_i} \cap K \) is \( F_\sigma \). The lemma follows. \( \square \)

**Theorem 3.4.** Let \( f : \mathbb{R} \to \mathbb{R} \). If \( f|_{D_f} \) is continuous on \( D_f \) then \( f \) is Baire class one.

**Proof.** Since \( D_f \) is \( F_\sigma \) then there exists a sequence of closed sets \( \{K_n\} \) in \( \mathbb{R} \) such that \( D_f = \bigcup_{n=1}^\infty K_n \). Since \( f|_{D_f} \) is continuous, \( f|_{K_n} \) is continuous for each \( n \). We apply Lemma 3.1 and Theorem 2.1 to conclude that \( f \) is Baire class one. \( \square \)

**Theorem 3.5.** Let \( f : \mathbb{R} \to \mathbb{R} \). If there is a sequence of closed sets \( \{E_n\}_{n=1}^\infty \) such that \( \mathbb{R} = \bigcup_{n=1}^\infty E_n \) and the restriction \( f|_{E_n} \) is continuous on \( E_n \) for each \( n \) then \( f \) is Baire class one.

**Proof.** Let \( \varepsilon > 0 \) be given. Since \( \{E_n\} \) covers \( \mathbb{R} \) it certainly covers \( D_f \). Since \( f|_{E_n} \) is continuous for each \( n \) by Lemma 3.1 there exists a sequence of closed sets \( \{F_n\} \) such that \( D_f \subset \bigcup_{n=1}^\infty F_n \) and \( \omega_f(F_n) < \varepsilon \) for each \( n \). Thus by Corollary 2.1, \( f \) is Baire class one. \( \square \)
It is also interesting to note that if there is a sequence \( \{D_n\} \) of closed sets such that \( D_f = \bigcup_{n=1}^{\infty} D_n \) and \( f|_{D_n} \) is continuous for each \( n \) then \( f \) is Baire one on the whole of \( \mathbb{R} \). Clearly, this class of functions lies between the class of all piecewise continuous functions and the class of Baire one functions. It is not clear whether this class of functions is the same as the class of all Baire one functions.

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References


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