Notes on strongly Whyburn spaces

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Abstract. We introduce the notion of a strongly Whyburn space, and show that a space $X$ is strongly Whyburn if and only if $X \times (\omega + 1)$ is Whyburn. We also show that if $X \times Y$ is Whyburn for any Whyburn space $Y$, then $X$ is discrete.

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1. Introduction

Throughout this paper, all spaces are assumed to be $T_2$, unless a specific separation axiom is indicated.

A space $X$ is said to be Fréchet-Urysohn if $A \subset X$ and $p \in \overline{A}$ imply that there is a sequence $\{p_n : n \in \omega\} \subset A$ converging to $p$. A space $X$ is said to be strongly Fréchet-Urysohn [14] (or, countably bi-sequential [9]) if for a decreasing sequence $\{A_n : n \in \omega\}$ of subsets of $X$, $p \in \bigcap \{\overline{A_n} : n \in \omega\}$ implies that there are points $p_n \in A_n$ converging to $p$. Every strongly Fréchet-Urysohn space is Fréchet-Urysohn. Michael [9, Proposition 4.D.5] showed that a space $X$ is strongly Fréchet-Urysohn if and only if $X \times \mathbb{I}$ is Fréchet-Urysohn, where $\mathbb{I}$ is the closed unit interval. In this result, $\mathbb{I}$ can be replaced by the convergent sequence $\omega + 1$: see the proof of [9, Proposition 4.D.5].

According to recent literature (e.g., [4], [12]), a space $X$ is said to be Whyburn if $A \subset X$ and $p \in \overline{A} \setminus A$ imply that there is a subset $B \subset A$ such that $\overline{B} = \{p\} \cup B$. Every Fréchet-Urysohn space is Whyburn, because the convergent sequence is closed in a $T_2$-space. This notion was considered in Whyburn [16], and was called property $H$. Whyburn showed in [16, Corollary 1] that every quotient map onto a $T_1$-space $Y$ having property $H$ is pseudo-open (=hereditarily quotient). Later, introducing the notion of an accessibility space [17] which is weaker than property $H$, he sharpened this result. He showed that for a $T_1$-space $Y$, every quotient map onto $Y$ is pseudo-open if and only if $Y$ is an accessibility space. A space having property $H$ is always an accessibility space, and conversely a regular accessibility space has property $H$. A Whyburn space is sometimes called an $AP$-space according to [13].
Even if a space \( X \) is Whyburn, \( X \times (\omega + 1) \) need not be Whyburn. Such examples are given in Bella and Yaschenko [5]. Aull [3, Theorem 11] showed that a \( T_2 \)-space \( X \) is a \( k \)-space and an accessibility space if and only if it is Fréchet-Urysohn.\(^1\) Hence we have:

**Proposition 1.1.** For a \( k \)-space \( X \), \( X \times (\omega + 1) \) is Whyburn if and only if \( X \) is strongly Fréchet-Urysohn.

**Proof:** Assume that \( X \times (\omega + 1) \) is Whyburn. Since \( X \times (\omega + 1) \) is a \( k \)-space [6, Theorem 3.3.27], by Aull’s result, \( X \times (\omega + 1) \) is Fréchet-Urysohn. Thus \( X \) is strongly Fréchet-Urysohn by Michael’s result. The converse immediately follows from Michael’s result mentioned above. \( \square \)

Let \( S_\omega \) be the space obtained by identifying the limits of countably many convergent sequences. This space is Fréchet-Urysohn (hence, a \( k \)-space), but not strongly Fréchet-Urysohn. Therefore, \( S_\omega \times (\omega + 1) \) is not Whyburn by the preceding proposition. One purpose of this paper is to make clear when \( X \times (\omega + 1) \) is Whyburn. Another topic is when \( X \times Y \) is Whyburn for any Whyburn space \( Y \).

### 2. Strongly Whyburn spaces

**Definition 2.1.** A space \( X \) is strongly Whyburn if for any sequence \( \{A_n : n \in \omega\} \) of subsets in \( X \) and a point \( p \in X \setminus \bigcup \{A_n : n \in \omega\} \), \( p \in \bigcap \{\bigcup_{m \geq n} A_m : n \in \omega\} \) implies that there is a sequence \( \{B_n : n \in \omega\} \) of closed subsets in \( X \) such that \( B_n \subset A_n \) and \( \{p\} = \bigcap \{\bigcup_{m \geq n} B_m : n \in \omega\} \).

In the definition above, some \( B_n \) may be empty, and note that the condition \( \{p\} = \bigcap \{\bigcup_{m \geq n} B_m : n \in \omega\} \) holds if and only if (a) the closed family \( \{B_n : n \in \omega\} \) in \( X \) is locally finite at any point in \( X \setminus \{p\} \), and (b) \( p \in \bigcup \{B_n : n \in \omega\} \) holds. If all \( A_n \)'s are identical with a set \( A \), there is an \( F_{\sigma} \)-subset \( F \subset A \) in \( X \) such that \( F = \{p\} \cup F \). Therefore, every strongly Whyburn space is Whyburn. Moreover, we can easily observe that every strongly Fréchet-Urysohn space is strongly Whyburn. Thus we have the implications below.

\[
\text{strongly Fréchet-Urysohn} \quad \rightarrow \quad \text{Fréchet-Urysohn} \\
\downarrow \\
\text{strongly Whyburn} \quad \rightarrow \quad \text{Whyburn}
\]

**Theorem 2.2.** For a space \( X \), the following are equivalent:

1. \( X \) is strongly Whyburn,
2. \( X \times (\omega + 1) \) is Whyburn.

**Proof:** (1)→(2) We have only to check the Whyburn property at a point \( (p, \omega) \in X \times (\omega + 1) \). Let \( A \subset X \times (\omega + 1) \) and assume \( (p, \omega) \in \overline{A} \setminus A \). If \( (p, \omega) \in A \cap (X \times \{\omega\}) \), using the Whyburn property of \( X \), we can take a subset \( B \subset A \) which...

\(^1\)In particular, every compact \( T_2 \) Whyburn space is Fréchet-Urysohn. This fact was given in [1, Proposition 1 and Theorem 1] and [8, Theorem 1].
The condition \((p, \omega) \in \overline{A}\) implies \(p \in \bigcap \{\overline{U_{m \geq n}} A_m : n \in \omega\}\), so there are closed subsets \(B_n\) in \(X\) such that \(B_n \subset A_n\) and \(\{p\} = \bigcap \{\overline{U_{m \geq n}} B_m : n \in \omega\}\). Let \(B = \bigcup \{B_n \times \{n\} : n \in \omega\}\). The condition \(p \in \bigcap \{\overline{U_{m \geq n}} B_m : n \in \omega\}\) obviously implies \((p, \omega) \not\in \overline{B}\). We observe that \(\{p, \omega\}\) is closed. Let \(q \in X \setminus \{p\}\). By \(\{p\} = \bigcap \{\overline{U_{m \geq n}} B_m : n \in \omega\}\), there are a neighborhood \(U\) of \(q\) and some \(n \in \omega\) such that \(U \cap (\bigcup \{B_m : m \geq n\}) = \emptyset\). Then we have \(U \times [n, \omega] \cap B = \emptyset\). Thus \((q, \omega) \not\in \overline{B}\).

(2)\to(1) Assume that \(A_n \subset X\), \(p \in X \setminus \bigcup \{A_n : n \in \omega\}\) and \(p \in \bigcap \{\overline{U_{m \geq n}} A_m : n \in \omega\}\). Let \(A = \bigcup \{A_n \times \{n\} : n \in \omega\}\). Then obviously \((p, \omega) \in \overline{A}\). Since \(X \times (\omega + 1)\) is Whyburn, there is a subset \(B \subset A\) such that \(\overline{B} = \{(p, \omega)\} \cup B\). We can put \(B = \bigcup \{B_n \times \{n\} : n \in \omega\}\) for some \(B_n \subset A_n\). Then each \(B_n\) is closed in \(X\), and the condition \((p, \omega) \in \overline{B}\) implies \(p \in \bigcap \{\overline{U_{m \geq n}} B_m : n \in \omega\}\). Let \(q \in X \setminus \{p\}\). By the condition \((q, \omega) \not\in \overline{B}\), there are a neighborhood \(U\) of \(q\) and some \(n \in \omega\) such that \((U \times [n, \omega]) \cap B = \emptyset\). Hence we have \(q \not\in \bigcup \{B_m : m \geq n\}\). Consequently we have \(\{p\} = \bigcap \{\overline{U_{m \geq n}} B_m : n \in \omega\}\). \(\square\)

**Corollary 2.3.** For a \(k\)-space \(X\), \(X\) is strongly Whyburn if and only if it is strongly Fréchet-Urysohn.

Unfortunately, the author does not know if for a strongly Whyburn space \(X\), \(X \times \mathbb{I}\) is Whyburn. A space \(X\) is said to have \textit{countable fan-tightness} [2] if whenever \(A_n \subset X\) and \(p \in \bigcap \{\overline{A_n} : n \in \omega\}\), there are finite subsets \(F_n \subset A_n\) such that \(p \in \bigcup \{F_n : n \in \omega\}\). It is known [5, Corollary 3.4] that if a regular space \(X\) has countable fan-tightness and every point of \(X\) is a \(G_\delta\)-set, then \(X\) is Whyburn. Note that if a space \(X\) has countable fan-tightness, so does \(X \times Y\) for any first-countable space \(Y\). Therefore we can say that if a regular space \(X\) has countable fan-tightness and every point of \(X\) is a \(G_\delta\)-set, then \(X \times Y\) is Whyburn for any first-countable space \(Y\) (in particular, \(X\) is strongly Whyburn). A space is said to be \textit{submaximal} if every dense subset is open (equivalently, every subset with the empty interior is closed and discrete). Every regular submaximal space is Whyburn [5, Proposition 1.3], but if \(X\) is a countable dense-in-itself submaximal space, \(X \times (\omega + 1)\) is not Whyburn [5, Theorem 2.3]. Hence, a countable submaximal dense-in-itself space cannot be strongly Whyburn. It looks interesting to give a direct proof of this fact, using the definition of the strong Whyburn property. Our idea owes to Bella and Yaschenko [5].

**Proposition 2.4.** If a space \(X\) is countable, dense-in-itself and submaximal, then it is not strongly Whyburn.

**Proof:** Fix a point \(p \in X\), and let \(X \setminus \{p\} = \{x_n : n \in \omega\}\). Let \(A_n = \{x_n\}\) for each \(n \in \omega\). Then obviously \(p \in \bigcap \{\overline{U_{m \geq n}} A_m : n \in \omega\}\). Assume that there is
a sequence \( \{B_n : n \in \omega\} \) of closed subsets in \( X \) such that \( B_n \subset A_n \) and \( \{p\} = \bigcap \{\bigcup_{m \geq n} B_m : n \in \omega\} \). Then \( B_n = \emptyset \), or \( B_n = \{x_n\} \). Let \( I = \{n \in \omega : B_n \neq \emptyset\} \).

Since the family \( \{B_n : n \in I\} \) is locally finite at each point in \( X \setminus \{p\} \), the set \( C = \{x_n : n \in I\} \) is a discrete subspace of \( X \), so \( C \) has the empty interior. Hence \( C \) is closed in \( X \). This is a contradiction, because of \( p \in \overline{C} \). \( \square \)

We give one application of Theorem 2.2. For a Tychonoff space \( X \), we denote by \( C_p(X) \) the space of all real-valued continuous functions with the topology of pointwise convergence.

**Lemma 2.5** ([11, Theorem 2.10]). If \( X \times Y \) contains a homeomorphic copy of \( S_\omega \) and \( X \) is first-countable, then \( Y \) contains a homeomorphic copy of \( S_\omega \).

**Proposition 2.6.** If \( C_p(X) \) is Whyburn, then \( S_\omega \) cannot be embedded into \( C_p(X) \).

**Proof:** Fix a point \( x \in X \). Note that \( C_p(X) \) is homeomorphic to \( C_p(X, x) \times \mathbb{R} \), where \( C_p(X, x) = \{f \in C_p(X) : f(x) = 0\} \) and \( \mathbb{R} \) is the real line. Since \( C_p(X) \) is Whyburn, \( C_p(X, x) \times (\omega + 1) \) is also Whyburn, so \( C_p(X, x) \) is strongly Whyburn. If \( C_p(X) \) has a homeomorphic copy of \( S_\omega \), by the preceding lemma, \( C_p(X, x) \) has a homeomorphic copy of \( S_\omega \). This is a contradiction. \( \square \)

The Whyburn property for \( C_p(X) \) were investigated in [5], [10] and [15]. So far the author knows, there is no precise characterization (in terms of \( X \)) for \( C_p(X) \) to be Whyburn.

Let \( \mathcal{F} \) be a filter on a set. Then \( \mathcal{F} \) is said to be free if \( \bigcap \mathcal{F} = \emptyset \) holds, and have the countable intersection property if for each countable subfamily \( \mathcal{G} \subset \mathcal{F} \), \( \bigcap \mathcal{G} \neq \emptyset \) holds. If \( \mathcal{F} \) is an ultrafilter, then \( \bigcap \mathcal{G} \neq \emptyset \) is equivalent to \( \bigcap \mathcal{G} \in \mathcal{F} \).

For the discrete space \( D(\kappa) \) of cardinality \( \kappa \geq \omega \), let \( p \in \beta D(\kappa) \setminus D(\kappa) \), where \( \beta D(\kappa) \) is the Stone-Čech compactification of \( D(\kappa) \) (i.e., \( p \) is a free ultrafilter on \( D(\kappa) \)). Let \( X(p) = \{p\} \cup D(\kappa) \) be the subspace of \( \beta D(\kappa) \). We examine whether \( X(p) \) is strongly Whyburn.

A space is said to be a \( P \)-space if every \( G_\delta \)-subset is open. There are many non-discrete Whyburn \( P \)-spaces, for example, consider the one-point Lindelöfication of the discrete space of cardinality \( \omega_1 \). In contrast with this fact, we have the following.

**Lemma 2.7.** Every strongly Whyburn \( P \)-space is discrete.

**Proof:** Let \( X \) be a strongly Whyburn space and assume that there is a non-isolated point \( p \in X \). Then \( p \notin X \setminus \{p\} \), so there is an \( F_\sigma \)-subset \( F \subset X \setminus \{p\} \) in \( X \) such that \( p \notin \overline{F} \). This implies that \( X \) is not a \( P \)-space. \( \square \)

**Theorem 2.8.** Let \( p \in \beta D(\kappa) \setminus D(\kappa) \). Then the following assertions are equivalent:

1. \( X(p) \) is strongly Whyburn,
2. \( p \) does not have the countable intersection property,
3. \( X(p) \times Y \) is Whyburn for any first-countable space \( Y \).
Proof: (1)→(2) If \( p \) has the countable intersection property, then \( X(p) \) is obviously a \( P \)-space. By Lemma 2.7, \( X(p) \) is not strongly Whyburn.

(3)→(1) is trivial.

(2)→(3) We have only to check the Whyburn property at \((p, y) \in X(p) \times Y\). Suppose \((p, y) \in \overline{A} \setminus A\) for some subset \( A \subset X(p) \times Y\). Without loss of generality, we may assume \( A \subset D(\kappa) \times Y\). We put \( A = \bigcup \{\{\alpha\} \times A_\alpha : \alpha < \kappa\} \), where \( A_\alpha \subset Y\) and some \( A_\alpha \) may be empty. Let \{\( U_n : n \in \omega \)\} be an open neighborhood base at \( y \) such that \( U_n \supset U_{n+1}\). For each \( n \in \omega \), we put \( P_n = \{\alpha < \kappa : A_\alpha \cap U_n \neq \emptyset\} \). Then \( P_n \supset P_{n+1} \), and \( P_n \in P \) by the condition \((p, y) \in \overline{A}\). Using (2), we can take subsets \( Q_n \subset P_n \) such that \( Q_n \in p \), \( Q_n \supset Q_{n+1} \) and \( \bigcap\{Q_n : n \in \omega\} = \emptyset \). For each \( n \in \omega \) and \( \alpha \in Q_n \setminus Q_{n+1} \), take a point \( y_{n, \alpha} \in U_n \cap A_\alpha \). We define a subset \( B \subset A \) as follows:

\[
B = \{(\alpha, y_{n, \alpha}) : n \in \omega, \alpha \in Q_n \setminus Q_{n+1}\}.
\]

First we observe \((p, y) \in \overline{B}\). Let \( N \) be a neighborhood of \((p, y)\) in \( X(p) \times Y\). Take \( R \in p \) and \( n \in \omega \) satisfying \((\{p\} \cup R) \times U_n \subset N\). Since \( R \cap Q_n \neq \emptyset \) and \( \bigcap\{Q_k : k \in \omega\} = \emptyset \), there is some \( k \geq n \) such that \( R \cap (Q_k \setminus Q_{k+1}) \neq \emptyset \). If \( \alpha \in R \cap (Q_k \setminus Q_{k+1}) \), then \((\alpha, y_{k, \alpha}) \in B\). If \( \alpha \notin Q_{k+1} \), then \( \alpha < \kappa \) and \((\alpha, y_{k, \alpha}) \in B\). Thus we have \((p, y) \in \overline{B}\). Next we observe \( \overline{B} = B \cup \{(p, y)\} \). For a point \( y' \in Y \setminus \{y\} \), we see \((p, y') \notin \overline{B}\). Since \( Y \) is \( T_2 \), there are an open neighborhood \( V \) of \( y' \) and \( n \in \omega \) such that \( V \cap U_n = \emptyset \). We consider the open neighborhood \((\{p\} \cup Q_n) \times V\) of \((p, y')\). Suppose \((\{p\} \cup Q_n) \times V) \cap B \neq \emptyset \). Then there are some \( k \in \omega \) and \( \alpha \in Q_k \setminus Q_{k+1} \) such that \((\alpha, y_{k, \alpha}) \in (\{p\} \cup Q_n) \times V\). The conditions \( \alpha \notin Q_{k+1} \) and \( \alpha \in Q_n \) imply \( n \leq k \). On the other hand, \( y_{k, \alpha} \in U_k \) and \( y_{k, \alpha} \notin U_n \) (because, \( y_{k, \alpha} \in V\)) imply \( k < n \). This is a contradiction. Thus we have \((p, y') \notin \overline{B}\). Therefore \( X(p) \times Y \) is Whyburn. \( \square \)

We refer to [7, Chapter 12] on measurable and non-measurable cardinals. What we have to recall is that for a set \( X \), every ultrafilter \( p \) on \( X \) with the countable intersection property satisfies \( \bigcap p \neq \emptyset \) if and only if the cardinality of \( X \) is non-measurable [7, 12.2]. By Theorem 2.8, we have the following.

Corollary 2.9. The following assertions hold.

1. If \( m \) is a measurable cardinal and \( p \) is a free ultrafilter on \( D(m) \) with the countable intersection property, then \( X(p) \) is not strongly Whyburn.
2. If \( n \) is a non-measurable cardinal and \( p \) is a free ultrafilter on \( D(n) \), then \( X(p) \) is strongly Whyburn.

3. \( \kappa \)-Whyburn spaces

Finally, in this section, we investigate when \( X \times Y \) is Whyburn for any Whyburn space \( Y \). If \( X \times Y \) is Fréchet-Urysohn for any Fréchet-Urysohn space \( Y \), then \( Y \) is discrete. Because, if \( X \) is not discrete, then \( X \) contains the convergent sequence \( \omega + 1 \), so the product \( X \times S_\omega \) is not Fréchet-Urysohn.
Temporarily, for an infinite cardinal $\kappa$, a space $X$ is said to be $\kappa$-Whyburn if $A \subset X$, $|A| \leq \kappa$ and $p \in \overline{A} \setminus A$ imply that there is a subset $B \subset A$ such that $\overline{B} = \{p\} \cup B$. Obviously a space is Whyburn if and only if it is $\kappa$-Whyburn for each infinite cardinal $\kappa$.

**Theorem 3.1.** For an infinite cardinal $\kappa$ and a space $X$, the following assertions are equivalent:

1. every subset $A \subset X$ with $|A| \leq \kappa$ is closed (equivalently, closed and discrete) in $X$,
2. $X \times Y$ is $\kappa$-Whyburn for any $\kappa$-Whyburn space $Y$,
3. $X \times Y$ is $\kappa$-Whyburn for any Whyburn space $Y$.

**Proof:** (1) $\rightarrow$ (2) Let $Y$ be a $\kappa$-Whyburn space, and assume that $A \subset X \times Y$, $|A| \leq \kappa$ and $(p, q) \in \overline{A} \setminus A$. Let $\pi_X : X \times Y \rightarrow X$ be the projection. Since the set $\pi_X (A \setminus \{(p) \times Y\})$ is closed in $X$, we have $(p, q) \in \overline{A} \cap \{(p) \times Y\}$. Applying the $\kappa$-Whyburn property of $Y$, we can take a subset $B \subset A$ such that $\overline{B} = \{(p, q)\} \cup B$.

(2) $\rightarrow$ (3) is trivial.

We show (3) $\rightarrow$ (1). Note that $X$ is, at least, $\kappa$-Whyburn. Assume the contrary of (1). Then there is a subset $A \subset X$ such that $A$ is not closed in $X$ and $|A| \leq \kappa$. Let $|A| = \lambda \leq \kappa$, and let $p \in \overline{A} \setminus A$. The subspace $S = \{p\} \cup A$ of $X$ is Whyburn, because of $|S| \leq \kappa$. For each $\alpha < \lambda$, let $Y_\alpha = \{p_\alpha\} \cup A_\alpha$ be a homeomorphic copy of $S$, where $p_\alpha = p$ and $A_\alpha = A$. Let $Y = \{\tilde{p}\} \cup (\bigcup_{\alpha < \lambda} A_\alpha)$ be the quotient space of the topological sum of $Y_\alpha$, obtained by collapsing the set $\{p_\alpha : \alpha < \lambda\}$ to one point $\tilde{p}$. It is not difficult to check that $Y$ is Whyburn. Since $|S \times Y| \leq \kappa$, we have only to see that $S \times Y$ is not Whyburn. Let $f : A \rightarrow \lambda$ be a bijection. We put $E = \bigcup\{(x) \times A_f(x) : x \in A\}$, then obviously $(p, \tilde{p}) \in \overline{E} \setminus E$. If $S \times Y$ is Whyburn, there is a subset $F \subset E$ such that $\overline{F} = \{(p, \tilde{p})\} \cup F$. The set $F$ is of the form $F = \bigcup\{(x) \times B_f(x) : x \in A\}$, where $B_f(x) \subset A_f(x)$. Since $\{p, \tilde{p}\} \cup F$ is closed, $\bigcup\{B_f(x) : x \in A\}$ is closed in $Y$. This is a contradiction, because of $(p, \tilde{p}) \in \overline{F}$. Thus $S \times Y$ is not Whyburn.

Applying the preceding theorem, we immediately have:

**Corollary 3.2.** For a space $X$, $X \times Y$ is Whyburn for any Whyburn space $Y$ if and only if $X$ is discrete.

**References**


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