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INTEGRABILITY FOR VERY WEAK SOLUTIONS TO BOUNDARY VALUE PROBLEMS OF p-HARMONIC EQUATION

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Abstract. The paper deals with very weak solutions $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p < n$, to boundary value problems of the p-harmonic equation

$$(*) \qquad \begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0, & x \in \Omega, \\ u(x) = \theta(x), & x \in \partial\Omega. \end{cases}$$

We show that, under the assumption $\theta \in W^{1,q}(\Omega)$, $q > r$, any very weak solution u to the boundary value problem $(*)$ is integrable with

$$u \in \begin{cases} \theta + L^{q^*}_{\rm weak}(\Omega) & \text{for } q < n, \\ \theta + L^{\tau}_{\rm weak}(\Omega) & \text{for } q = n \text{ and any } \tau < \infty, \\ \theta + L^{\infty}(\Omega) & \text{for } q > n, \end{cases}$$

provided that r is sufficiently close to p.

Keywords: integrability; very weak solution; boundary value problem; p-harmonic equation

MSC 2010: 35J25, 35D30

1. Introduction and preliminary lemmas

Throughout this paper Ω will stand for a bounded regular domain in $\mathbb{R}^n$, $n \geqslant 2$. By a regular domain we understand any domain of finite measure for which the estimates (2.4) and (2.5) below for the Hodge decomposition are satisfied, see [11], [12]. A Lipschitz domain, for example, is regular.

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Let $1 < p < n$. We shall examine the boundary value problem of the p-harmonic equation

$$\tag{1.1} \begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0, & x \in \Omega, \\ u(x) = \theta(x), & x \in \partial\Omega, \end{cases}$$

where $\theta(x) \in W^{1,q}(\Omega)$, $q > r$.

This paper deals with *very weak* solutions to (1.1).

Definition 1.1. A function $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p$, is called a very weak solution to the boundary value problem (1.1) if

$$\tag{1.2} \int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \,\mathrm{d}x = 0$$

holds true for all $\varphi \in W_0^{1,r/(r-p+1)}(\Omega)$.

Recall that a function $u \in \theta + W_0^{1,p}(\Omega)$ is called a weak solution to the boundary value problem (1.1) if (1.2) holds true for all $\varphi \in W_0^{1,p}(\Omega)$. This is what we call the natural setting of problem (1.1). The words *very weak* in Definition 1.1 mean that the integrable exponent r of u can be smaller than the natural one p. We refer the readers to [11], Theorem 1, page 602, and [9], Theorems 1 and 2, page 251, for some results related to very weak solutions to the p-harmonic equation, although up to now, the existence and uniqueness of such solutions remain unclear.

In this paper we will need the definition of weak L^t-spaces or Marcinkiewicz spaces (see [2], Chapter 1, Section 2, [10], Chapter 2, Section 5 or [16], Chapter 2, Section 18): for $t > 0$, the weak L^t-space, $L^t_{\mathrm{weak}}(\Omega)$, consists of all measurable functions f such that

$$|\{x \in \Omega\colon\ |f(x)| > s\}| \leqslant \frac{k}{s^t}$$

for some positive constant $k = k(f)$ and every $s > 0$, where $|E|$ is the n-dimensional Lebesgue measure of E. Note that if $f \in L^t_{\mathrm{weak}}(\Omega)$ for some $t > 1$, then $f \in L^\tau(\Omega)$ for every $1 \leqslant \tau < t$.

Integrability property is important in the regularity theories of nonlinear elliptic PDEs and systems, see [1], [3]–[8], [17]–[19], [21], [22]. In [14], [15], the authors considered regularity properties of the p-harmonic type equations with r sufficiently close to p. In [9], Greco et al. were concerned with the nonhomogeneous p-harmonic equation

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = -\operatorname{div} f,$$

and obtained an estimate for the operator H which carries a given vector function f into the gradient field ∇u. In the present paper, we consider very weak solutions to boundary value problems of (1.1). The main result of this paper is the following theorem.

Theorem 1.1. *Let $\theta \in W^{1,q}(\Omega)$, $q > r$. There exists $\varepsilon_0 = \varepsilon_0(n,p) > 0$ such that for every very weak solution $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p < n$, to the boundary value problem* (1.1), *we have*

$$u \in \begin{cases} \theta + L^{q^*}_{\mathrm{weak}}(\Omega) & \text{for } q < n, \\ \theta + L^{\tau}_{\mathrm{weak}}(\Omega) & \text{for } q = n \text{ and any } \tau < \infty, \\ \theta + L^{\infty}(\Omega) & \text{for } q > n, \end{cases} \tag{1.3}$$

provided that $|p - r| < \varepsilon_0$.

Note that we have restricted ourselves to the case $r < n$ since otherwise any function in $W^{1,r}(\Omega)$ is in the space $L^t(\Omega)$ for any $t < \infty$ by the Sobolev embedding theorem. Note also that very weak solutions u to the boundary value problem (1.1) are taken from the Sobolev space $W^{1,r}(\Omega)$. The embedding theorem guarantees that the integrability of u reaches r^*. Our result (1.3) improves such integrability. We remark that the key point in the proof of Theorem 1.1 is the choice of appropriate test functions. We will use the stability estimate of Hodge decomposition used in [9], [11], [12], see (2.3)–(2.5) below.

In order to prove Theorem 1.1, we need the following two lemmas.

Lemma 1.1. *For $1 < p < 2$ and any $X, Y \in \mathbb{R}^n$, one has*

$$\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle \geqslant |X - Y|((|X - Y| + |Y|)^{p-1} - |Y|^{p-1}).$$

Proof. It is no loss of generality to assume that $X \neq Y$. For $0 \leqslant t \leqslant 1$ we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle |tX - tY + Y|^{p-2}(tX - tY + Y), X - Y\rangle \\ &\qquad = (p-2)|tX - tY + Y|^{p-4}\langle tX - tY + Y, X - Y\rangle^2 \\ &\qquad\quad + \langle |tX - tY + Y|^{p-2}(X - Y), X - Y\rangle. \end{aligned}$$

This implies

$$\begin{aligned} &\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle \\ &\quad = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\langle |tX - tY + Y|^{p-2}(tX - tY + Y), X - Y\rangle \,\mathrm{d}t \\ &\quad = (p-2)\int_0^1 |tX - tY + Y|^{p-4}\langle tX - tY + Y, X - Y\rangle^2 \,\mathrm{d}t \\ &\qquad + |X - Y|^2 \int_0^1 |tX - tY + Y|^{p-2}\,\mathrm{d}t. \end{aligned} \tag{1.4}$$

Note that

$$|tX - tY + Y|^{p-4}\langle tX - tY + Y, X - Y\rangle^2 \leqslant |tX - tY + Y|^{p-2}|X - Y|^2,$$

which together with (1.4) and the fact $1 < p < 2$ implies

$$\begin{aligned}\langle |X|^{p-2}X - |Y|^{p-2}Y, X - Y\rangle &\geqslant (p-1)|X - Y|^2 \int_0^1 |tX - tY + Y|^{p-2}\,\mathrm{d}t \\ &\geqslant (p-1)|X - Y|^2 \int_0^1 (t|X - Y| + |Y|)^{p-2}\,\mathrm{d}t \\ &= |X - Y| \int_0^1 \mathrm{d}(t|X - Y| + |Y|)^{p-1}\,\mathrm{d}t \\ &= |X - Y|\big((|X - Y| + |Y|)^{p-1} - |Y|^{p-1}\big).\end{aligned}$$

This completes the proof of Lemma 1.1. □

The following technical result can be found in [23], Lemma 4.1.

Lemma 1.2. *Let $s_0 > 0$ and let $\phi\colon (s_0, \infty) \to [0, \infty)$ be a decreasing function such that for every r, s with $r > s > s_0$,*

$$\phi(r) \leqslant \frac{c}{(r - s)^{\alpha}}(\phi(s))^{\beta},$$

where c, α, β are positive constants. Then

(i) *if $\beta > 1$ we have that $\phi(s_0 + d) = 0$, where*

$$d^{\alpha} = c2^{\alpha\beta/(\beta-1)}(\phi(s_0))^{\beta-1};$$

(ii) *if $\beta < 1$ we have that*

$$\phi(s) \leqslant 2^{\mu/(1-\beta)}(c^{1/(1-\beta)} + (2s_0)^{\mu}\phi(s_0))s^{-\mu},$$

where $\mu = \alpha/(1 - \beta)$.

2. Proof of Theorem 1.1

In the following, $C(*,\dots,*)$ will denote a constant that depends only on the quantities $*,\dots,*$, whose value may vary from line to line.

For any $L>0$ we take

$$v=\begin{cases} u-\theta+L & \text{for } u-\theta<-L,\\ 0 & \text{for } -L\leqslant u-\theta\leqslant L,\\ u-\theta-L & \text{for } u-\theta>L,\end{cases} \tag{2.1}$$

so that, by our assumptions, we have $v\in W_0^{1,r}(\Omega)$ and

$$\nabla v=(\nabla u-\nabla\theta)\cdot 1_{\{|u-\theta|>L\}}, \tag{2.2}$$

where 1_E is the characteristic function for the set E, that is, $1_E=1$ if $x\in E$ and $1_E=0$ otherwise. We introduce the Hodge decomposition of the vector field $|\nabla v|^{p-2}\nabla v\in L^{r/(r-p+1)}(\Omega)$. Accordingly,

$$|\nabla v|^{r-p}\nabla v=\nabla\varphi+h, \tag{2.3}$$

where φ is in $W_0^{1,r/(r-p+1)}(\Omega)$ and h is a divergence free vector field of class $L^{r/(r-p+1)}(\Omega,\mathbb{R}^n)$. The reader is referred to [9], [11], [12] for estimates concerning such decomposition. We have

$$\|\nabla\varphi\|_{r/(r-p+1)}\leqslant C(n,p)\|\nabla v\|_r^{r-p+1} \tag{2.4}$$

and

$$\|h\|_{r/(r-p+1)}\leqslant C(n,p)|p-r|\|\nabla v\|_r^{r-p+1}. \tag{2.5}$$

In particular, φ can be used as a test function for the integral identity (1.2), namely

$$\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x=\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u,h\rangle\,\mathrm{d}x.$$

This implies

$$\begin{aligned}\int_{\{|u-\theta|>L\}}&\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x\\ &=\int_{\{|u-\theta|>L\}}\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,h\rangle\,\mathrm{d}x+\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,h\rangle\,\mathrm{d}x\\ &\quad-\int_{\{|u-\theta|>L\}}\langle|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x=I_1+I_2+I_3.\end{aligned} \tag{2.6}$$

Now we shall distinguish between two cases.

Case 1: $p \geqslant 2$. Since for any $X, Y \in \mathbb{R}^n$ (see [20], page 72)

$$2^{2-p}|X-Y|^p \leqslant \langle |X|^{p-2}X - |Y|^{p-2}Y, X-Y\rangle,$$

the left-hand side of (2.6) can be estimated as

$$\begin{aligned}(2.7)\qquad &\int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u - \nabla\theta|^{r-p}(\nabla u - \nabla\theta)\rangle \,\mathrm{d}x \\ &\qquad \geqslant 2^{2-p}\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r \,\mathrm{d}x.\end{aligned}$$

We now estimate $|I_1|$, $|I_2|$ and $|I_3|$. By an elementary inequality (see [13]): for any $X, Y \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$(2.8)\qquad \big||X|^\varepsilon X - |Y|^\varepsilon Y\big| \leqslant \begin{cases} (1+\varepsilon)(|Y|+|X-Y|)^\varepsilon|X-Y| & \text{for } \varepsilon > 0,\\ \dfrac{1-\varepsilon}{2^\varepsilon(1+\varepsilon)}|X-Y|^{1+\varepsilon} & \text{for } -1 < \varepsilon \leqslant 0,\end{cases}$$

and using Hölder inequality, (2.5) and Young inequality, we obtain

$$\begin{aligned}(2.9)\qquad |I_1| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, h\rangle \,\mathrm{d}x\right| \\ &\leqslant (p-1)\int_{\{|u-\theta|>L\}} (|\nabla\theta| + |\nabla u - \nabla\theta|)^{p-2}|\nabla u - \nabla\theta||h| \,\mathrm{d}x \\ &\leqslant 2^{p-2}(p-1)\bigg(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-2}|\nabla u - \nabla\theta||h| \,\mathrm{d}x \\ &\qquad + \int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^{p-1}|h| \,\mathrm{d}x\bigg) \\ &\leqslant 2^{p-2}(p-1)(\|\nabla\theta\|_r^{p-2}\|\nabla u - \nabla\theta\|_r\|h\|_{r/(r-p+1)} \\ &\qquad + \|\nabla u - \nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)}) \\ &\leqslant 2^{p-2}(p-1)C(n,p)|p-r|(\|\nabla\theta\|_r^{p-2}\|\nabla u - \nabla\theta\|_r^{r-p+2} + \|\nabla u - \nabla\theta\|_r^r) \\ &\leqslant 2^{p-2}(p-1)C(n,p)|p-r|(C(\varepsilon)\|\nabla\theta\|_r^r + (1+\varepsilon)\|\nabla u - \nabla\theta\|_r^r);\end{aligned}$$

here and in the sequel, $\|\cdot\|_r = \|\cdot\|_{r,\{|u-\theta|>L\}}$, we omit the subscript for the sake of simplicity.

Using the Hölder inequality, (2.5) and Young inequality again, $|I_2|$ and $|I_3|$ can be estimated as

$$\begin{aligned}(2.10)\qquad |I_2| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, h\rangle \,\mathrm{d}x\right| \leqslant \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|h| \,\mathrm{d}x \\ &\leqslant \|\nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)} \leqslant C(n,p)|p-r|\|\nabla\theta\|_r^{p-1}\|\nabla u - \nabla\theta\|_r^{r-p+1} \\ &\leqslant C(n,p)|p-r|[C(\varepsilon)\|\nabla\theta\|_r^r + \varepsilon\|\nabla u - \nabla\theta\|_r^r],\end{aligned}$$

$$
\begin{aligned}
(2.11)\qquad |I_3| &= \left|-\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle \,\mathrm{d}x\right| \\
&\leqslant \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|\nabla u-\nabla\theta|^{r-p+1}\,\mathrm{d}x \leqslant \|\nabla\theta\|_r^{p-1}\|\nabla u-\nabla\theta\|_r^{r-p+1} \\
&\leqslant C(\varepsilon)\|\nabla\theta\|_r^r + \varepsilon\|\nabla u-\nabla\theta\|_r^r.
\end{aligned}
$$

Combining (2.6)–(2.7), (2.9)–(2.11) we arrive at

$$
(2.12)\qquad \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r\,\mathrm{d}x \leqslant C(n,p,\varepsilon)\|\nabla\theta\|_r^r + (C(n,p)|p-r|+\varepsilon)\|\nabla u-\nabla\theta\|_r^r.
$$

Case 2: $1<p<2$. Lemma 1.1 yields

$$
\begin{aligned}
\int_{\{|u-\theta|>L\}} &\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x \\
&\geqslant \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1} - |\nabla\theta|^{p-1})\,\mathrm{d}x.
\end{aligned}
$$

This implies

$$
\begin{aligned}
(2.13)\qquad \int_{\{|u-\theta|>L\}} &|\nabla u-\nabla\theta|^r\,\mathrm{d}x \\
&\leqslant \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1})\,\mathrm{d}x \\
&\leqslant \int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x \\
&\quad + \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}|\nabla\theta|^{p-1}\,\mathrm{d}x \\
&\leqslant \int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle\,\mathrm{d}x \\
&\quad + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r\,\mathrm{d}x + C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^r\,\mathrm{d}x.
\end{aligned}
$$

By (2.8) and (2.5), $|I_1|$ can be estimated as

$$
\begin{aligned}
(2.14)\qquad |I_1| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, h\rangle\,\mathrm{d}x\right| \\
&\leqslant \frac{3-p}{2^{p-2}(p-1)}\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{p-1}|h|\,\mathrm{d}x \\
&\leqslant \frac{3-p}{2^{p-2}(p-1)}\|\nabla u-\nabla\theta\|_r^{p-1}\|h\|_{r/(r-p+1)} \\
&\leqslant \frac{3-p}{2^{p-2}(p-1)}C(n,p)|p-r|\|\nabla u-\nabla\theta\|_r^r.
\end{aligned}
$$

For the case $1 < p < 2$, $|I_2|$ and $|I_3|$ can also be estimated by (2.10) and (2.11). Combining (2.6), (2.13), (2.14), (2.10) and (2.11), we arrive at (2.12).

Let $\varepsilon_0 = 1/C(n,p)$. Then for $|p-r| < \varepsilon_0$ we have $C(n,p)|p-r| < 1$. Taking ε small enough, such that $C(n,p)|p-r|+\varepsilon < 1$, then the second term on the right-hand side of (2.12) can be absorbed by the left-hand side; thus we obtain

$$\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r \,\mathrm{d}x \leqslant C(n,p) \int_{\{|u-\theta|>L\}} |\nabla\theta|^r \,\mathrm{d}x. \tag{2.15}$$

Since $\theta \in W^{1,q}(\Omega)$, $q > r$, we obtain using the Hölder inequality

$$\begin{aligned} \int_{\{|u-\theta|>L\}} |\nabla\theta|^r \,\mathrm{d}x &\leqslant \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^q \,\mathrm{d}x\Big)^{r/q} |\{|u-\theta|>L\}|^{(q-r)/q} \\ &= \|\nabla\theta\|_q^r |\{|u-\theta|>L\}|^{(q-r)/q}. \end{aligned} \tag{2.16}$$

We now turn our attention back to the function $v \in W_0^{1,r}(\Omega)$. By the Sobolev embedding theorem, and using (2.2), we have

$$\begin{aligned} \Big(\int_\Omega |v|^{r^*} \,\mathrm{d}x\Big)^{1/r^*} &\leqslant C(n,r)\Big(\int_\Omega |\nabla v|^r \,\mathrm{d}x\Big)^{1/r} \\ &= C(n,r)\Big(\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r \,\mathrm{d}x\Big)^{1/r}. \end{aligned} \tag{2.17}$$

Since $|v| = (|u-\theta| - L)\cdot 1_{\{|u-\theta|>L\}}$, we have

$$\Big(\int_{\{|u-\theta|>L\}} (|u-\theta|-L)^{r^*} \,\mathrm{d}x\Big)^{1/r^*} = \Big(\int_\Omega |v|^{r^*} \,\mathrm{d}x\Big)^{1/r^*}, \tag{2.18}$$

and for $\widetilde{L} > L$,

$$\begin{aligned} (\widetilde{L}-L)^{r^*} |\{|u-\theta|>\widetilde{L}\}| &= \int_{\{|u-\theta|>L\}} (\widetilde{L}-L)^{r^*} \,\mathrm{d}x \\ &\leqslant \int_{\{|u-\theta|>\widetilde{L}\}} (|u-\theta|-L)^{r^*} \,\mathrm{d}x \leqslant \int_{\{|u-\theta|>L\}} (|u-\theta|-L)^{r^*} \,\mathrm{d}x. \end{aligned} \tag{2.19}$$

By collecting (2.15)–(2.19), we deduce that

$$((\widetilde{L}-L)^{r^*} |\{|u-\theta|>\widetilde{L}\}|)^{1/r^*} \leqslant C(n,r)\|\theta\|_q |\{|u-\theta|>L\}|^{1/r-1/q}.$$

Thus

$$|\{|u-\theta|>\widetilde{L}\}| \leqslant \frac{1}{(\widetilde{L}-L)^{r^*}} (C(n,r)\|\theta\|_q)^{r^*} |\{|u-\theta|>L\}|^{r^*(1/r-1/q)}. \tag{2.20}$$

Let $\phi(s) = |\{|u-\theta| > s\}|$, $\alpha = r^*$, $C = (C(n,r)\|\nabla\theta\|_q)^{r^*}$, $\beta = r^*(1/r - 1/q)$ and $s_0 > 0$. Then (2.20) becomes

$$\phi(\widetilde{L}) \leqslant \frac{C}{(\widetilde{L}-L)^\alpha}\phi(L)^\beta, \tag{2.21}$$

for $\widetilde{L} > L > 0$.

For the case $q < n$, one has $\beta < 1$. In this case, if $s \geqslant 1$, we get from Lemma 1.2 that

$$|\{|u-\theta| > s\}| \leqslant C(\alpha,\beta,s_0)s^{-t},$$

where $t = \alpha/(1-\beta) = q^*$. For $0 < s < 1$, one has

$$|\{|u-\theta| > s\}| \leqslant |\Omega| = |\Omega| s^{q^*} s^{-q^*} \leqslant |\Omega| s^{-q^*}.$$

Thus $u \in \theta + L^{q^*}_{\rm weak}(\Omega)$.

For the case $q = n$, one has $\beta = 1$. For any $\tau < \infty$, (2.21) implies

$$\phi(\widetilde{L}) \leqslant \frac{C}{(\widetilde{L}-L)^\alpha}\phi(L) = \frac{C}{(\widetilde{L}-L)^\alpha}\phi(L)^{1-\alpha/\tau}\phi(L)^{\alpha/\tau} \leqslant \frac{C|\Omega|^{\alpha/\tau}}{(\widetilde{L}-L)^\alpha}\phi(L)^{1-\alpha/\tau}.$$

As above, we derive

$$u \in \theta + L^\tau_{\rm weak}(\Omega).$$

For the case $q > n$, one has $\beta > 1$. Lemma 1.2 implies $\phi(d) = 0$ for some $d = d(\alpha,\beta,s_0,r,\|\nabla\theta\|_q)$. Thus $|\{|u-\theta| > d\}| = 0$, which means $u - \theta \leqslant d$ a.e. in Ω. Therefore

$$u \in \theta + L^\infty(\Omega),$$

completing the proof of Theorem 1.1. □

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References

[1] *A. Bensoussan, J. Frehse*: Regularity Results for Nonlinear Elliptic Systems and Applications. Applied Mathematical Sciences 151, Springer, Berlin, 2002.

[2] *S. Campanato*: Sistemi ellittici in forma divergenza. Regolarità all'interno. Pubblicazioni della Classe di Scienze: Quaderni, Scuola Normale Superiore Pisa, Pisa, 1980. (In Italian.)

[3] *H. Gao*: Regularity for solutions to anisotropic obstacle problems. Sci. China, Math. *57* (2014), 111–122.

[4] *H. Gao, Y. M. Chu*: Quasiregular Mappings and A-harmonic Equations. Science Press, Beijing, 2013. (In Chinese.)

[5] *H. Gao, Q. Huang*: Local regularity for solutions of anisotropic obstacle problems. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 4761–4765.
[6] *H. Gao, C. Liu, H. Tian*: Remarks on a paper by Leonetti and Siepe. J. Math. Anal. Appl. *401* (2013), 881–887.
[7] *M. Giaquinta*: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Annals of Mathematics Studies 105, Princeton University Press, Princeton, 1983.
[8] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Grundlehren der mathematischen Wissenschaften, Springer, Berlin, 1977.
[9] *L. Greco, T. Iwaniec, C. Sbordone*: Inverting the p-harmonic operator. Manuscr. Math. *92* (1997), 249–258.
[10] *E. Giusti*: Metodi diretti nel calcolo delle variazioni. Unione Matematica Italiana, Bologna, 1994. (In Italian.)
[11] *T. Iwaniec*: p-harmonic tensors and quasiregular mappings. Ann. Math. (2) *136* (1992), 589–624.
[12] *T. Iwaniec, G. Martin*: Geometric Function Theory and Nonlinear Analysis. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2001.
[13] *T. Iwaniec, L. Migliaccio, L. Nania, C. Sbordone*: Integrability and removability results for quasiregular mappings in high dimensions. Math. Scand. *75* (1994), 263–279.
[14] *T. Iwaniec, C. Sbordone*: Weak minima of variational integrals. J. Reine Angew. Math. *454* (1994), 143–161.
[15] *T. Iwaniec, C. Scott, B. Stroffolini*: Nonlinear Hodge theory on manifolds with boundary. Ann. Mat. Pura Appl. (4) *177* (1999), 37–115.
[16] *A. Kufner, O. John, S. Fučík*: Function Spaces. Monographs and Textsbooks on Mechanics of Solids and Fluids; Mechanics: Analysis, Noordhoff International Publishing, Leyden; Academia, Praha, 1977.
[17] *O. A. Ladyzhenskaya, N. N. Ural'tseva*: Linear and Quasilinear Elliptic Equations. Mathematics in Science and Engineering, Academic Press, New York, 1968.
[18] *F. Leonetti, F. Siepe*: Global integrability for minimizers of anisotropic functionals. Manuscr. Math. *144* (2014), 91–98.
[19] *F. Leonetti, F. Siepe*: Integrability for solutions to some anisotropic elliptic equations. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 2867–2873.
[20] *P. Lindqvist*: Notes on the p-Laplace Equation. Report. University of Jyväskylä Department of Mathematics and Statistics 102, University of Jyväskylä, Jyväskylä, 2006.
[21] *N. G. Meyers, A. Elcrat*: Some results on regularity for solutions of nonlinear elliptic systems and quasi-regular functions. Duke Math. J. *42* (1975), 121–136.
[22] *C. B. Morrey,* Jr.: Multiple Integrals in the Calculus of Variations. Die Grundlehren der mathematischen Wissenschaften 130, Springer, New York, 1966.
[23] *G. Stampacchia*: Èquations elliptiques du second ordre à coefficients discontinus. Séminaire de mathématiques supérieures 16 (été 1965), Les Presses de l'Université de Montréal, Montreal, Que., 1966. (In French.)

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