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Some results on spaces with $\aleph_1$-calibre

Wei-Feng Xuan, Wei-Xue Shi

Abstract. We prove that, assuming CH, if $X$ is a space with $\aleph_1$-calibre and a zeroset diagonal, then $X$ is submetrizable. This gives a consistent positive answer to the question of Buzyakova in Observations on spaces with zeroset or regular $G_\delta$-diagonals, Comment. Math. Univ. Carolin. 46 (2005), no. 3, 469–473. We also make some observations on spaces with $\aleph_1$-calibre.

Keywords: $\aleph_1$-calibre; star countable; zeroset diagonal
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1. Introduction

H. Martin in [6] proved that a separable space with a zeroset diagonal is submetrizability. However, having a zeroset diagonal does not guarantee submetrizable. Recall that a space has $\aleph_1$-calibre if every uncountable family of open sets contains an uncountable subfamily with non-emptyset intersection. It is clear that every separable space has $\aleph_1$-calibre. Naturally, Buzyakova in [1] posted the following question.

Question 1.1. Let $X$ have $\aleph_1$-calibre and a zeroset diagonal. Is $X$ submetrizable?

In this paper, we prove that, assuming CH, if $X$ is a space with $\aleph_1$-calibre and a zeroset diagonal, then $X$ is submetrizable. This gives a consistent positive answer to the Question 1.1. We also make some observations on spaces with $\aleph_1$-calibre.

2. Notation and terminology

All the spaces are assumed to be Hausdorff unless otherwise stated.

Definition 2.1. A space $X$ has a zeroset diagonal if there is a continuous mapping $f : X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x,x) : x \in X\}$.

Definition 2.2. A space $X$ is called submetrizable if there exists a continuous injection of $X$ into a metrizable space.

Clearly, every submetrizable space has a zeroset diagonal. Note that there is a space which has a zeroset diagonal but not submetrizable [7].

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Definition 2.3. A space $X$ is star countable if whenever $U$ is an open cover of $X$, there is a countable subset $A$ of $X$ such that $\text{St}(A,U) = X$, where $\text{St}(A,U) = \bigcup\{U \in U : U \cap A \neq \emptyset\}$.

This notation was first introduced and studied by S. Ikenaga in [5]. Sometimes a star countable space is also called that it has countable weak extent.

Lemma 2.4. $\Delta$-system Lemma states that every uncountable collection of finite sets contains an uncountable $\Delta$-system, i.e., a collection of sets whose pairwise intersection is constant.

All notation and terminology not explained here is given in [4].

3. Results

Theorem 3.1. Assume CH. If $X$ is a space with $\aleph_1$-calibre and a zeroset diagonal, then $X$ is submetrizable.

Proof: In [3], it has been proved that if $X$ has a zeroset diagonal and $X^2$ is star countable, then $X$ is submetrizable. So, it is sufficient to prove that $X^2$ is star countable. Notice that a space with $\aleph_1$-calibre has countable Souslin number and a zeroset diagonal implies a regular $G_\delta$-diagonal. We can apply a known result from [2] that the cardinality of a space with a regular $G_\delta$-diagonal and countable Souslin number is at most $\mathfrak{c}$ to conclude that $|X| \leq \mathfrak{c}$. Clearly, $|X^2| \leq \mathfrak{c}$, and hence $|X^2| \leq \aleph_1$ since CH. Assume $|X^2| = \aleph_1$. Enumerate $X^2$ as $\{x_\alpha : \alpha < \aleph_1\}$.

Suppose that $X^2$ is not star countable. Then there exists an open cover $U$ of $X^2$ such that for any countable subset $A$ of $X^2$, $X^2 \setminus \text{St}(A,U) \neq \emptyset$. It is clear that $\overline{A} \subset \text{St}(A,U)$. In fact, for any $x \in \overline{A}$, there exists an open set $U \in \mathcal{U}$ which contains $x$, satisfying that $U \cap A \neq \emptyset$, and hence $x \in U \subset \text{St}(A,U)$. So, $X^2 \setminus \overline{A} \neq \emptyset$. For each $\alpha < \aleph_1$, let $U_\alpha = X^2 \setminus \{x_\beta : \beta < \alpha\}$. Then $\{U_\alpha : \alpha < \aleph_1\}$ is an uncountable decreasing family of non-empty open sets of $X^2$ and $\bigcap\{U_\alpha : \alpha < \aleph_1\} = \emptyset$. However, since $X$ has $\aleph_1$-calibre hence $X^2$ also has $\aleph_1$-calibre [4, p. 116], which implies that $\bigcap\{U_\alpha : \alpha < \aleph_1\} \neq \emptyset$. This is a contradiction! □

Theorem 3.1 gives a consistent positive answer to the Question 1.1. A natural question then arises: Assume $\neg$ CH. Let $X$ have $\aleph_1$-calibre and $|X| \leq \mathfrak{c}$. Is $X^2$ star countable? The answer to this question is negative. The following examples will show that we cannot drop the assumption of CH.

Example 3.2. Assume $2^{\aleph_1} = \mathfrak{c}$. There is a space $X$ having $\aleph_1$-calibre and $|X| = \mathfrak{c}$, however, $X^2$ is not star countable.

Proof: Let $X = \{x \in D^\mathfrak{c} : 0 < |\{\alpha < \mathfrak{c} : x(\alpha) = 1\}| \leq \aleph_1\}$, where $D = \{0,1\}$. Clearly, since $2^{\aleph_1} = \mathfrak{c}$, then $|X| = 2^{\aleph_1} = (2^{\aleph_1})^{\aleph_1} = 2^{\aleph_1} = \mathfrak{c}$.

We firstly prove that $X$ has $\aleph_1$-calibre. For any finite partial function $\varphi : \mathfrak{c} \rightarrow D$, let $B(\varphi) = \{x \in X : x|\text{dom} \varphi = \varphi\}$; then the sets $B(\varphi)$ are a base of $X$. Let $\mathcal{U} = \{U_\alpha : \alpha < \aleph_1\}$ be a family of open sets in $X$. For $\alpha < \aleph_1$ let $\varphi_\alpha$ be a finite partial function from $\mathfrak{c}$ to $D$ such that $B(\varphi_\alpha) \subseteq U_\alpha$, and let $S_\alpha = \text{dom} \varphi_\alpha$. By
the $\Delta$-system Lemma, there is an uncountable subset $\Lambda \subset \aleph_1$ and a finite $S \subset c$ such that $S_\xi \cap S_\eta = S$ whenever $\xi, \eta \in \Lambda$ and $\xi \neq \eta$. Since $S$ is finite, there is an uncountable $\Lambda_0 \subset \Lambda$ such that $\varphi_\xi|_S = \varphi_\eta|_S$ whenever $\xi, \eta \in \Lambda_0$, and hence $\bigcap_{\alpha \in \Lambda_0} U_\alpha \supset \bigcap_{\alpha \in \Lambda_0} B(\varphi_\alpha) \neq \emptyset$. Thus, $X$ has $\aleph_1$-calibre.

To show that $X^2$ is not star countable, we only need to prove that $X$ is not star countable. For $\alpha < c$ let $\varphi_\alpha = \langle \alpha, 1 \rangle$ and $U_\alpha = B(\varphi_\alpha)$; clearly $U = \{U_\alpha : \alpha < c\}$ is an open cover of $X$. Let $A$ be any countable subset of $X$, and let $S = \bigcup_{x \in A} \{\alpha < c : x(\alpha) = 1\}$. It is easy to see that $|S| \leq \aleph_1 < 2^\aleph_1 = c$, so there is some $\gamma \in c \setminus S$. Let $x$ be the unique point of $X$ such that $x(\gamma) = 1$ and $x(\alpha) = 0$ for any other $\alpha < c$. Suppose that there exists $U_\alpha$ of $U$ such that $U_\alpha \cap A \neq \emptyset$ and $x \in U_\alpha$. Then $x(\alpha) = 1$ and hence $\alpha = \gamma \notin S$. However, let $y \in U_\alpha \cap A$; clearly, $y(\alpha) = 1$, and hence $\alpha \in S$. This is a contradiction. Thus $St(A, U) \neq X$. This shows $X$ is not star countable. 

**Example 3.3.** Assume $\text{MA}+ \neg \text{CH}$. There is a first countable space $X$ with $\aleph_1$-calibre, however, $X^2$ is not star countable.

**Proof:** Let $X$ be the space of all nonempty compact nowhere dense subsets of $\mathbb{R}$ with the Pixley-Roy topology. A neighbourhood for $x \in X$ is obtained by taking a neighbourhood $U$ of $x$ on the real line and letting $[x, U] = \{y \in X : x \subset y \subset U\}$. Clearly, $|X| = c$. It is shown in [8] that $X$ is a first countable space with $\aleph_1$-calibre.

To show that $X^2$ is not star countable, we only need to prove that $X$ is not star countable. Let $U = \{[r, \mathbb{R}] : r \in \mathbb{R}\}$ be an open cover of $X$. Let $A$ be any countable subset of $X$. It was established in Baire category theorem that a non-empty complete metric space is not the countable union of nowhere-dense closed sets so $\mathbb{R} \setminus \bigcup A \neq \emptyset$. We pick some $r_0 \in \mathbb{R} \setminus \bigcup A$. Hence, $r_0 \notin St(A, U)$, since $[r_0, \mathbb{R}]$ is the only element of $U$ containing $r_0$ and $[r_0, \mathbb{R}] \cap A = \emptyset$. This shows $X$ is not star countable. 

We say that $X$ has countable tightness if $x \in \overline{A}$ for any $A$ of $X$, then there exists a countable subset $A_0$ of $A$ such that $x \in \overline{A_0}$; it is denoted by $t(X) = \aleph_0$.

**Proposition 3.4.** Let $X$ be a space with $\aleph_1$-calibre and $t(X) = \aleph_0$. If $d(X) \leq \aleph_1$, then $X$ is separable.

**Proof:** Since $d(X) \leq \aleph_1$, there exists a dense subset $A$ of $X$ with $|A| \leq \aleph_1$. If $|A| < \aleph_1$, it is obvious that $X$ is separable. We assume that $|A| = \aleph_1$. Enumerate $A$ as $\{x_\alpha : \alpha < \aleph_1\}$ and let $F_\alpha = \overline{\{x_\beta : \beta < \alpha\}}$ for each $\alpha < \aleph_1$. Then we have an $\aleph_1$-sequence $\mathcal{F} = \{F_\alpha : \alpha < \aleph_1\}$ of increasing closed separable subsets of $X$.

For any point $x \in X$, $x \in \overline{A}$. Since $t(X) = \aleph_0$, there exists a countable subset $A_0$ of $A$ such that $x \in \overline{A_0}$, and hence there exists some $F_\alpha$ such that $x \in A_0 \subset F_\alpha$. Thus $\bigcup \mathcal{F} = X$. We prove that there exists some $F_\alpha = X$. If $F_\alpha \neq X$ for any $\alpha < \aleph_1$ then the family $\{X \setminus F_\alpha : \alpha < \aleph_1\}$ is point-countable and uncountable which is a contradiction. Therefore $F_\alpha = X$ for some $\alpha < \aleph_1$, and hence $X$ is separable. 

□
References


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