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ON THE DOUBLE LUSIN CONDITION AND CONVERGENCE  
THEOREM FOR KURZWEIL-HENSTOCK TYPE INTEGRALS

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*Dedicated to Professor Jaroslav Kurzweil on the occasion of his 90th birthday*

*Abstract.* Equiintegrability in a compact interval  $E$  may be defined as a uniform integrability property that involves both the integrand  $f_n$  and the corresponding primitive  $F_n$ . The pointwise convergence of the integrands  $f_n$  to some  $f$  and the equiintegrability of the functions  $f_n$  together imply that  $f$  is also integrable with primitive  $F$  and that the primitives  $F_n$  converge uniformly to  $F$ . In this paper, another uniform integrability property called uniform double Lusin condition introduced in the papers E. Cabral and P. Y. Lee (2001/2002) is revisited. Under the assumption of pointwise convergence of the integrands  $f_n$ , the three uniform integrability properties, namely equiintegrability and the two versions of the uniform double Lusin condition, are all equivalent. The first version of the double Lusin condition and its corresponding uniform double Lusin convergence theorem are also extended into the division space.

*Keywords:* Kurzweil-Henstock integral;  $g$ -integral; double Lusin condition; uniform double Lusin condition

*MSC 2010:* 26A39

## 1. INTRODUCTION

It is now known that a function  $f$  on a closed and bounded interval  $E$  in  $\mathbb{R}^n$  is Kurzweil-Henstock integrable with primitive  $F$  if and only if  $f$  and  $F$  satisfy the following: for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that

$$(D) \sum |f(x)||I| < \varepsilon \quad \text{and} \quad (D) \sum |F(I)| < \varepsilon$$

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whenever  $D$  is a  $\delta$ -fine partial division of  $E$  in  $\Gamma_\varepsilon$ , where

$$\Gamma_\varepsilon = \{(x \cdot I): I \subset E, x \text{ is a vertex of } I \text{ and } |F(I) - f(x)|I| \geq \varepsilon|I|\}.$$

This condition was introduced in [3] and called the *double Lusin condition* in [4]. A sequence  $\{f_n\}$  of Kurzweil-Henstock integrable functions with the corresponding primitives  $\{F_n\}$  is said to satisfy the *uniform double Lusin condition* or simply  $UI_1$  if given  $\varepsilon > 0$  there is a common gauge  $\delta$  for all  $f_n$  such that

$$(D) \sum |f_n(x)|I| < \varepsilon \quad \text{and} \quad (D) \sum |F_n(I)| < \varepsilon$$

whenever  $D$  is a  $\delta$ -fine partial division of  $E$  in  $\Gamma_{\varepsilon,n}$ , where

$$\Gamma_{\varepsilon,n} = \{(x \cdot I): I \subset E, x \text{ is a vertex of } I \text{ and } |F_n(I) - f_n(x)|I| \geq \varepsilon|I|\}.$$

It was shown in [3] that if the functions  $f_n$  satisfy the  $UI_1$  condition and  $f_n \rightarrow f$  pointwise everywhere then  $f$  is Kurzweil-Henstock integrable and

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

The proof of this convergence theorem makes use of the fact that  $UI_1$  implies equi-integrability of the functions  $f_n$ . In this paper this convergence theorem is proved by looking at the behavior of  $\Gamma_{\varepsilon,n}$  as  $n$  approaches infinity and by comparing it with  $\Gamma_\varepsilon$ . Furthermore, supposing the functions  $f_n$  are integrable with primitives  $F_n$  and  $f_n \rightarrow f$  pointwise everywhere then the following will be shown to be equivalent:

- (i) the functions  $f_n$  satisfy the  $UI_1$  condition;
- (ii) the functions  $f_n$  satisfy the  $UI_2$  condition, that is, for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for all  $n$

$$(D) \sum |I| < \varepsilon \quad \text{and} \quad (D) \sum |F_n(I)| < \varepsilon$$

whenever  $D$  is a  $\delta$ -fine partial division of  $E$  in  $\Gamma_{\varepsilon,n}$ ;

- (iii) the functions  $f_n$  are equiintegrable, that is, for every  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for all  $n$

$$(D) \sum |f_n(x)|I| - F_n(I)| < \varepsilon$$

whenever  $D$  is a  $\delta$ -fine partial division of  $E$ .

An axiomatic approach to the Kurzweil-Henstock integral can be found in [5]. This general theory is called the division space. In the division space we have defined an integral called the  $g$ -integral, which includes the Kurzweil-Henstock integral,

the McShane integral, and the approximate Perron integral. The double Lusin characterization of the  $g$ -integral and the corresponding convergence theorem will also be given.

## 2. THE KURZWEIL-HENSTOCK INTEGRAL IN $\mathbb{R}^n$

Let  $E$  be a compact interval in  $\mathbb{R}^n$  described by  $E = \prod_{j=1}^n [a_j, b_j]$ , where  $[a_j, b_j]$ ,  $j = 1, 2, \dots, n$ , is a compact interval in  $\mathbb{R}$ . From this point onwards,  $E$  will always refer to an interval while  $I$  will denote any subinterval of  $E$  in general. It is also useful to denote  $E = [a, b]$ , where  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$ . The measure of  $E$  is its outer Lebesgue measure  $|E|$  given by  $|E| = \prod_{j=1}^n (b_j - a_j)$ . In general, the measure of any interval  $I$  is equal to its outer Lebesgue measure  $|I|$ .

In our discussion,  $\mathbb{R}^m$  will be equipped with the norm  $\|\cdot\|$  defined by

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$$

for  $x = (x_1, x_2, \dots, x_n)$  in  $\mathbb{R}^n$ . Given  $x \in \mathbb{R}^n$  and  $r > 0$  we set

$$B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\},$$

where  $x - y = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$ .

A partial division  $D = \{(x, I)\}$  of  $E$  is any finite set of point-interval pairs with  $x$  a vertex of the corresponding subinterval  $I \subset E$  and with the interiors of the subintervals  $I$  disjoint. If for some partial division  $D = \{(x, I)\}$ ,

$$\bigcup_{(x, I) \in D} I = E,$$

then  $D$  is said to be a division of  $E$ . A gauge on  $E$  is a function  $\delta: E \rightarrow (0, \infty)$ . A partial division  $D = \{(x, I)\}$  is said to be  $\delta$ -fine if for each pair  $(x, I)$  in  $D$ ,  $I \subset B(x, \delta(x))$ .

A function  $f: E \rightarrow \mathbb{R}$  is said to be Kurzweil-Henstock integrable if there is a real number  $A$  such that given  $\varepsilon > 0$  there is a gauge  $\delta$  on  $E$  such that for any  $\delta$ -fine division of  $E$  we have

$$\left| (D) \sum f(x)|I| - A \right| < \varepsilon,$$

where  $(D) \sum f(x)|I|$  denotes the sum over all the pairs  $(x, I)$  in  $D$ . The number  $A$  is called the integral of  $f$  over  $E$  and we write

$$\int_E f = A.$$

If  $f$  is Kurzweil-Henstock integrable on  $E$  then  $f$  is Kurzweil-Henstock integrable on any subinterval  $I$  of  $E$ . Hence, we can define an additive interval function  $F$  by

$$F(I) = \int_I f.$$

We call  $F$  the primitive of  $f$ . Then  $F$  is an additive interval function in the sense that for any finite collection  $\{I_i : i = 1, \dots, n\}$  whose union is a subinterval  $I$  of  $E$  we have

$$F\left(\bigcup_{i=1}^n I_i\right) = \sum_{i=1}^n F(I_i).$$

Note that if we have a primitive interval function  $F$ , we can define a point function corresponding to  $F$ , and conversely. Let  $x \in E$ , where  $x = (x_1, x_2, \dots, x_m)$ . If for some  $i$ ,  $x_i = a_i$ , set  $F(x) = 0$ . In case  $x_i \neq a_i$  for all  $i$ , set  $F(x) = F([a, x])$ . Then we have a unique point function. Conversely, given a point  $F$  on  $E$ , we can define an interval function as follows: let  $I = [\alpha, \beta]$ , where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ . Write  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$ , where  $\gamma_i = \alpha_i$  or  $\beta_i$  and  $n(\gamma)$  denotes the number of terms for which  $\gamma_i = \alpha_i$ . Define

$$F(I) = \sum_{\gamma} (-1)^{n(\gamma)} F(\gamma),$$

where the summation is over all the vertices  $\gamma$ . We have a unique interval function  $F$ . Given a primitive point function, we may recover the primitive interval function as described above. Hence, we may identify them with each other.

The following theorems were proved in [3].

**Theorem 2.1.** *Let  $f$  and  $F$  be functions defined on  $E$ , where the interval function corresponding to  $F$  is additive. Then  $f$  is Kurzweil-Henstock integrable on  $E$  with primitive  $F$  if and only if given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  we have*

$$(D \cap \Gamma_{\varepsilon}) \sum |f(x)||I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon}) \sum |F(I)| < \varepsilon.$$

**Theorem 2.2.** *Let  $f$  and  $F$  be functions defined on  $E$  where the interval function corresponding to  $F$  is additive. Then  $f$  is Kurzweil-Henstock integrable on  $E$  with primitive  $F$  if and only if given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  we have*

$$(D \cap \Gamma_{\varepsilon}) \sum |I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon}) \sum |F(I)| < \varepsilon.$$

### 3. CONVERGENCE THEOREMS INVOLVING UNIFORM INTEGRABILITY

Let  $\{f_n\}$  be a sequence of Kurzweil-Henstock integrable functions on  $E$  with the corresponding primitives  $F_n$  and  $\Gamma_{\varepsilon,n}$  being as defined above.

**Theorem 3.1.** *Suppose the functions  $f_n$  satisfy the  $UI_1$  condition. If  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ , then  $f$  is Kurzweil-Henstock integrable on  $E$  and*

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n.$$

**Theorem 3.2.** *Suppose the functions  $f_n$  satisfy the  $UI_2$  condition. If  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ , then  $f$  is Kurzweil-Henstock integrable on  $E$  and*

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n.$$

We shall show alternative proofs of Theorem 3.1 and Theorem 3.2 in the succeeding discussions.

**Lemma 3.1.** *Suppose the functions  $f_n$  are Kurzweil-Henstock integrable on  $E$  with the corresponding primitives  $F_n$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ . If given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  there exists a positive integer  $N_D$  such that for  $n > N_D$*

$$(D \cap \Gamma_{\varepsilon,n}) \sum |f_n(x)||I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon,$$

then the following hold:

- (i) *there exists a function  $F$  on  $E$  such that  $F_n(x) \rightarrow F(x)$  for all  $x \in E$ ,*
- (ii) *the function  $f$  is integrable with primitive  $F$  and  $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$ .*

**Proof.** (i) Let  $\varepsilon > 0$  and  $x \in E$ , where  $[a, x]$  is a nondegenerate interval. We will show that  $\{F_n(x)\}$  is a Cauchy sequence. Hence it is convergent.

There exists a gauge  $\delta_x$  such that for every  $\delta_x$ -fine division  $D$  of  $[a, x]$  there exists a positive integer  $N_D$  for which for  $n \geq N_D$

$$(D \cap \Gamma_{\varepsilon/k,n}) \sum |f_n(\xi)||I| < \frac{\varepsilon}{k} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon/k,n}) \sum |F_n(I)| < \frac{\varepsilon}{k},$$

where  $k = 5 + 2|E|$ .

Let  $D_x$  be a fixed  $\delta_x$ -fine division of  $[a, x]$ . We can find  $M_x \geq N_{D_x}$  such that for all  $n, m \geq M_x$

$$|f_n(\xi) - f_m(\xi)| < \frac{\varepsilon}{k|E|}.$$

It follows that for  $n, m \geq N_x$ ,

$$\begin{aligned} |F_n(x) - F_m(x)| &= |F_n([a, x]) - F_m([a, x])| \\ &\leq \left| F_n([a, x]) - (D_x) \sum f_m(\xi)|I| \right| \\ &\quad + \left| (D_x) \sum f_n(\xi)|I| - (D_x) \sum f_m(\xi)|I| \right| \\ &\quad + \left| (D_x) \sum f_m(\xi)|I| - F_m([a, x]) \right| < \varepsilon. \end{aligned}$$

Choose  $F(x) = \lim_{n \rightarrow \infty} F_n(x)$ .

We will now show that  $f$  is integrable and  $F$  is its primitive.

(ii) Given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that if  $D$  is a  $\delta$ -fine partial division of  $E$ , there is a positive integer  $N_D$  such that for  $n > N_D$

$$(D \cap \Gamma_{\varepsilon/2, n}) \sum |f(x)||I| < \frac{\varepsilon}{2} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon/2, n}) \sum |F_n(I)| < \frac{\varepsilon}{2}.$$

Since  $f_n(x) \rightarrow f(x)$  and  $F_n(x) \rightarrow F(x)$  for all  $x \in E$ , there exists a positive integer  $M_1 > N_D$  such that for  $n > M_1$  the inequalities

$$(D) \sum |f_n(x) - f(x)||I| < \frac{\varepsilon}{2} \quad \text{and} \quad (D) \sum |F_n(I) - F(I)| < \frac{\varepsilon}{2}$$

hold. Furthermore, there exists a positive integer  $M_2 > M_1$  such that for  $n > M_2$ ,

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2, n}.$$

Then for  $n > M_2$ ,

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum |f(x)||I| &\leq (D \cap \Gamma_\varepsilon) \sum |f(x) - f_n(x)||I| + (D \cap \Gamma_\varepsilon) \sum |f_n(x)||I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum |F(I)| &\leq (D \cap \Gamma_\varepsilon) \sum |F(I) - F_n(I)| + (D \cap \Gamma_\varepsilon) \sum |F_n(I)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The proof is complete. □

Correspondingly, we have the following result for  $UI_2$ .

**Lemma 3.2.** Suppose the functions  $f_n$  are Kurzweil-Henstock integrable on  $E$  with the corresponding primitives  $F_n$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ . If given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  there exists a positive integer  $N_D$  such that for  $n > N_D$

$$(D \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon,$$

then  $f$  is Kurzweil-Henstock integrable and  $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$ .

Theorem 3.1 and Theorem 3.2 follow from Lemma 3.1 and Lemma 3.2, respectively. We will now show the equivalence that we described in the introduction.

**Theorem 3.3.** Let  $\{f_n\}$  be a sequence of integrable functions on  $E$  with the corresponding primitives  $F_n$  and suppose that  $f_n(x)$  converges for all  $x \in E$ . Then the following statements are equivalent:

- (1)  $f_n$  satisfy the  $\text{UI}_1$  condition,
- (2)  $f_n$  satisfy the  $\text{UI}_2$  condition,
- (3)  $f_n$  are equiintegrable.

Given  $f_n$  pointwise convergent everywhere to a function  $f$ , each of the statements above implies that  $f$  is Kurzweil-Henstock integrable and  $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$ .

**Proof.** (1)  $\Rightarrow$  (2): Suppose the functions  $f_n$  satisfy the  $\text{UI}_1$  condition. Given  $\varepsilon > 0$  there exists a gauge  $\delta$  on  $E$  independent of  $n$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  we have

$$(D \cap \Gamma_{\varepsilon^2/2,n}) \sum |f_n(x)||I| < \frac{\varepsilon^2}{2} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon^2/2,n}) \sum |F_n(I)| < \frac{\varepsilon^2}{2}.$$

We may assume  $\varepsilon < 1$ . Then  $\Gamma_{\varepsilon,n} \subset \Gamma_{\varepsilon^2/2,n}$ . Hence, for any  $\delta$ -fine partial division  $D$  of  $E$ , we have

$$(D \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon.$$

(2)  $\Rightarrow$  (1): Suppose the functions  $f_n$  satisfy the  $\text{UI}_2$  condition. Given  $\varepsilon > 0$  and a positive integer  $i$ , there exists a gauge  $\delta_i$  on  $E$  independent of  $n$  such that for any  $\delta_i$ -fine partial division  $D_i$  of  $E$  we have

$$(D_i \cap \Gamma_{\varepsilon_i,n}) \sum |I| < \varepsilon_i,$$

where  $\varepsilon_i = \varepsilon/(i2^i)$ . Since for every  $x$  the sequence  $\{f_n(x)\}$  is convergent,  $\{f_n(x)\}$  is bounded. Let

$$X_i = \{x \in E: |f_n(x)| < i, \forall n \in \mathbb{N}\}$$

and put  $Y_1 = X_1$  and for  $i = 2, 3, \dots$

$$Y_i = X_i \setminus \bigcup_{k=1}^{i-1} X_k.$$

Define  $\delta(x) = \delta_i(x)$  when  $x \in Y_i$ . Let  $D$  be a  $\delta$ -fine partial division of  $E$ . Split  $D$  into  $D_i$ ,  $i = 1, 2, \dots$ , where  $D_i$  contains those pairs with tags in  $Y_i$ . For any  $n$  and  $i$

$$(D_i \cap \Gamma_{\varepsilon_i, n}) \sum |I| < \varepsilon_i.$$

Thus, for any  $n$ ,

$$(D \cap \Gamma_{\varepsilon, n}) \sum |f_n(x)||I| \leq \sum_{i=1}^{\infty} i(D_i \cap \Gamma_{\varepsilon, n}) \sum |I| < \varepsilon.$$

Furthermore, since  $f_n$  satisfy the  $UI_2$  condition, we may choose  $\delta$  appropriately so that

$$(D \cap \Gamma_{\varepsilon, n}) \sum |F_n(I)| < \varepsilon.$$

Hence  $f_n$  satisfy  $UI_1$ .

In [3], it was shown that (1) implies (3).

(3)  $\Rightarrow$  (1): Suppose the functions  $f_n$  are equiintegrable. Then for any positive number  $\varepsilon$  and positive integer  $i$ , there exists a gauge  $\delta_i$  on  $E$  independent of  $n$  such that for any  $\delta_i$ -fine partial division  $D$  of  $E$  we have

$$(D) \sum |F_n(I) - f_n(x)||I| < \varepsilon_i^2,$$

where  $\varepsilon_i = \varepsilon/(i2^{i+1})$ .

Then for any  $\delta_i$ -fine partial division  $D$  of  $E$

$$(D \cap \Gamma_{\varepsilon_i}) \sum |I| \leq \frac{1}{\varepsilon_i} (D \cap \Gamma_{\varepsilon_i, n}) \sum |F_n(I) - f_n(x)||I| < \varepsilon_i.$$

Following the argument we used in the proof that (2) implies (1), there exists a gauge  $\delta$  on  $E$  independent of  $n$  such that for any  $\delta$ -fine partial division  $D$  of  $E$  we have

$$(D \cap \Gamma_{\varepsilon/2, n}) \sum |f_n(x)||I| < \frac{\varepsilon}{2}.$$

We may assume that  $\varepsilon < 1$ . Then for any  $\delta$ -fine partial division  $D$  and any  $n$  we have  $(D \cap \Gamma_{\varepsilon, n}) \sum |I| < \varepsilon$  and

$$\begin{aligned} (D \cap \Gamma_{\varepsilon, n}) \sum |F_n(I)| &\leq (D \cap \Gamma_{\varepsilon, n}) \sum |F_n(I) - f_n(x)||I| \\ &\quad + (D \cap \Gamma_{\varepsilon, n}) \sum |f_n(x)||I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The proof is complete. □

#### 4. THE $g$ -INTEGRAL

To describe the division space we isolate the properties of  $\delta$ -fine divisions that make integration work. These are properties (i) to (iv) given below. In addition to these four properties, a fifth property called decomposability (v) has to be added in order to prove convergence theorems. A decomposable division space consists of three mathematical objects: a space  $T$ , a family  $\mathcal{I}$  of intervals  $I$  in  $T$ , and a collection  $\mathcal{A}$  of families of some point-interval pairs  $(x, I)$  satisfying certain conditions. A set  $E$  is an elementary set of  $T$  if it is a finite union of intervals in  $\mathcal{I}$ . We shall agree that for any elementary set  $E$  and any  $I \in \mathcal{I}$ ,  $E \setminus I$  is an elementary set or it is empty. A division  $D$  of  $E$  is the family of a finite number of mutually disjoint intervals  $I$  with union  $E$ . A subfamily  $\mathcal{I}_1$  of  $\mathcal{I}$  divides  $E$  if a division  $D$  of  $E$  exists with the intervals of  $D$  belonging to  $\mathcal{I}_1$ . Let  $U = \{(x, I)\} \in \mathcal{A}$ . Then  $U$  divides  $E$  if  $\{I: (x, I) \in U\}$  divides  $E$ .

The triple  $(T, \mathcal{I}, \mathcal{A})$  is called a decomposable division space if the following conditions are satisfied:

- (i) For every elementary set  $E$  of  $T$  there is  $U \in \mathcal{A}$  dividing  $E$ .
- (ii) If both  $U_1, U_2 \in \mathcal{A}$  are dividing  $E$  there is  $U_3 \in \mathcal{A}$  dividing  $E$  with  $U_3 \subset U_1 \cap U_2$ .
- (iii) If  $U_0 \in \mathcal{A}$  divides the union of two disjoint elementary sets  $E_1$  and  $E_2$  then a family  $U_1 = \{(x, I)\} \subset U_0$  with  $I \subset E_1$  belongs to  $\mathcal{A}$  and divides  $E_1$ .
- (iv) Given disjoint elementary sets  $E_1$  and  $E_2$ , if  $U_1 \in \mathcal{A}$  divides  $E_1$  with  $I \subset E_1$  for all  $(x, I) \in U_1$ , and  $U_2 \in \mathcal{A}$  divides  $E_2$  with  $I \subset E_2$  for all  $(x, I) \in U_2$ , then there is  $U_3 \in \mathcal{A}$  dividing  $E_1 \cup E_2$  with  $U_3 \subset U_1 \cup U_2$ .
- (v) (*Decomposability*) For all elementary sets  $K$ , all sequences  $U_j$  dividing  $K$ , and all sequences  $\{E_j\}$  of mutually disjoint subsets of  $E$  there is  $U_0 \in \mathcal{A}$  dividing  $K$  such that

$$U_0[E_j] \subset U_j[E_j], \quad j = 1, 2, \dots,$$

where

$$U[X] = \{(x, I) \in U: x \in X\}.$$

**Definition 4.1.** Let  $X \subset E$  and  $g: \bigcup_{U \in \mathcal{A}} U \rightarrow \mathbb{R}$ . We say that  $g$  has a bounded Riemann sums on  $X$  (BRS( $X$ )) if there exists  $U \in \mathcal{A}$  such that

$$(D) \sum |g(x, I)| < M, \quad \text{for some } M > 1$$

whenever  $D$  is a partial division of  $E$  from  $U[X]$ . The function  $g(x, I)$  is said to have countably bounded Riemann sums on  $E$  (CBRS( $E$ )) if

▷ there exists a sequence of set-number pairs  $\{(X_i, M_i)\}$  with  $\bigcup_{i=1}^{\infty} X_i = E$  and

$$1 < M_1 \leq M_2 \leq M_3 \dots$$

▷ and for every  $i$ , there exists  $U_i \in \mathcal{A}$  dividing  $E$  such that

$$(D) \sum |g(x, I)| < M_i$$

whenever  $D$  is a partial division of  $E$  from  $U_i[X_i]$ .

Supposing  $g(x, I)$  has countably bounded Riemann sums (using  $X_i, M_i, U_i$ ) we denote

$$V_H(g, X_i, U_i) = \sup_{D \in U_i[X_i]} \sum |g(x, I)| \leq M_i.$$

In what follows, let  $g: \bigcup_{U \in \mathcal{A}} U \rightarrow \mathbb{R}$  have countably bounded Riemann sums on  $E$  (using  $X_i, M_i, U_i$ ).

**Definition 4.2.** A function  $h: \bigcup_{U \in \mathcal{A}} U \rightarrow \mathbb{R}$  is said to be  $g$ -integrable with primitive  $\mathcal{H}$  if for every positive integer  $i$ , for every  $\varepsilon > 0$  there exists  $U_i^\varepsilon \in \mathcal{A}$  dividing  $E$  with  $U_i^\varepsilon[X_i] \subset U_i[X_i]$  such that

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \varepsilon V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U_i^\varepsilon$ .

The family  $U_i^\varepsilon$  in Definition 4.2 is a function of  $i$ . That is, the inequality stated in the definition holds when the sum is taken over all  $x \in X_i$  using  $U_i^\varepsilon$  but may not hold when the sum is taken over all  $x \in X_j$ ,  $j \neq i$ . However, there is a family  $U^\varepsilon$  which is uniform with respect to  $i$ . That is, inequality in Definition 4.2 holds regardless of, where the sum is being taken over. This is what the next theorem says. The advantage of saying Definition 4.2 with  $U_i^\varepsilon$  instead of  $U^\varepsilon$  is in the proof of the double Lebesgue formulation (Theorem 4.3) where  $U_i^\varepsilon$  is more quickly obtained than  $U^\varepsilon$ .

**Theorem 4.1.** *If  $h(x, I)$  is  $g$ -integrable on  $E$  with primitive  $\mathcal{H}$  then for every  $\varepsilon > 0$  there exists  $U^\varepsilon \in \mathcal{A}$  dividing  $E$  with  $U^\varepsilon[X_i] \subset U_i[X_i]$  for each  $i$ , such that for each  $i$ ,*

$$(D) \sum_{x \in Y_i} |h(x, I) - \mathcal{H}(I)| < \varepsilon V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U^\varepsilon$ .

**P r o o f.** Let  $Y_1 = X_1$ . For  $i > 1$ , let

$$Y_i = X_i \setminus \bigcup_{j=1}^{i-1} X_j.$$

The sets  $Y_i$  are pairwise disjoint. In Definition 4.1, we can actually let the sequence  $\{X_i\}$  to be nondecreasing and

$$U[X_{i-1}] = U_{i-1}[X_{i-1}]$$

so that for all  $i$ ,

$$V_H(g; X_i, U_i) \geq V_H(g; X_{i-1}, U_{i-1}).$$

By Definition 4.2, for every  $i \in \mathbb{N}$ , for every  $\varepsilon > 0$  there exists  $U_i^\varepsilon \in \mathcal{A}$  dividing  $E$  with  $U_i^\varepsilon[X_i] \subset U_i[X_i]$  such that

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \frac{\varepsilon}{2^i} V_H(g; X_i, U_i)$$

for all partial divisions  $D$  in  $U_i^\varepsilon$ . By the decomposability property, we can choose  $U^\varepsilon \in \mathcal{A}$  dividing  $E$  such that

$$U^\varepsilon[Y_i] \subset U_i^\varepsilon[Y_i].$$

Then for each  $i \in \mathbb{N}$ ,

$$\begin{aligned} (D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| &= \sum_{k=1}^i (D) \sum_{x \in Y_k} |h(x, I) - \mathcal{H}(I)| \\ &< \sum_{k=1}^i \frac{\varepsilon}{2^k} V_H(g; X_k, U_k) \\ &\leq V_H(g; X_i, U_i) \sum_{k=1}^i \frac{\varepsilon}{2^k} < \varepsilon V_H(g; X_i, U_i) \end{aligned}$$

for all partial divisions  $D$  in  $U^\varepsilon$ . The proof is complete.  $\square$

The next theorem says that Definition 4.2 is just a special case of the generalized Riemann integral defined in ([5], page 165).

**Theorem 4.2.** *If  $h(x, I)$  is  $g$ -integrable with primitive  $\mathcal{H}$  then  $h(x, I)$  is generalized Riemann integrable and the two integrals coincide.*

*Proof.* Let  $Y_i$  be as in the proof of the preceding theorem.

By Definition 4.2, for every positive integer  $i$ , for every  $\varepsilon > 0$  there exists  $U_i^\varepsilon \in \mathcal{A}$  dividing  $E$  with  $U_i^\varepsilon[X_i] \subset U_i[X_i]$  such that,

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \frac{\varepsilon}{2^i M_i} V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U_i^\varepsilon$ . By the decomposability property, we can choose  $U^\varepsilon \in \mathcal{A}$  dividing  $E$  such that  $U^\varepsilon[Y_i] \subset U_i^\varepsilon[Y_i]$ .

Then for all divisions  $D$  of  $E$  from  $U^\varepsilon$ , we have

$$\begin{aligned} (D) \sum |h(x, I) - \mathcal{H}(I)| &= \sum_{i=1}^{\infty} (D) \sum_{x \in Y_i} |h(x, I) - \mathcal{H}(I)| \\ &< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i M_i} V_H(g; X_i, U_i) \leq \varepsilon. \end{aligned}$$

This is precisely the definition of the generalized Riemann integral.  $\square$

**Definition 4.3.** We say that a function  $h(x, I)$  satisfies the  $\alpha g$ -condition on  $E$  if for every  $x \in E$  there is a minimum number  $\alpha(x) \in \mathbb{Z}^+$  such that for any  $(x, I)$  in the division space we have

$$|h(x, I)| \leq \alpha(x) |g(x, I)|.$$

Given functions  $\mathcal{H}$  and  $h(x, I)$ , and  $U \in \mathcal{A}$ , we denote

$$\Gamma_\varepsilon = \{(x, I) \in U : |\mathcal{H}(I) - h(x, I)| \geq \varepsilon |g(x, I)|\}.$$

We are now ready to present the double Lusin formulation for the  $g$ -integral.

**Theorem 4.3.** Let  $h: \bigcup_{U \in \mathcal{A}} U \rightarrow \mathbb{R}$  satisfy the  $\alpha g$ -condition and  $H: \mathcal{I}(E) \rightarrow \mathbb{R}$ . Then  $h(x, I)$  is  $g$ -integrable on  $E$  with primitive  $\mathcal{H}$  if and only if for every  $i$ , for every  $\varepsilon > 0$  there exists  $U_i^\varepsilon \in \mathcal{A}$  with  $U_i^\varepsilon[X_i] \subset U_i[X_i]$  such that

$$(D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x, I)| < \varepsilon V_H(g; X_i, U_i)$$

and

$$(D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |H(I)| < \varepsilon V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U_i^\varepsilon$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $h(x, I)$  is  $g$ -integrable with primitive  $\mathcal{H}$ . Let

$$E_k = \{x \in E : \alpha(x) = k\}.$$

Given  $0 < \varepsilon < 1$  and a positive integer  $i$ , for every  $k$  there exists  $U_i^{\varepsilon, k} \in \mathcal{A}$  dividing  $E$  with  $U_i^{\varepsilon, k}[X_i] \subset U_i[X_i]$  such that

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \frac{\varepsilon^2}{k 2^{k+1}} V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U_i^{\varepsilon, k}$ .

By decomposability, there exists  $U_i^\varepsilon \in \mathcal{A}$  dividing  $E$  with

$$U_i^\varepsilon[E_k] \subset U_i^{\varepsilon, k}[E_k].$$

Then for all partial divisions  $D$  of  $E$  from  $U_i^\varepsilon$ , we have

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x, I)| &\leq \sum_{k=1}^{\infty} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i \cap E_k} k |g(x, I)| \\ &\leq \sum_{k=1}^{\infty} \frac{k}{\varepsilon} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i \cap E_k} |h(x, I) - \mathcal{H}(I)| \\ &< \sum_{k=1}^{\infty} \frac{k}{\varepsilon} \frac{\varepsilon^2}{k 2^{k+1}} V_H(g; X_i, U_i) \\ &< \varepsilon V_H(g; X_i, U_i) \end{aligned}$$

and

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I)| &\leq (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I) - h(x, I)| + (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x, I)| \\ &< \frac{\varepsilon}{2} V_H(g; X_i, U_i) + \frac{\varepsilon}{2} V_H(g; X_i, U_i) = \varepsilon V_H(g; X_i, U_i). \end{aligned}$$

( $\Leftarrow$ ) For the converse, given  $i$ , for every  $\varepsilon > 0$  choose  $U_i^\varepsilon$  such that

$$(D \cap \Gamma_{\varepsilon/3}) \sum_{x \in X_i} |h(x, I)| < \frac{\varepsilon}{3} V_H(g; X_i, U_i)$$

and

$$(D \cap \Gamma_{\varepsilon/3}) \sum_{x \in X_i} |\mathcal{H}(I)| < \frac{\varepsilon}{3} V_H(g; X_i, U_i)$$

for all partial divisions  $D$  of  $E$  from  $U_i^\varepsilon$ . Then for a partial division  $D$  of  $E$  from  $U_i^\varepsilon$  we have

$$\begin{aligned} (D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| &\leq (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x, I)| + (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I)| \\ &\quad + (D \setminus \Gamma_\varepsilon) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| \\ &< \frac{\varepsilon}{3} V_H(g; X_i, U_i) + \frac{\varepsilon}{3} V_H(g; X_i, U_i) \\ &\quad + (D \setminus \Gamma_\varepsilon) \sum_{x \in X_i} \frac{\varepsilon}{3} |g(x, I)| \\ &< \varepsilon V_H(g; X_i, U_i). \end{aligned}$$

The proof is complete. □

We now present a generalization of Theorem 3.1 to the decomposable division space.

**Theorem 4.4.** *Let  $\{h_n(x, I)\}$  be a sequence of  $g$ -integrable functions satisfying the  $\alpha g$ -condition with the corresponding primitives  $\mathcal{H}_n$ . If the following conditions are satisfied:*

- (1)  $h_n(x, I) \rightarrow h(x, I)$  for all  $(x, I)$  in the division space, where  $h(x, I): \bigcup_{U \in \mathcal{A}} U \rightarrow \mathbb{R}$ ,
- (2) for every  $i$ , for every  $\varepsilon > 0$  there exists  $U_i^\varepsilon \in \mathcal{A}$  independent of  $n$  with  $U_i^\varepsilon[X_i] \subset U_i[X_i]$  such that for any partial division  $D$  of  $E$  in  $U_i^\varepsilon[X_i]$  we have

$$(D \cap \Gamma_{\varepsilon, n}) \sum |h_n(x, I)| < \varepsilon V_H(g; X_i, U_i)$$

and

$$(D \cap \Gamma_{\varepsilon, n}) \sum |\mathcal{H}_n(I)| < \varepsilon V_H(g; X_i, U_i).$$

Then the following statements hold:

- (i) there exists function  $\mathcal{H}$  such that  $\mathcal{H}_n(I) \rightarrow \mathcal{H}(I)$  for all  $I \in \mathcal{I}(E)$ ,
- (ii)  $h(x, I)$  is  $g$ -integrable with

$$\int_E h(x, I) = \lim_{n \rightarrow \infty} \int_E h_n(x, I).$$

**Proof.** Let  $I_0 \in \mathcal{I}$ ,  $U \in \mathcal{A}$  dividing  $I_0$  and  $D$  a division of  $I_0$ . By the triangle inequality

$$\begin{aligned} (D) \sum |\mathcal{H}_m(I) - \mathcal{H}_n(I)| &\leq (D) \sum |\mathcal{H}_m(I) - h_m(x, I)| \\ &\quad + (D) \sum |h_m(x, I) - h_n(x, I)| \\ &\quad + (D) \sum |h_n(x, I) - \mathcal{H}_n(I)|. \end{aligned}$$

In view of condition (2), the fact that  $h_n(x, I)$  converges for all  $(x, I)$ , and the inequality above it follows that  $\{\mathcal{H}_n(I)\}$  is Cauchy and hence convergent. We let  $\mathcal{H}(I)$  be its limit.

Consider  $X_i$ . Let  $X_i = X$  and  $\varepsilon > 0$ . In view of condition (2), there exists  $U^{\varepsilon/2} \in \mathcal{A}$  independent of  $n$  with  $U^{\varepsilon/2}[X] \subset U[X]$  such that for any partial division  $D$  of  $E$  in  $U^{\varepsilon/2}[X]$  we have

$$(D \cap \Gamma_{\varepsilon/2, n}) \sum |h_n(x, I)| < \frac{\varepsilon}{2} V_H(g; X, U)$$

and

$$(D \cap \Gamma_{\varepsilon/2, n}) \sum |\mathcal{H}_n(I)| < \frac{\varepsilon}{2} V_H(g; X, U).$$

Let  $D$  be any partial division of  $E$  in  $V^\varepsilon[X]$ . By condition (3) there exists a positive integer  $N_1$  such that for any  $n > N_1$  we have

$$|\mathcal{H}(I) - \mathcal{H}_n(I)| < \frac{\varepsilon}{2}|g(x, I)| \quad \forall (x, I) \in D.$$

From condition (1), given the same there exists a positive integer  $N_2 > N_1$  such that for any  $n > N_2$  we have

$$|h(x, I) - h_n(x, I)| < \frac{\varepsilon}{2}|g(x, I)| \quad \forall (x, I) \in D.$$

Finally, in view of conditions (1) and (3) there exists a positive integer  $N_D > N_2$  such that for  $n > N_D$  we have

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2, n}.$$

Then for  $n > N_D$ ,

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum |h(x, I)| &\leq (D \cap \Gamma_\varepsilon) \sum |h(x, I) - h_n(x, I)| + (D \cap \Gamma_\varepsilon) \sum |h_n(x, I)| \\ &< \frac{\varepsilon}{2}(D \cap \Gamma_\varepsilon) \sum |g(x, I)| + \frac{\varepsilon}{2}V_H(g; X, U) \\ &\leq \varepsilon V_H(g; X, U) \end{aligned}$$

and

$$\begin{aligned} (D \cap \Gamma_\varepsilon) \sum |\mathcal{H}(I)| &\leq (D \cap \Gamma_\varepsilon) \sum |\mathcal{H}(I) - \mathcal{H}_n(I)| + (D \cap \Gamma_\varepsilon) \sum |\mathcal{H}_n(I)| \\ &< \frac{\varepsilon}{2}(D \cap \Gamma_\varepsilon) \sum |g(x, I)| + \frac{\varepsilon}{2}V_H(g; X, U) \\ &\leq \varepsilon V_H(g; X, U). \end{aligned}$$

Therefore  $h(x, I)$  is  $g$ -integrable and  $\mathcal{H}(I)$  is its primitive.  $\square$

The double Lusin condition that is used to characterize the Kurzweil-Henstock integral uses point-interval pairs  $(x, I)$  such that  $x$  is contained in  $I$ . It can be shown that the McShane integral has a similar double Lusin characterization with tags  $x$  floating around  $I$  (see [2]). The approximate Perron (AP) integral was given in [5]. A double Lusin characterization of AP integral can also be given. The Henstock-Stieltjes integral was given in [1] with a function  $g$  of bounded variation (BV). In our characterization of the  $g$ -integral it is possible for  $g$  to be  $BVG^*$ , where  $BVG^*$  is defined as follows: Given a subset  $X$  of  $E$ , an interval function  $g$  on  $E$  is said to be  $BV^*(X)$  if there exists a nonnegative number  $M$  such that for any  $X$ -tagged partial division  $D$  of  $E$  we have  $(D) \sum |g(I)| \leq M$  and  $g$  is  $BVG^*$  on  $E$  if  $E = \bigcup_{i=1}^{\infty} X_i$  such that for each  $i$ ,  $g$  is  $BV^*(X_i)$ . It remains to be explored what happens when  $g$  is  $BVG^*$ .

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