Czechoslovak Mathematical Journal

Xi Fu; Bowen Lu

Some characterizations of harmonic Bloch and Besov spaces

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 2, 417–430

Persistent URL: `http://dml.cz/dmlcz/145733`

Czechoslovak Mathematical Journal, 66 (141) (2016), 417–430

SOME CHARACTERIZATIONS OF HARMONIC BLOCH AND BESOV SPACES

Xi Fu, Shaoxing, Bowen Lu, Hangzhou

(Received May 14, 2015)

Abstract. The relationship between weighted Lipschitz functions and analytic Bloch spaces has attracted much attention. In this paper, we define harmonic ω-α-Bloch space and characterize it in terms of

$$\omega((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta})\Big|\frac{f(x)-f(y)}{x-y}\Big|$$

and

$$\omega((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta})\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big|$$

where ω is a majorant. Similar results are extended to harmonic little ω-α-Bloch and Besov spaces. Our results are generalizations of the corresponding ones in G. Ren, U. Kähler (2005).

Keywords: harmonic function; Bloch space; Besov space; majorant

MSC 2010: 32A18, 31B05, 30C20

1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ with $n \geqslant 2$, where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}$ and $\mathrm{d}\sigma$ is the normalized surface measure on the unit sphere $S=\partial\mathbb{B}$. We denote the class of all harmonic functions on the unit ball by $H(\mathbb{B})$. The ball centered at x with radius r will be denoted by $\mathbb{B}(x,r)$.

For each $\alpha>0$, the *harmonic α-Bloch space* $\mathcal{B}^{\alpha}$ consists of all functions $f\in H(\mathbb{B})$ such that

$$\|f\|_{\alpha}=\sup_{x\in\mathbb{B}}(1-|x|^2)^{\alpha}|\nabla f(x)|<\infty,$$

The research was partly supported by NSF of China (No. 11501374) and NSF of Zhejiang (No. LQ14A010006).

and the *little α-Bloch space* $\mathcal{B}_0^\alpha$ consists of the functions $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1^-} \sup_{x\in\mathbb{B}} (1-|x|^2)^\alpha |\nabla f(x)| = 0.$$

The *harmonic Besov space* $\mathcal{B}_p$ is the space of all functions in $H(\mathbb{B})$ for which

$$\int_{\mathbb{B}} (1-|x|^2)^p |\nabla f(x)|^p \,\mathrm{d}\tau(x) < \infty,$$

where $p > n-1$ and $\mathrm{d}\tau(x) = (1-|x|^2)^{-n}\,\mathrm{d}v(x)$ is the invariant measure on $\mathbb{B}$.

Let f be a continuous function in $\mathbb{B}$. If there exists a constant C such that

$$\mathcal{L}_f(x,y) = (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\left|\frac{f(x)-f(y)}{x-y}\right| \leqslant C$$

for any $x, y \in \mathbb{B}$, then we say that f satisfies the *weighted Lipschitz condition*. By means of it, Ren and Kähler [11] obtained the following:

Theorem 1.1. *Let $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}^1$ if and only if it satisfies the weighted Lipschitz condition.*

Moreover, they characterized the spaces $\mathcal{B}_0^1$ and $\mathcal{B}_p$ as follows:

Theorem 1.2. *Let $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}_0^1$ if and only if*

$$\lim_{|x|\to 1^-} \sup_{x,y\in\mathbb{B}, x\neq y} \mathcal{L}_f(x,y) = 0.$$

Theorem 1.3. *Let $f \in H(\mathbb{B})$ and $p \in (2(n-1),\infty)$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \mathcal{L}_f^p(x,y)\,\mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty.$$

Note that if $p \in (1, 2(n-1))$, then the integral condition in Theorem 1.3 forces the function f to be a constant (see [11]). We refer to [5], [6], [8], [12], [13], [17] for the corresponding results in the complex unit ball for holomorphic or $\mathcal{M}$-harmonic functions. See [7], [10], [16], [15], [18] for various characterizations of the Bloch, little Bloch, and Besov spaces in the unit ball of $\mathbb{C}^n$.

Let $\omega\colon [0,\infty) \to [0,\infty)$ be a continuous increasing function with $\omega(0) = 0$. We call ω *a majorant* if $\omega(t)/t$ is non-increasing for $t > 0$ (see [4]). Following [1], the *harmonic ω-α-Bloch space* $\mathcal{B}_\omega^\alpha$ consists of all functions $f \in H(B)$ such that

$$\|f\|_{\omega,\alpha} = \sup_{x\in\mathbb{B}} \omega((1-|x|^2)^\alpha)|\nabla f(x)| < \infty,$$

and the *little ω-α-Bloch space* $\mathcal{B}^\alpha_{\omega,0}$ consists of the functions $f \in \mathcal{B}^\alpha_\omega$ such that

$$\lim_{|x|\to 1^-} \sup_{x\in\mathbb{B}} \omega((1-|x|^2)^\alpha)|\nabla f(x)| = 0.$$

In particular, when $\omega(t) = t$, we remark that the space $\mathcal{B}^\alpha_\omega$ (or $\mathcal{B}^\alpha_{\omega,0}$) is $\mathcal{B}^\alpha$ (or $\mathcal{B}^\alpha_0$).

The main purpose of this paper is to give some characterizations for the spaces $\mathcal{B}^\alpha_\omega, \mathcal{B}^\alpha_{\omega,0}$, and $\mathcal{B}_p$. In Section 2, we collect some known results that will be needed in the proof of our results. Our main results and their proofs are presented in Sections 3 and 4.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means there is a positive constant C such that $C/B \leqslant A \leqslant CB$.

2. Preliminaries

We shall use the following notation: we write $x, y \in \mathbb{R}^n$ in polar coordinates as $x = |x|x'$ and $y = |y|y'$. For any $a, b \in \mathbb{R}^n$, the symmetry lemma shows that

$$\big||a|b - a'\big| = \big||b|a - b'\big|.$$

For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in $\mathbb{B}$. It is an involution of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, and has the form

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{\big||a|x-a'\big|^2}, \quad x \in \mathbb{B}.$$

By a simple computation, we have

$$|\varphi_a(x)| = \frac{|x-a|}{\big||x|a-x'\big|},$$
$$1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{\big||a|x-a'\big|^2},$$

and

$$|J\varphi_a(x)| = \frac{(1-|a|^2)^n}{\big||x|a-x'\big|^{2n}},$$

where $J\varphi_a$ denotes the Jacobian of φ_a.

For any $a \in \mathbb{B}$ and $r \in (0,1)$, we define the *pseudo-hyperbolic ball* with center a and radius r as

$$E(a, r) = \{w \in \mathbb{B} \colon |\varphi_a(w)| < r\}.$$

Clearly, $E(a,r)=\varphi_a(\mathbb{B}(a,r))$.

The following result comes from [11], Lemma 2.1.

Lemma 2.1. *Let* $r\in(0,1)$ *and* $y\in E(x,r)$. *Then* $1-|x|^2\asymp 1-|y|^2\asymp \big||x|y-x'\big|$.

As an application of Lemma 1, we easily get the following:

Corollary 2.1. *Let* $r\in(0,1)$ *and* $\eta=\inf\limits_{a\in\mathbb{B},\,x,y\in E(a,r)}(1-|x|^2)/(1-|y|^2)$. *Then* $\eta\in(0,1)$.

For $f\in H(\mathbb{B})$, we define $\widetilde{\nabla} f(x)$ of f at x by

$$\widetilde{\nabla} f(x)=\nabla(f\circ\varphi_x)(0)$$

for $x\in\mathbb{B}$. We call $|\widetilde{\nabla} f(x)|$ the invariant gradient of f at x by the following proposition, which is proved in [3].

Proposition 2.1. *Let* $f\in H(\mathbb{B})$ *and* $x\in\mathbb{B}$. *Then* $|\widetilde{\nabla} f(x)|=(1-|x|^2)|\nabla f(x)|$ *and*

$$|\widetilde{\nabla}(f\circ\varphi)(x)|=|(\widetilde{\nabla} f)\circ\varphi(x)|$$

for any Möbius transformation φ.

In order to prove our main results, we need the following lemma, which follows from a discussion similar to the proof of [9], Lemma 2.5.

Lemma 2.2. *Suppose that* $f\colon \overline{\mathbb{B}}(a,r)\to\mathbb{R}$ *is continuous and harmonic in* $\mathbb{B}(a,r)$. *Then*

$$|\nabla f(a)|\leqslant\frac{n^{3/2}}{r}\int_S|f(a+r\zeta)-f(a)|\,\mathrm{d}\sigma(\zeta).$$

P r o o f. Without loss of generality, we assume that $a=0$ and $f(0)=0$. Then for $x\in\mathbb{B}$

$$f(x)=\int_S K(x,\zeta)f(r\zeta)\,\mathrm{d}\sigma(\zeta),$$

where

$$K(x,\zeta)=\frac{r^{n-2}(r^2-|x|^2)}{|x-r\zeta|^n}$$

is the Poisson kernel for the ball. By a simple calculation,

$$\frac{\partial}{\partial x_j}K(x,\zeta)=r^{n-2}\Big[\frac{-2x_j}{|x-r\zeta|^n}-\frac{n(r^2-|x|^2)(x_j-r\zeta_j)}{|x-r\zeta|^{n+2}}\Big].$$

Hence, we have

$$
\begin{aligned}
|\nabla f(0)| &= \left[\sum_{j=1}^{n}\left|\int_S \frac{\partial}{\partial x_j}K(0,\zeta)f(r\zeta)\,\mathrm{d}\sigma(\zeta)\right|^2\right]^{1/2} \\
&\leqslant \sum_{j=1}^{n}\left|\int_S \frac{\partial}{\partial x_j}K(0,\zeta)f(r\zeta)\,\mathrm{d}\sigma(\zeta)\right| \\
&\leqslant \int_S |f(r\zeta)|\sum_{j=1}^{n}\left|\frac{\partial}{\partial x_j}K(0,\zeta)\right|\mathrm{d}\sigma(\zeta) \\
&\leqslant \frac{n^{3/2}}{r}\int_S |f(r\zeta)|\,\mathrm{d}\sigma(\zeta).
\end{aligned}
$$

□

Lemma 2.3 ([11])**.** *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$
\int_{\mathbb{B}} \frac{(1-|y|^2)^{\alpha}}{|x|y-x'|^{n+\alpha+\beta}}\,\mathrm{d}v(y) \asymp \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log\dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}
$$

We end this section with two inequalities which will be used in the sequel.

Lemma 2.4. *Let $\omega(t)$ be a majorant and $u \in (0,1]$, $v \in (1,\infty)$. Then for $t \in (0,\infty)$,*

$$
\begin{aligned}
\omega(ut) &\geqslant u\omega(t), \\
\omega(vt) &\leqslant v\omega(t).
\end{aligned}
$$

Lemma 2.5. *Let $a, b > 0$, $0 < s < 1$. Then $sa + (1-s)b \geqslant a^s b^{1-s}$.*

3. Harmonic Bloch space

In this section, we give some characterizations of the spaces $\mathcal{B}_{\omega}^{\alpha}$ and $\mathcal{B}_{\omega,0}^{\alpha}$.

Theorem 3.1. *Let $r \in (0,1)$, $f \in H(\mathbb{B})$, $0 < \beta \leqslant \alpha$. Then $f \in \mathcal{B}_{\omega}^{\alpha}$ if and only if*

$$
L_{\omega,f} = \sup_{y \in E(x,r), x \neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\left|\frac{f(x)-f(y)}{x-y}\right| < \infty.
$$

Proof. We follow the proof of Theorem 3.1 in [11]. First, we prove the sufficiency. Let $f \in H(\mathbb{B})$. For each $x \in \mathbb{B}$, by Lemma 2.2,

$$|\nabla f(x)| \leqslant \frac{C}{\varrho} \int_S |f(x+\varrho\zeta) - f(x)| \,\mathrm{d}\sigma(\zeta),$$

where $\varrho < r(1-|x|^2)/2$. A straightforward calculation shows that $E(x,r)$ is a Euclidean ball with center $(1-r^2)x/(1-r^2|x|^2)$ and radius $(1-|x|^2)r/(1-r^2|x|^2)$. Note that if $y \in \overline{\mathbb{B}(x,\varrho)}$, then

$$|\varphi_x(y)| = \frac{|x-y|}{\big||x|y - x'\big|} \leqslant \frac{|x-y|}{1-|x|} \leqslant \frac{2|x-y|}{1-|x|^2} < r.$$

It follows immediately that $\overline{\mathbb{B}(x,\varrho)} \subset E(x,r)$ and

$$|\nabla f(x)| \leqslant \frac{CL_{\omega,f}}{\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)}.$$

By letting $y \to x$, $f \in \mathcal{B}^{\alpha}_{\omega}$.

Conversely, let $f \in \mathcal{B}^{\alpha}_{\omega}$ and for any $y \in E(x,r)$, $y \neq x$,

$$\begin{aligned}
|f(x) - f(y)| &= \left|\int_0^1 \frac{\mathrm{d}f}{\mathrm{d}s}(sx+(1-s)y)\,\mathrm{d}s\right| \\
&\leqslant \sum_{k=1}^{n} \left|(x_k - y_k)\int_0^1 \frac{\partial f}{\partial x_k}(sx+(1-s)y)\,\mathrm{d}s\right| \\
&\leqslant \sqrt{n}\,|x-y| \int_0^1 |\nabla f(sx+(1-s)y)|\,\mathrm{d}s \\
&\leqslant C|x-y|\|f\|_{\omega,\alpha} \int_0^1 \frac{\mathrm{d}s}{\omega((1-|sx+(1-s)y|^2)^{\alpha})}.
\end{aligned}$$

Since for $s \in [0,1]$,

$$\begin{aligned}
1 - |sx+(1-s)y|^2 &\geqslant 1 - |sx+(1-s)y| \\
&\geqslant s(1-|x|) + (1-s)(1-|y|) \\
&\geqslant s\Big(\frac{1-|x|^2}{2}\Big) + (1-s)\Big(\frac{1-|y|^2}{2}\Big) \\
&\geqslant \frac{1}{2}(1-|x|^2)^s(1-|y|^2)^{1-s}
\end{aligned}$$

and $1-|y|^2 \geqslant \eta(1-|x|^2)$ by Lemma 2.5 and Corollary 2.1, we infer that

$$\begin{aligned}
\Big|\frac{f(x)-f(y)}{x-y}\Big| &\leqslant C\int_0^1 \frac{\mathrm{d}s}{\omega\big(2^{-\alpha}(1-|x|^2)^{\alpha s}(1-|y|^2)^{\alpha-\alpha s}\big)}\\
&\leqslant C\int_0^1 \frac{\mathrm{d}s}{\omega\big(2^{-\alpha}(1-|x|^2)^{\alpha}\eta^{\alpha-\alpha s}\big)}\\
&\leqslant \frac{C}{\omega\big((1-|x|^2)^{\alpha}\big)}\int_0^1 \frac{2^{\alpha}\,\mathrm{d}s}{\eta^{\alpha-\alpha s}}\\
&\leqslant \frac{C}{\omega\big((1-|x|^2)^{\alpha}\big)}
\end{aligned}$$

by Lemma 2.4 for $y \in E(x,r)$, $y \neq x$. Thus,

$$\sup_{y\in E(x,r), x\neq y} \omega\big((1-|x|^2)^{\alpha}\big)\Big|\frac{f(x)-f(y)}{x-y}\Big| < \infty.$$

For each $y \in E(x,r)$,

$$(1-|x|^2)^{\alpha} = (1-|x|^2)^{\beta}(1-|x|^2)^{\alpha-\beta} \geqslant (1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\eta^{\alpha-\beta}.$$

By Lemma 2.4, we deduce that

$$\omega\big((1-|x|^2)^{\alpha}\big) \geqslant \eta^{\alpha-\beta}\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big),$$

from which we see that $L_{\omega,f} < \infty$. □

Theorem 3.2. *Let $r \in (0,1)$, $f \in \mathcal{B}_{\omega}^{\alpha}$. Then $f \in \mathcal{B}_{\omega,0}^{\alpha}$ if and only if*

$$\text{(1)}\qquad \lim_{|x|\to 1^-} \sup_{y\in E(x,r),\, x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{x-y}\Big| = 0.$$

P r o o f. We follow the proof of Theorem 3.2 in [11]. To prove sufficiency, assume that (1) holds. Then for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sup_{y\in E(x,r),\, x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{x-y}\Big| < \varepsilon,$$

whenever $\delta < |x| < 1$. By an argument similar to that in the proof of Theorem 3.1, we have

$$\omega\big((1-|x|^2)^{\alpha}\big)|\nabla f(x)| < C \sup_{y\in E(x,r),\, x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{x-y}\Big| < C\varepsilon,$$

whenever $\delta < |x| < 1$. Hence

$$\lim_{|x|\to 1^-} \omega\big((1-|x|^2)^\alpha\big)|\nabla f(x)| = 0.$$

To prove necessity, we assume that $f \in \mathcal{B}^\alpha_{\omega,0}$. For $\lambda \in (0,1)$, let $f_\lambda = f(\lambda x)$. By the proof of Theorem 3.1, we have

$$\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{(f-f_\lambda)(x)-(f-f_\lambda)(y)}{x-y}\Big| \leqslant C\|f-f_\lambda\|_{\omega,\alpha}$$

and

$$\begin{aligned}
&\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f_\lambda(x)-f_\lambda(y)}{x-y}\Big| \\
&\quad < \frac{\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)}{\omega\big((1-|\lambda x|^2)^\alpha\big)}\omega\big((1-|\lambda x|^2)^\alpha\big)\Big|\frac{f(\lambda x)-f(\lambda y)}{\lambda x-\lambda y}\Big| \\
&\quad \leqslant C\frac{\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)}{\omega((1-|\lambda x|^2)^\alpha)}\|f\|_{\omega,\alpha}
\end{aligned}$$

for all $y \in E(x,r)$. By the triangle inequality, we have

$$\begin{aligned}
&\sup_{y\in E(x,r),\, x\neq y} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{x-y}\Big| \\
&\qquad\qquad \leqslant C\|f-f_\lambda\|_{\omega,\alpha} + \frac{\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)}{\omega\big((1-|\lambda x|^2)^\alpha\big)}\|f\|_{\omega,\alpha}.
\end{aligned}$$

In the above inequality, first by letting $|x| \to 1^-$ and then letting $\lambda \to 1^-$, we obtain the desired result. $\square$

Remark 3.1. It is worth noting that for $\omega(t) = t$, Li and Wulan [8] obtained the holomorphic version of Theorems 3.1 and 3.2 in the unit ball of $\mathbb{C}^n$.

Section 2 gives that for any $x, y \in \mathbb{B}$, $|x-y| \leqslant \big||x|y - x'\big|$. Motivated by this fact, we remove the restriction $y \in E(x,r)$, replace $x-y$ by $|x|y-x'$ in Theorem 3.1 and obtain the following:

Theorem 3.3. *Let $f \in H(\mathbb{B})$, $0 \leqslant \beta < 1$, $\beta \leqslant \alpha < 1+\beta$. Then $f \in \mathcal{B}^\alpha_\omega$ if and only if*

$$L = \sup_{x,y\in\mathbb{B},\, x\neq y} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big| < \infty. \tag{2}$$

P r o o f. Assume that (2) holds. Fix $r \in (0,1)$. It follows from [11] that

$$|\nabla f(x)| \leqslant \frac{C}{(1-|x|^2)} \int_{E(x,r)} |f(y)| \,\mathrm{d}\tau(y).$$

Fixing $x \in \mathbb{B}$ and replacing f by $f - f(x)$, we have

$$\omega\big((1-|x|^2)^\alpha\big)|\nabla f(x)| \leqslant \frac{C\omega\big((1-|x|^2)^\alpha\big)}{(1-|x|^2)} \int_{E(x,r)} |f(y)-f(x)| \,\mathrm{d}\tau(y).$$

By Lemmas 2.1, 2.4 and Corollary 2.1,

$$\begin{aligned}\omega\big((1-|x|^2)^\alpha\big)|\nabla f(x)| &\leqslant C\eta^{\beta-\alpha} \int_{E(x,r)} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big| \,\mathrm{d}\tau(y)\\ &\leqslant CL\eta^{\beta-\alpha}\int_{E(x,r)} \mathrm{d}\tau = CL\eta^{\beta-\alpha}\tau(\mathbb{B}(0,r)).\end{aligned}$$

Since $\tau(\mathbb{B}(0,r)) = n\int_0^r t^{n-1}(1-t^2)^{-n}\,\mathrm{d}t$ is a constant, we see that

$$\sup_{x\in\mathbb{B}} \omega\big((1-|x|^2)^\alpha\big)|\nabla f(x)| < \infty.$$

Hence $f \in \mathcal{B}^\alpha_\omega$.

Conversely, assume that $f \in \mathcal{B}^\alpha_\omega$. We argue as in the proof of Theorem 3.1. Since for $x, y \in \mathbb{B}$, $s \in [0,1]$,

$$\begin{aligned}\big(1-|sx+(1-s)y|^2\big)^\alpha &\geqslant \Big(s\Big(\frac{1-|x|^2}{2}\Big) + (1-s)\Big(\frac{1-|y|^2}{2}\Big)\Big)^\alpha\\ &\geqslant \Big(\frac{s}{2}\Big)^\beta\Big(\frac{1-s}{2}\Big)^{\alpha-\beta}(1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta},\end{aligned}$$

we get

$$\begin{aligned}\Big|\frac{f(x)-f(y)}{x-y}\Big| &\leqslant C\int_0^1 \frac{\mathrm{d}s}{\omega\big((1-|sx+(1-s)|y|^2)^\alpha\big)}\\ &\leqslant C\int_0^1 \frac{\mathrm{d}s}{\omega\big((s/2)^\beta((1-s)/2)^{\alpha-\beta}(1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)}\\ &\leqslant \frac{C}{\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)}\int_0^1 \frac{\mathrm{d}s}{s^\beta(1-s)^{\alpha-\beta}}\\ &\leqslant \frac{C}{\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)},\end{aligned}$$

where the last integral converges since $\alpha < 1+\beta$. Thus

$$\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big| < \infty.$$

$\square$

Similarly, we can prove the following.

Theorem 3.4. *Let $f \in \mathcal{B}^{\alpha}_{\omega}$, $0 \leqslant \beta < 1$, $\beta \leqslant \alpha < 1+\beta$. Then $f \in \mathcal{B}^{\alpha}_{\omega,0}$ if and only if*

$$\lim_{|x|\to 1^-} \sup_{x,y\in\mathbb{B},\, x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big| = 0.$$

4. Harmonic Besov space

In this section, we show some characterizations of Besov space $\mathcal{B}_p$ for $H(\mathbb{B})$. First, we generalize Theorem 1.3 into the following form.

Theorem 4.1. *Let $f \in H(\mathbb{B})$ and $p \in (2(n-1),\infty)$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}} (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big|^p \,\mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty. \tag{3}$$

Proof. Assume that $f \in \mathcal{B}_p$. Since

$$\frac{1}{\big||x|y-x'\big|} \leqslant \frac{1}{|x-y|}$$

for all $x, y \in \mathbb{B}$, it follows from Theorem 1.3 that (3) holds.

Conversely, assume that (3) holds. Fix $r \in (0,1)$. From the proof of Theorem 3.3, by taking $\omega(t)=t$, $\alpha = 1$ and $\beta = 1/2$, we have

$$(1-|x|^2)|\nabla f(x)| \leqslant C \int_{E(x,r)} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big| \,\mathrm{d}\tau(y).$$

As an application of Hölder's inequality,

$$\begin{aligned}(1-|x|^2)^p|\nabla f(x)|^p &\leqslant C \int_{E(x,r)} (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big|^p \,\mathrm{d}\tau(y)\\ &\leqslant C \int_{\mathbb{B}} (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big|^p \,\mathrm{d}\tau(y).\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\mathbb{B}} (1-|x|^2)^p|\nabla f(x)|^p \,\mathrm{d}\tau(x)\\ &\quad\leqslant C \int_{\mathbb{B}}\int_{\mathbb{B}} (1-|x|^2)^{p/2}(1-|y|^2)^{p/2}\Big|\frac{f(x)-f(y)}{|x|y-x'}\Big|^p \,\mathrm{d}\tau(x)\,\mathrm{d}\tau(y).\end{aligned}$$

The result follows. □

An immediate consequence of Theorems 3.1 and 4.3 is the following corollary.

Corollary 4.1. *Let $f \in H(\mathbb{B})$, $p \in (2(n-1), \infty)$ and $0 \leqslant k \leqslant p$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}} \int_{\mathbb{B}} (1-|x|^2)^{p/2} (1-|y|^2)^{p/2} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^k |x-y|^{p-k}} \,\mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty.$$

Second, we give a new characterization of $\mathcal{B}_p$ in terms of a double integral of the function $|f(x)-f(y)|^p / \big||x|y-x'\big|^{2n}$.

Theorem 4.2. *Let $f \in H(\mathbb{B})$ and $p \in (n-1, \infty)$. Then $f \in \mathcal{B}_p$ if and only if*

$$I = \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{2n}} \,\mathrm{d}v(x)\,\mathrm{d}v(y) < \infty. \tag{4}$$

P r o o f. Assume that $f \in \mathcal{B}_p$. By the formula $\big||x|\varphi_x(u)-x'\big|^2 = (1-|x|^2) \times (1-|\varphi_x(u)|^2)/(1-|u|^2)$ and by making the change of variables $y = \varphi_x(u)$ we have

$$\begin{aligned} I &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f\circ\varphi_x(0) - f\circ\varphi_x(u)|^p}{\big||x|\varphi_x(u)-x'\big|^{2n}} |J\varphi_x(u)| \,\mathrm{d}v(x)\,\mathrm{d}v(u) \\ &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f\circ\varphi_x(0) - f\circ\varphi_x(u)|^p}{(1-|x|^2)^n(1-|\varphi_x(u)|^2)^n} \frac{(1-|u|^2)^n(1-|x|^2)^n}{\big||x|u-x'\big|^{2n}} \,\mathrm{d}v(u)\,\mathrm{d}v(x) \\ &= \int_{\mathbb{B}} \int_{\mathbb{B}} |f\circ\varphi_x(0) - f\circ\varphi_x(u)|^p \,\mathrm{d}v(u)\,\mathrm{d}\tau(x) \\ &\leqslant C \int_{\mathbb{B}} \mathrm{d}\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla}(f\circ\varphi_x)(u)|^p \,\mathrm{d}v(u). \end{aligned}$$

The last inequality follows from [2], Theorem 1.3, and Proposition 2.1.

Since $|\widetilde{\nabla}(f\circ\varphi_x)(u)| = |\widetilde{\nabla} f(\varphi_x(u))|$, changing variables again leads to

$$\begin{aligned} I &\leqslant C \int_{\mathbb{B}} \mathrm{d}\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla}(f\circ\varphi_x)(u)|^p \,\mathrm{d}v(u) \\ &\leqslant C \int_{\mathbb{B}} \mathrm{d}\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla} f(w)|^p \frac{(1-|x|^2)^n}{\big||x|w-x'\big|^{2n}} \,\mathrm{d}v(w). \end{aligned}$$

It follows from Fubini's theorem and Lemma 2.3 that

$$I \leqslant C \int_{\mathbb{B}} |\widetilde{\nabla} f(w)|^p \,\mathrm{d}\tau(w) = C \int_{\mathbb{B}} (1-|w|^2)^p |\nabla f(w)|^p \,\mathrm{d}\tau(w).$$

For the converse, we assume that (4) holds. For $x \in \mathbb{B}$, from [11],

$$(1-|x|^2)|\nabla f(x)| \leqslant C \int_{E(x,r)} |f(x)-f(y)| \,\mathrm{d}\tau(y).$$

Applying Hölder's inequality and Lemma 2.2,

$$\int_{\mathbb{B}} (1-|x|^2)^p |\nabla f(x)|^p \,\mathrm{d}\tau(x) \leqslant C \int_{\mathbb{B}} \int_{E(x,r)} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{2n}} \,\mathrm{d}v(x)\,\mathrm{d}v(y) \leqslant I,$$

from which we see that $f \in \mathcal{B}_p$. $\square$

As an application of Theorem 4.2, we end this section with the following result, which can be regarded as an extension of [6], Theorem 1, into the harmonic setting.

Theorem 4.3. *Let $f \in H(\mathbb{B})$ and $p \in (n-1,\infty)$, $n \leqslant \alpha, \beta < \infty$. Then $f \in \mathcal{B}_p$ if and only if*

$$J = \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{\alpha+\beta}} (1-|x|^2)^{\alpha} (1-|y|^2)^{\beta} \,\mathrm{d}\tau(x)\,\mathrm{d}\tau(y) < \infty. \tag{5}$$

Proof. Similarly as in the proofs of the above theorems, we have

$$\begin{aligned}
(1-|x|^2)^p |\nabla f(x)|^p &\leqslant C \int_{E(x,r)} |f(x)-f(y)|^p \,\mathrm{d}\tau(y) \\
&\leqslant C \int_{E(x,r)} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{\alpha+\beta}} (1-|x|^2)^{\alpha}(1-|y|^2)^{\beta} \,\mathrm{d}\tau(y) \\
&\leqslant C \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{\alpha+\beta}} (1-|x|^2)^{\alpha}(1-|y|^2)^{\beta} \,\mathrm{d}\tau(y),
\end{aligned}$$

from which $f \in \mathcal{B}_p$.

Now, we prove the converse. Suppose that $f \in \mathcal{B}_p$. Then

$$\begin{aligned}
J &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{2n}} \frac{(1-|x|^2)^{\alpha-n}(1-|y|^2)^{\beta-n}}{\big||x|y-x'\big|^{\alpha+\beta-2n}} \,\mathrm{d}v(x)\,\mathrm{d}v(y) \\
&\leqslant C \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{\big||x|y-x'\big|^{2n}} \,\mathrm{d}v(x)\,\mathrm{d}v(y)
\end{aligned}$$

by Lemma 2.3 applied twice. Following Theorem 4.2, we get that $J < \infty$, as desired. The proof is completed. $\square$

Remark 4.1. After submission, the authors have learned of the interesting paper [14], in which some more general results on the characterizations of harmonic Besov spaces are presented. Specifically, Theorem 4.1 in this note is Corollary 1.6 of [14], which is a special case of Theorem 1.2 there. Theorem 4.2 is contained in Corollary 6.2 of [14], which is a special case of Theorem 6.1 there. Theorem 4.3 is closely related to Theorem 6.6 of [14]. However, the methods of proof in these two works are quite different from each other.

Acknowledgment. The authors are grateful to the referee for drawing their attention to the results in [14] as well as for many helpful comments.

References

[1] *S. Chen, S. Ponnusamy, A. Rasila*: On characterizations of Bloch-type, Hardy-type and Lipschitz-type spaces. Math. Z. *279* (2015), 163–183.
[2] *B. R. Choe, H. Koo, H. Yi*: Derivatives of harmonic Bergman and Bloch functions on the ball. J. Math. Anal. Appl. *260* (2001), 100–123.
[3] *E. S. Choi, K. Na*: Characterizations of the harmonic Bergman space on the ball. J. Math. Anal. Appl. *353* (2009), 375–385.
[4] *K. M. Dyakonov*: Equivalent norms on Lipschitz-type spaces of holomorphic functions. Acta Math. *178* (1997), 143–167.
[5] *M. Jevtić, M. Pavlović*: On $\mathcal{M}$-harmonic Bloch space. Proc. Amer. Math. Soc. *123* (1995), 1385–1392.
[6] *S. Li*: Characterizations of Besov spaces in the unit ball. Bull. Korean Math. Soc. *49* (2012), 89–98.
[7] *S. Li, S. Stević*: Some characterizations of the Besov space and the α-Bloch space. J. Math. Anal. Appl. *346* (2008), 262–273.
[8] *S. Li, H. Wulan*: Characterizations of α-Bloch spaces on the unit ball. J. Math. Anal. Appl. *343* (2008), 58–63.
[9] *M. Mateljević, M. Vuorinen*: On harmonic quasiconformal quasi-isometries. J. Inequal. Appl. 2010 (2010), Article ID 178732, 19 pages.
[10] *M. Nowak*: Bloch space and Möbius invariant Besov spaces on the unit ball of $\mathbb{C}^n$. Complex Variable Theory Appl. *44* (2001), 1–12.
[11] *G. Ren, U. Kähler*: Weighted Lipschitz continuity and harmonic Bloch and Besov spaces in the real unit ball. Proc. Edinb. Math. Soc., II. Ser. *48* (2005), 743–755.
[12] *G. Ren, C. Tu*: Bloch space in the unit ball of $\mathbb{C}^n$. Proc. Am. Math. Soc. *133* (2005), 719–726.
[13] *W. Rudin*: Function Theory in the Unit Ball of $\mathbb{C}^n$. Grundlehren der mathematischen Wissenschaften 241, Springer, New York, 1980.
[14] *A. E. Üreyen*: An estimate of the oscillation of harmonic reproducing kernels with applications. J. Math. Anal. Appl. *434* (2016), 538–553.
[15] *R. Yoneda*: A characterization of the harmonic Bloch space and the harmonic Besov spaces by an oscillation. Proc. Edinb. Math. Soc., II. Ser. *45* (2002), 229–239.
[16] *R. Yoneda*: Characterizations of Bloch space and Besov spaces by oscillations. Hokkaido Math. J. *29* (2000), 409–451.
[17] *R. Zhao*: A characterization of Bloch-type spaces on the unit ball of $\mathbb{C}^n$. J. Math. Anal. Appl. *330* (2007), 291–297.

[18] *K. Zhu*: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Mathematics 226, Springer, New York, 2005.

Authors' addresses: Xi Fu, Department of Mathematics, Shaoxing University, No. 508, West Huancheng Road, Shaoxing 312000, Zhejiang, P. R. China, e-mail: fuxi1984@hotmail.com; Bowen Lu, School of International Business, Zhejiang International Studies University, No. 140, Wensan Road, Hangzhou 310012, Zhejiang, P. R. China, e-mail: sxlujingjing@163.com.