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Normability of gamma spaces

FILIP SOUDSKÝ

Abstract. We give a full characterization of normability of Lorentz spaces $\Gamma^p_w$. This result is in fact known since it can be derived from Kamińska A., Maligranda L., On Lorentz spaces, Israel J. Funct. Anal. 140 (2004), 285–318. In this paper we present an alternative and more direct proof.

Keywords: Lorentz space; weight; normability
Classification: 46E30

1. Introduction and the main result

In this paper we present a complete characterization of those parameters $p$ and $w$, where $p \in (0, 1)$ and $w$ is a nonnegative measurable function (weight), for which the corresponding classical Lorentz space $\Gamma^p_w$ (the precise definition is given below) is normable. By this we mean that the functional $\| \cdot \|_{\Gamma^p_w}$ is equivalent to a norm. We in fact prove two characterizations, quite different in nature. One of them is a certain integrability condition on the weight while the other states that the corresponding space coincides with the space $L^1 + L^\infty$. The proofs are based on a combination of discretization and weighted norm inequalities.

This result is in fact known as it can be derived from Theorem 2.1 in [4] characterizing isomorphic copies of $l^p$ in the space $\Gamma$. We present here a new elementary proof which does not go beyond the scope of the classical Lorentz spaces.

We recall that classical Lorentz spaces of type $\Lambda$ were first introduced by Lorentz in 1951 ([5]) while their modification of type $\Gamma$ was developed first in 1990 by Sawyer ([6]) in connection with their crucial duality properties. These spaces proved to be extremely useful for a wide range of applications and have been studied ever since by many authors (e.g., [1], [3], [8], [7]). Normability of spaces of type $\Lambda$ has been characterized long time ago (see [6] and [2]).

The result is a contribution to the long-standing research of functional properties such as linearity, (quasi)-normability etc., of classical Lorentz spaces of various types (see, e.g. [5], [1], [6], [3]).

During the whole paper, the underlying measure space $(\mathcal{R}, \mu)$ is always nonatomic and $\sigma$-finite with $\mu(\mathcal{R}) = \infty$. We shall also use the symbol $\mathcal{M}(\mathcal{R})$ for

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the set of all real-valued measurable functions defined on \( \mathcal{R} \). For a measurable, real-valued function \( f \) on such a space, a non-increasing rearrangement of \( f \) is defined by

\[
f^*(t) := \inf \{ s : \mu(\{|f| > s\}) \leq t \},
\]

while the maximal function of \( f \) is given by

\[
f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)ds.
\]

Throughout all of this paper the expression weight will always be used for positive, measurable function defined on \((0, \infty)\).

**Definition 1.** Let \( 0 < p < \infty \) and let \( w \) be a weight. Set

\[
\Lambda^p_w := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Lambda^p_w} := \left( \int_0^\infty f^*(s)^p w(s)ds \right)^{\frac{1}{p}} < \infty \right\}
\]

and

\[
\Gamma^p_w := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Gamma^p_w} := \left( \int_0^\infty f^{**}(s)^p w(s)ds \right)^{\frac{1}{p}} < \infty \right\}.
\]

Furthermore in the following text we shall use notation \( X := \Gamma^p_w \). In order to avoid the technical difficulties, we shall assume that \( w \) is locally integrable and

\[
\int_0^\infty w(s)s^{-p}ds < \infty,
\]

for all \( a > 0 \). We may also assume this without loss of generality, since if \( w \notin L^1_{\text{loc}} \) or (1.1) is not satisfied, then \( \Gamma^p_w = \{0\} \). In the following text function \( W \) will be defined as

\[
W(t) := \int_0^t w(s)ds.
\]

We recall that the space \( L^1 + L^\infty \) consists of all functions \( f \in \mathcal{M}(\mathcal{R}) \) for which there exists a decomposition \( f = g + h \) such that \( g \in L^1 \) and \( f \in L^\infty \), and it is equipped with the norm

\[
\|f\|_{L^1 + L^\infty} := \int_0^1 f^*(s)ds.
\]

Let us also recall the definition of norm in weighted Lebesgue space on \((0, \infty)\) which shall be also used in the proof, namely

\[
\|f\|_{L^p_w} := \left( \int_0^\infty |f(s)|^p w(s)ds \right)^{\frac{1}{p}}.
\]
Remark 1. The equivalence of condition (ii) and (iii) in the following theorem can be obtained from [4, Proposition 1.4], while the equivalence of (i) and (iii) from [4, Theorem 2.1].

Theorem 1. Let $0 < p < 1$ and let $w$ be a weight. Then the following conditions are equivalent.

(i) The space $\Gamma^p_w$ is normable.
(ii) Both $w(s)$ and $w(s)s^{-p}$ are integrable on $(0, \infty)$.
(iii) The identity

$$\Gamma^p_w = L^1 + L^\infty$$

holds in the sense of equivalent norms.

2. Proof of Theorem 1

Lemma 1. Let $X$ be a linear vector space. Let $\sigma : X \to [0, \infty)$ be a positively homogenous functional. Then the following conditions are equivalent:

(i) $\sigma$ is equivalent to a norm;
(ii) there exists a constant $C$, independent on $N$, such that

$$\sigma\left(\sum_{k=1}^N f_k\right) \leq C \sum_{k=1}^N \sigma(f_k),$$

for all $f_k \in X$.

Proof of Lemma 1: First let us suppose that (i) holds. Denote the equivalent norm by $\rho$. Then we have

$$\sigma\left(\sum_{k=1}^N f_k\right) \leq C \rho\left(\sum_{k=1}^N f_k\right) \leq C \sum_{k=1}^N \rho(f_k) \leq C \sum_{k=1}^N \sigma(f_k).$$

Now, suppose that (2) holds. Denote

$$\rho(f) := \inf \left(\sum_{k=1}^N \sigma(f_k)\right),$$

where the infimum on the right-hand side is taken over all finite decompositions of $f$, i.e.,

$$\sum_{k=1}^N f_k = f.$$  \tag{2.1}

Then obviously

$$\rho(f) \leq \sigma(f),$$
for all \( f \in X \). On the other hand, for all \( f_k \) satisfying (2.1) we have

\[
C \left( \sum_{k=1}^{N} \sigma(f_k) \right) \geq \sigma(f).
\]

Passing to the infimum on the left-hand side gives

\[
C \varrho(f) \geq \sigma(f).
\]

Now, take \( f_1, f_2 \in X \). Let

\[
\sum_{k=1}^{N_1} f^1_k = f_1, \quad \sum_{k=1}^{N_2} f^2_k = f_2,
\]

then

\[
\varrho(f_1 + f_2) \leq \sum_{k=1}^{N_1} \sigma(f^1_k) + \sum_{k=1}^{N_2} \sigma(f^2_k).
\]

By passing to the infimum on the right-hand side we obtain the triangle inequality for \( \varrho \).

**Proof of Theorem 1:** Let us first prove that (i) implies (ii). We shall give an indirect proof. Suppose that (ii) is not true. Then either

\[
\int_0^\infty w(s)ds = \infty \tag{2.2}
\]
or

\[
\int_0^\infty s^{-p}w(s)ds = \infty \tag{2.3}
\]

First, note that if \( w \in \mathcal{B}_p \) then \( \| \cdot \|_X \approx \| \cdot \|_{\Lambda_w} \). Since the functional \( \| \cdot \|_{\Lambda_w} \) is not normable for \( p < 1 \) (as was shown in [2]), neither is \( \| \cdot \|_X \). This allows us to focus on the case when \( w \notin \mathcal{B}_p \). Therefore we may suppose that there exists a sequence \( \{a_n\}_{n=1}^\infty \) such that

\[
a_n^p \int_{a_n}^\infty w(s)s^{-p}ds \geq 2^n W(a_n). \tag{2.4}
\]

Now let us define

\[
H(t) := \frac{t^p \int_t^\infty w(s)s^{-p}ds}{W(t)}. \tag{2.5}
\]

Since \( H \) is continuous on \((0, \infty)\) and therefore bounded on every \([c, d] \subset (0, \infty)\), we may without loss of generality (by choosing appropriate sub-sequence) assume that either \( a_n \downarrow 0 \) or \( a_n \uparrow \infty \). Now, let us consider three cases:

1. \( a_n \uparrow \infty \);
(2) $a_n \downarrow 0$ and (2.3) holds;
(3) $a_n \downarrow 0$, (2.2) holds and $\sup_{t>1} H(t) < \infty$ (We can assume this otherwise it is in fact Case 1).

Case 1. Now, if $a_n \uparrow \infty$, we may again without loss of generality suppose that

\begin{equation}
\int_{a_n+1}^{\infty} w(s)s^{-p}ds \leq \frac{1}{2} \int_{a_n}^{\infty} w(s)s^{-p}ds.
\end{equation}

Fix $N \in \mathbb{N}$. Pick $\{f_k\}_{k=1}^{N}$, such that

1. $\text{supp}(f_{k+1}) \subset \text{supp}(f_k)$,
2. $f_k^*(s) = q_k \chi(0,a_k)$, where

$$q_k = \left( a_k^p \int_{a_k}^{\infty} w(s)s^{-p}ds \right)^{-\frac{1}{p}}.$$

Then (2.6) gives

\begin{equation}
\int_{a_n}^{\infty} w(s)s^{-p}ds \leq 2 \int_{a_n}^{a_{n+1}} w(s)s^{-p}ds.
\end{equation}

Note that

$$f_k^{**}(s) = q_k \left( \chi(0,a_k) + a_k^{-1} \chi(a_k,\infty) \right).$$

Now, by (2.4) we have

\begin{equation}
\|f_k\|_X = q_k \left( W(a_k) + a_k^p \int_{a_k}^{\infty} w(s)s^{-p}ds \right)^{\frac{1}{p}} \\
\leq q_k \left( 2a_k^p \int_{a_k}^{\infty} w(s)s^{-p}ds \right)^{\frac{1}{p}} = 2^{\frac{1}{p}}.
\end{equation}

Calculate

$$\left\| \sum_{k=1}^{N} f_k \right\|_X \geq \left\| \sum_{k=1}^{N} f_k^{**} \chi(a_k,a_{k+1}) \right\|_{L_w^p}$$

$$= \left( \sum_{k=1}^{N} q_k^p a_k^p \int_{a_k}^{a_{k+1}} w(s)s^{-p}ds \right)^{\frac{1}{p}}$$

$$\geq 2^{\frac{1}{p}} \left( \sum_{k=1}^{N} q_k^p a_k^p \int_{a_k}^{\infty} w(s)s^{-p}ds \right)^{\frac{1}{p}}$$

$$= 2^{\frac{1}{p}} \left( \sum_{k=1}^{N} 1 \right)^{\frac{1}{p}} \approx N^{\frac{1}{p}}.$$
The third inequality follows from (2.7), while the next one from (2.4). Therefore by Lemma 1 we obtain that \( \| \cdot \|_X \) cannot be equivalently normed.

**Case 2.** Suppose (2.3) holds. If \( a_n \downarrow 0 \), define \( a_0 = \infty \). We may without loss of generality suppose that

\[
(2.9) \quad \int_{a_{n+1}}^{\infty} w(s)s^{-p}ds \geq 2 \int_{a_n}^{\infty} w(s)s^{-p}ds.
\]

Fix \( N \in \mathbb{N} \). Now, let us pick \( \{f_k\}_{k=1}^N \) with the following properties

1. \( \text{supp}(f_{k+1}) \subset \text{supp}(f_k) \),
2. \( f_k^* = q_k \chi_{(0,a_k)} \), where

\[
q_k = \left( a_k^p \int_{a_k}^{\infty} w(s)s^{-p}ds \right)^{-\frac{1}{p}}.
\]

The same calculation as in (2.8) gives

\[
\|f_k\| \leq 2^{\frac{1}{p}}.
\]

Now, by (2.9), we have

\[
(2.10) \quad \int_{a_{n+1}}^{a_n} w(s)s^{-p}ds \geq \frac{1}{2} \int_{a_{n+1}}^{a_n} w(s)s^{-p}ds.
\]

Calculate

\[
\left\| \sum_{k=1}^{N} f_k \right\|_X \geq \left\| \sum_{k=1}^{N-1} f_{k+1}^* \chi_{(a_{k+1},a_k)} \right\|_{L_p^w} \approx \left( \sum_{k=1}^{N-1} q_{k+1}^p a_{k+1}^p \int_{a_{k+1}}^{\infty} w(s)s^{-p}ds \right)^{-\frac{1}{p}} \approx N^{\frac{1}{p}},
\]

where the third inequality follows from (2.10). Therefore, by Lemma 1, the functional is not normable.

**Case 3.** Now, suppose that the condition (2.2) holds. Again, if we can choose \( \{a_n\}_{n=1}^\infty \) satisfying (2.4) and such that \( a_n \uparrow \infty \), we may use the same calculation as in the previous one. Now if there is no such sequence, then the function \( H(t) \)
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(where $H$ is defined in (2.5)) is bounded on $[1, \infty)$. Set

$$C := 1 + \sup_{t > 1} H(t).$$

Fix $N \in \mathbb{N}$. Since $w$ is not in $L^1$, we may choose $\{a_k\}_{k=1}^\infty$ such that

$$W(a_{k+1}) \geq 2W(a_k),$$

and $a_1 > 1$. Observe that

$$\int_{a_{k-1}}^{a_k} w(s) ds \geq \frac{1}{2} W(a_k),$$

for $k = 1, \ldots, N$. Find a sequence $\{f_k\}_{k=1}^N$ such that

1. $\text{supp}(f_k) \subset \text{supp}(f_{k+1}),$
2. $f_k^*(s) = b_k \chi_{(0,a_k)}$, where $b_k = W^{-\frac{1}{p}}(a_k)$.

For technical reasons, set $a_0 := 0$. We have

$$\|f_k\|_X = W^{-\frac{1}{p}}(a_k) \left( W(a_k) + a_k \int_{a_k}^{\infty} w(s)s^{-p} ds \right)^{\frac{1}{p}} \leq W^{-\frac{1}{p}}(a_k) \left[ W(a_k)(1 + \sup_{t > 1} H(t)) \right]^{\frac{1}{p}} = C^{\frac{1}{p}}.$$

Calculate

$$\left\| \sum_{k=1}^N f_k \right\|_X \geq \left\| \sum_{k=1}^N \chi_{(a_{k-1},a_k)} b_k \right\|_{L^p_w} \geq \left( \sum_{k=1}^N b_k^p \int_{a_k}^{a_{k+1}} w(s) ds \right)^{\frac{1}{p}} \geq 2^{-\frac{1}{p}} \left( \sum_{k=1}^N b_k^p W(a_k) \right)^{\frac{1}{p}} = 2^{-\frac{1}{p}} \left( \sum_{k=1}^N 1 \right)^{\frac{1}{p}} = N^{\frac{1}{p}}.$$

The third inequality follows from (2.12).

Now, let us prove that (ii) implies (iii). We shall prove that if (ii) is satisfied then

$$B \int_0^1 f^*(s) ds \leq \|f\|_X \leq A \int_0^1 f^*(s) ds,$$

(2.13)
where

\[ A := \left[ \int_0^1 w(s)s^{-p} ds \left( 1 + \frac{\int_1^\infty w(s) ds}{\int_1^1 w(s) ds} \right) \right]^{\frac{1}{p}} \]

and

\[ B := \left( \int_0^1 w(s) ds \right)^{-\frac{1}{p}}. \]

We have

\[ \|f\|_X^p = \int_0^1 f^{**}(s)^p w(s) ds + \int_1^\infty f^{**}(s)^p w(s) ds =: I+II. \]

Let us first estimate the second term by the first one

\[ \int_1^\infty f^{**}(s)^p w(s) ds \leq f^{**}(1)^p \int_0^1 w(s) ds \left( \frac{\int_1^\infty w(s) ds}{\int_0^1 w(s) ds} \right) \leq \left( \frac{\int_1^\infty w(s) ds}{\int_0^1 w(s) ds} \right) \int_0^1 f^{**}(s)^p w(s) ds. \]

Now estimate

\[ \int_0^1 f^{**}(s)^p w(s) ds = \int_0^1 w(s)s^{-p} \left( \int_0^s f^*(z) dz \right)^p ds \leq \int_0^1 w(s)s^{-p} ds \left( \int_0^1 f^*(z) dz \right)^p. \]

Due to this two estimates we have

\[ \|f\|_X^p \leq A^p \left( \int_0^1 f^*(s) ds \right)^p. \]

On the other hand note that

\[ \left( \int_0^1 f^*(s) ds \right)^p = f^{**}(1)^p = \left( \int_0^1 w(s) ds \right)^{-1} f^{**}(1)^p \int_0^1 w(s) ds \leq B^p \int_0^1 f^{**}(s)^p w(s) ds \leq B^p \|f\|_X^p. \]

Therefore the desired equivalence (2.13) holds. \(\square\)

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