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On a question of $C_c(X)$
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Abstract. In this short article we answer the question posed in Ghadermazi M., Karamzadeh O.A.S., Namdari M., On the functionally countable subalgebra of $C(X)$, Rend. Sem. Mat. Univ. Padova 129 (2013), 47–69. It is shown that $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $\nu_0X$ is functionally countable. For a strongly zero-dimensional space $X$, this is equivalent to say that $X$ is functionally countable. Hence for every $P$-space it is equivalent to pseudo-$\aleph_0$-compactness.

Keywords: zero-dimensional space; strongly zero-dimensional space; $\mathbb{N}$-compact space; Banaschewski compactification; character; ring homomorphism; functionally countable subring; functional separability

Classification: Primary 54C30, 54D35, 46E25; Secondary 54D60, 54C40

1. Introduction

For topological spaces $X$ and $E$, the space $X$ is called $E$-completely regular provided that it can be topologically embedded into the product space $E^\kappa$, for some cardinal number $\kappa$. If we consider the particular case where $E = \mathbb{N}$ (i.e., the set of all natural numbers with the discrete topology), then one can verify that $X$ is zero-dimensional and Hausdorff (i.e., a $T_2$-space with a base consisting of clopen sets) if and only if $X$ is $\mathbb{N}$-completely regular. We also recall that a topological space $X$ is $E$-compact if it is embeddable as a closed subset into the product space $E^\tau$ for some cardinal number $\tau$. The notions of $E$-completely regular and $E$-compact spaces were introduced by Mrówka and Engelking in [7], Mrówka continued to investigate the properties of such spaces in [17], [19]. For a special case $E = \mathbb{N}$, see [15], [16]. He also wrote a survey on $E$-compact spaces in [18]. The reader is referred to [22] for terminology and notions about $E$-compactness. The following theorem is needed in the sequel, see e.g., [18, Theorem 4.14].

Theorem 1.1. For every $E$-completely regular space $X$ there exists a space $v_E(X)$ such that:

(a) $v_E(X)$ is $E$-compact and it contains $X$ as a dense subspace;

(b) every continuous function $f : X \to Y$, where $Y$ is an arbitrary $E$-compact space, admits a continuous extension $f^* : v_E(X) \to Y$.

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Any $E$-compact space $Z$ containing $X$ as a dense subspace with the properties (a) and (b) of Theorem 1.1, is homeomorphic with $v_E(X)$. As a special case, for every zero-dimensional space $X$ there exists an $\mathbb{N}$-compact space $v_0X$ such that every continuous function $f : X \to Y$, with $Y$ an $\mathbb{N}$-compact space, has a unique extension $f^* : v_0X \to Y$. Also, we could replace an arbitrary $\mathbb{N}$-compact space $Y$ with the fixed discrete space $Z$ (i.e., the set of all integer numbers), and have the following characterization of $\mathbb{N}$-compactification of a zero-dimensional space, see e.g., [22, 5.4(d)].

**Theorem 1.2.** An $\mathbb{N}$-compact extension $T$ of a zero-dimensional space $X$ is homeomorphic with $v_0X$ if and only if for each continuous function $f : X \to \mathbb{Z}$, there exists $F : T \to \mathbb{Z}$ such that $F|_X = f$.

A topological space $X$ is called strongly zero-dimensional if $X$ is a nonempty completely regular Hausdorff space and every finite cozero-set cover $\{U_i\}_{i=1}^k$ of the space $X$ has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$, whenever $i \neq j$. Equivalently, a nonempty completely regular Hausdorff space $X$ is strongly zero-dimensional if and only if for every pair $A, B$ of completely separated subsets of the space $X$, there exists a clopen set $U$ in $X$ such that $A \subseteq U \subseteq X \setminus B$, see e.g., [10].

It is well-known that every strongly zero-dimensional realcompact space is $\mathbb{N}$-compact. It is also easy to see that every countable subset of $\mathbb{R}$ is Lindelöf and zero-dimensional and hence is strongly zero-dimensional. Therefore every countable subset of $\mathbb{R}$ is $\mathbb{N}$-compact. So we have the following trivial lemma.

**Lemma 1.3.** Let $X$ be a zero-dimensional Hausdorff space. For each continuous function $f : X \to \mathbb{R}$ with countable image, there exists an extension $f^* : v_0X \to \mathbb{R}$ such that the image of $f^*$ is equal to the image of $f$.

For an arbitrary completely regular Hausdorff space $X$, we denote by $C_c(X)$ the set of all continuous real-valued functions on $X$ with countable image. The set $C_c(X)$ forms a subring of $C(X)$ (i.e., the set of all continuous real valued functions on $X$) with pointwise addition and multiplication. Ghadermazi, Karamzadeh and Namdari showed in [9] that for a completely regular Hausdorff space $X$ there exists a zero-dimensional space $Y$ such that $C_c(X) \cong C_c(Y)$. In view of this fact, in the present article we restrict our attention to zero-dimensional spaces. In the same article, the authors required to find an example of a zero-dimensional space $X$ for which $C_c(X)$ is not isomorphic to any $C(Y)$. They remarked that for an uncountable discrete space $X$, the ring $C_c(X)$ is not isomorphic to any ring of continuous functions. They also remained an unsettled question to determine completely regular Hausdorff spaces $X$ for which $C_c(X)$ is isomorphic to some ring of continuous functions. The reader may consult [9] for all prerequisites and unfamiliar notions for the functionally countable algebras.

In the present article, we give a complete answer to the aforementioned question and by virtue of it, we will show that not only for discrete spaces but for various kinds of zero-dimensional spaces $X$ the algebra $C_c(X)$ is not isomorphic to any ring of continuous functions.
We remind the reader that $\beta_0 X$ is the unique (up to homeomorphism) zero-dimensional compact space which contains $X$ as a dense subset such that every continuous two-valued function $f : X \to \{0, 1\}$ has a unique extension to $\beta_0 X$.

Bhattacharjee, Knox and McGovern have found that the maximal ideal space of $C_c(X)$ is homeomorphic with $\beta_0 X$, see [4]. They remarked that the proof of this fact can be modeled after [11, Theorem 5.1].

We conclude this section with the following proposition. We also remind the reader that a subset $Y \subseteq X$ is $C_c$-embedded in $X$ if for each $f \in C_c(Y)$, there exists $F \in C_c(X)$ such that $F|_Y = f$.

**Proposition 1.4.** An $\mathbb{N}$-compact extension $T$ of a zero-dimensional space $X$ is homeomorphic with $\upsilon_0 X$ if and only if $X$ is $C_c$-embedded in $T$.

**Proof:** By Lemma 1.3, the necessity is trivial. For sufficiency, it is enough to show that for each continuous function $f : X \to \mathbb{N}$, there exists a continuous function $F : T \to \mathbb{N}$ such that $F|_X = f$. Since $f \in C_c(X)$, there exists $h \in C_c(T)$ such that $h|_X = f$. The subset $h(T)$ of $\mathbb{R}$ is countable. Hence there exists an increasing sequence $r_1 < r_2 < \cdots < r_n < \cdots$ such that for each $n \in \mathbb{N}$, $r_n \not\in h(T)$ and

$$r_n < n < r_{n+1}.$$  

Define $W_1 = F^{-1}(-\infty, r_2)$ and for each $n > 2$, $W_n = F^{-1}(r_n, r_{n+1})$. Each $W_n$ is clopen and $T = \bigcup_{n \in \mathbb{N}} W_n$. Define the map $k : T \to \mathbb{N}$ to be such that for each $n \in \mathbb{N}$, $k|_{W_n} = n$. Clearly $k$ is continuous and $k|_X = f$. So we are done. \qed

2. When is $C_c(X) \cong C(Y)$?

We recall that a completely regular Hausdorff topological space $X$ is functionally countable if each continuous real-valued function on $X$ has countable image. It was mentioned in [14] that $X$ is functionally countable if and only if every second countable continuous image of $X$ is countable. Moreover, $X$ is functionally countable if and only if every metrizable image of $X$ is countable. All functionally countable spaces are zero-dimensional and pseudo-$\aleph_1$-compact space (i.e., every discrete family of non empty open sets is at most countable). In the literature of rings of continuous functions, functional countability appeared in [1], [2], [3], [13], [21], [23]. To achieve our main theorem we need the following proposition. Before, we recall that a character on $C_c(X)$ is a non zero algebra homomorphism from $C_c(X)$ onto $\mathbb{R}$. For example, the evaluation $\delta_x$ at a point $x \in X$, which is defined by $\delta_x(f) = f(x)$, for all $f \in C_c(X)$, is a character on $C_c(X)$. The following proposition is basic for the rest of this section. We remind the reader that there are several proofs for determining all the characters on $C(X)$, whenever $X$ is a realcompact space, see e.g., [8], [5]. We adapt the latest proof which appeared in [5] to determine all the characters on $C_c(X)$, whenever $X$ is an $\mathbb{N}$-compact space. The referee noted that the procedure of the proof of the following proposition is somehow similar to the proof of [12, Proposition 2.7].
Proposition 2.1. Let $X$ be an $\mathbb{N}$-compact space and $\Phi$ be a character on $C_c(X)$. There exists a unique $x \in X$ such that $\Phi(f) = f(x)$, for all $f \in C_c(X)$.

Proof: (Uniqueness) Since for each two distinct elements $x, y$ in $X$ there exists a clopen set $U$ such that $x \in U$ and $y \notin U$, the uniqueness of the point $x$ is trivial.

(Existence) For every $f \in C_c(X)$, $f(X)$ is a countable subset of $\mathbb{R}$ and therefore is $\mathbb{N}$-compact. The space $X$ is $\mathbb{N}$-compact and since $C_c(X)$ separates points from closed sets, $X$ can be embedded as a closed subset of the product space $\prod_{f \in C_c(X)} f(X)$. Thus, we can identify each point of $X$ with the point $(f(x))_{f \in C_c(X)}$ of the product space. For every $f \in C_c(X)$ consider the projection $\pi_f \in C_c(X)$ where

$$\pi_f(x) = f(x),$$

for $x = (f(x))_{f \in C_c(X)}$. Note that for each $f \in C_c(X)$, $\Phi(f) \in f(X)$. To see this, we have $\Phi(f - \Phi(f)) = 0$ and therefore $f - \Phi(f)$ is a nonunit in $C_c(X)$ and hence there exists $t \in X$ such that $f(t) = \Phi(f)$.

Consider the point

$$z = (\Phi(f))_{f \in C_c(X)} \in \prod_{f \in C_c(X)} f(X).$$

First we claim that $z \in X$. Otherwise, since $X$ is a closed set in $\prod_{f \in C_c(X)} f(X)$, there would exist $\epsilon > 0$ and a nonempty finite subset $J \subseteq C_c(X)$ such that the set

$$\Omega = \bigcap_{g \in J} \{(x_f)_{f \in C_c(X)} : |x_g - \Phi(\pi_g)| < \epsilon\}$$

is empty. Define

$$k = \sum_{g \in J} (\pi_g - \Phi(\pi_g))^2 \in C_c(X),$$

and observe that $\Phi(k) = 0$. Hence, $k$ is a nonunit in $C_c(X)$ and thus $k(z_k) = 0$ for some $z_k \in X$. Then, $|\pi_g(z_k) - \Phi(\pi_g)| = 0$ for all $g \in J$ and so $z_k \in \Omega = \emptyset$, which is a contradiction. We derive that $z \in X$, as desired. Now pick $f \in C_c(X)$ and $\epsilon > 0$. Since $z \in X$ and $f$ is continuous on $X$, there exists $\delta > 0$ and a nonempty finite subset $J \subseteq C_c(X)$ such that, for $x \in X$,

$$|\pi_g(x) - \Phi(\pi_g)| < \delta \quad \forall g \in J \quad \implies \quad |f(x) - f(z)| < \epsilon.$$

Now define

$$h = (f - \Phi(f))^2 + \sum_{g \in J} (\pi_g - \Phi(\pi_g))^2 \in C_c(X).$$

Clearly $\Phi(h) = 0$ and hence there exists $z_h \in X$ such that $f(z_h) = \Phi(f)$ and $\pi_g(z_h) = \Phi(\pi_g)$ for all $g \in J$. These equalities together with $(\ast)$ yield that

$$|\Phi(f) - f(z)| = |f(z_h) - f(z)| < \epsilon.$$
Therefore $\Phi(f) = f(z)$ and the proof is complete.

We recall that a maximal ideal $M$ in $C_c(X)$ is real if $\frac{C_c(X)}{M}$ is isomorphic with the field $\mathbb{R}$.

**Theorem 2.2.** Let $X$ be a zero-dimensional space. Then $X$ is $\mathbb{N}$-compact if and only if every real maximal ideal $M$ in $C_c(X)$ is fixed (i.e., there exists $p \in X$ such that $M = \{f \in C_c(X) : f(p) = 0\}$).

**Proof:** Let $X$ be $\mathbb{N}$-compact (i.e., $v_0X = X$) and $M$ be a real maximal ideal in $C_c(X)$. Then $M$ is the kernel of some character on $C_c(X)$. Hence by Proposition 2.1, the ideal $M$ must be fixed. Conversely, assume that every real maximal ideal of $C_c(X)$ is fixed and $v_0X \neq X$. Consider a point $p \in v_0X \setminus X$. By Lemma 1.3, we have the ring isomorphism $\Phi$ from $C_c(X)$ to $C_c(v_0X)$, which maps each $f \in C_c(X)$ to its unique extension on $v_0X$. Clearly the ideal

$$M^p_c = \{f \in C_c(v_0X) : f(p) = 0\}$$

is a fixed maximal ideal of $C_c(v_0X)$. Therefore the ideal $\Phi^{-1}(M^p_c)$ is a real maximal ideal of $C_c(X)$. But we have

$$\Phi^{-1}(M^p_c) = \{f \in C_c(X) : f^*(p) = 0\},$$

which clearly is not a fixed ideal of $C_c(X)$ and this is a contradiction.

We remind the reader that an ideal $I$ of $C_c(X)$ is a contraction of an ideal $J$ of $C(X)$ provided that $I = J \cap C_c(X)$.

**Remark 2.3.** For a zero-dimensional space $X$, Theorem 2.2 shows that every real maximal ideal of $C_c(X)$ is a contraction of a unique fixed maximal ideal in $C(X)$ if and only if $X$ is $\mathbb{N}$-compact. P. Nyikos gave an example of a realcompact and zero-dimensional space which is not $\mathbb{N}$-compact, see [20]. Therefore we infer that there exists a zero-dimensional space $X$ for which the subring $C_c(X)$ has a real maximal ideal that is not a contraction of a real maximal ideal of the ring $C(X)$.

**Theorem 2.4.** Let $X$ be an $\mathbb{N}$-compact space. Then $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $X$ is functionally countable.

**Proof:** Suppose that there exists a topological space $Y$ such that $C_c(X) \cong C(Y)$. We denote the maximal ideal spaces of $C_c(X)$ and $C(Y)$ by $\mathcal{M}_c(X)$ and $\mathcal{M}(Y)$, respectively. Since $C_c(X)$ and $C(Y)$ are isomorphic, $\mathcal{M}_c(X)$ must be homeomorphic with $\mathcal{M}(Y)$. The Gelfand-Kolmogoroff theorem states that $\mathcal{M}(Y)$ is homeomorphic with the Stone-Čech compactification of $Y$, denoted by $\beta Y$. On the other hand $\mathcal{M}_c(X)$ is homeomorphic with the Banaschewski compactification of $X$, see [11], [4]. Hence $\beta_0X$ is homeomorphic with $\beta Y$. Therefore $Y$ must be strongly zero-dimensional. Without loss of generality we can assume that $Y$ is also realcompact. Now let $\Phi : C(Y) \to C_c(X)$ be our ring isomorphism. For each $x \in X$, define the character $\Phi_x : C(Y) \to \mathbb{R}$ such that for each $f \in C(Y)$,
\( \Phi_x(f) = \Phi(f)(x) \). Since \( Y \) is real compact, by \([10, 10.5(c)]\), there exists a unique point \( \pi(x) \in Y \) such that

\[
\Phi_x(f) = f(\pi(x)),
\]

for each \( f \in C(Y) \). The mapping \( \pi \) from \( X \) into \( Y \), thus defined, evidently satisfies

\[
\Phi(f) = f \circ \pi,
\]

for each \( f \in C(Y) \). We need to show that \( \pi \) is a homeomorphism.

The map \( \pi \) is one to one. For \( p \neq q \in X \) there exists a clopen set \( U \) such that \( p \in U \) and \( q \notin U \). Consider the characteristic function \( \chi_U \). There exists \( f \in C(Y) \) such that \( \Phi(f) = f \circ \pi = \chi_U \). Hence \( f(\pi(p)) = 1 \) and \( f(\pi(q)) = 0 \). Therefore \( \pi(p) \neq \pi(q) \).

The map \( \pi \) is continuous. Suppose that \( V \) is a clopen subset of \( Y \). Consider the characteristic function \( \chi_V \in C(Y) \). Since the function \( \Phi(\chi_V) = \chi_V \circ \pi \) is continuous and two valued, the subset

\[
(\chi_V \circ \pi)^{-1}(1) = \pi^{-1}(V),
\]

is open in \( X \). Therefore the map \( \pi \) is continuous.

The image \( \pi(X) \) is dense in \( Y \). For if \( y \in Y \setminus \text{cl}_V \pi(X) \), there exists a clopen set \( V \) such that \( y \in V \) and \( V \cap \text{cl}_V \pi(X) = \emptyset \). Evidently \( \Phi \) takes the characteristic function \( \chi_V \) to zero and this is a contradiction, for \( \Phi \) is one to one.

Now we show that the map \( \pi : X \to \pi(X) \) is open. If \( W \) is a clopen subset of \( X \), for the characteristic function \( \chi_W \in C_c(X) \), there exists \( f \in C(Y) \) such that

\[
\Phi(f) = f \circ \pi = \chi_W.
\]

Note that \( f(\pi(W)) = \{1\} \) and \( f(\pi(X \setminus W)) = \{0\} \). Hence \( \pi(W) \) and \( \pi(X \setminus W) \) are two closed and disjoint subsets of \( \pi(X) \) and their union is \( \pi(X) \). Therefore \( \pi(W) \) is open in \( \pi(X) \). Thus \( X \) is homeomorphic with \( \pi(X) \).

We claim that \( Y \) is functionally countable. Consider the ring homomorphism

\[
\Phi^{-1} : C_c(X) \to C(Y).
\]

For each \( y \in Y \), define the character \( \Phi^{-1}_y : C_c(X) \to \mathbb{R} \) by

\[
\Phi^{-1}_y(g) = \Phi^{-1}(g)(y).
\]

Since \( X \) is \( \mathbb{N} \)-compact, there exists a unique point \( \sigma(y) \in X \) such that

\[
\Phi^{-1}_y(g) = \Phi^{-1}(g)(y) = g(\sigma(y)).
\]

Hence the function \( \sigma : Y \to X \), thus defined, satisfies \( \Phi^{-1}(g) = g \circ \sigma \), for all \( g \in C_c(X) \). Since \( \Phi^{-1} \) is onto, for each \( f \in C(Y) \) there exists \( g \in C_c(X) \) such that \( \Phi^{-1}(g) = g \circ \sigma = f \). Note that \( g \) has a countable image and hence \( f \in C_c(Y) \).
If we show that the image $\pi(X)$ is $C_c$-embedded in $Y$, then by Proposition 1.4, we observe that $\pi(X) = Y$. To see this, let $h \in C_c(\pi(X))$. Then $h \circ \pi \in C_c(X)$. There exists $T \in C_c(Y) = C(Y)$ (for $Y$ is functionally countable) such that

$$\Phi(T) = T \circ \pi = h \circ \pi.$$ 

The restriction of $T$ to $\pi(X)$ is $h$. Hence by Proposition 1.4, $\pi(X) = Y$. Therefore $X$ is homeomorphic with $Y$. For completing the proof we observe that $X$ and $Y$ are homeomorphic, $Y$ is functionally countable and hence $X$ is functionally countable. $\square$

By Lemma 1.3, one can deduce that for a zero-dimensional space $X$ we have $C_c(X) \cong C_c(\nu_0X)$. Therefore the following result is immediate.

**Corollary 2.5.** For a zero-dimensional space $X$, the ring $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $\nu_0X$ is functionally countable.

**Remark 2.6.** It is well-known that a realcompact strongly zero-dimensional space is $\mathbb{N}$-compact. Also since each continuous real valued function on a completely regular Hausdorff space $X$ has an extension to $\nu X$ with the same image, we have $\nu_0X = \nu X$.

Theorem 2.4 together with Remark 2.6, imply the following corollary.

**Corollary 2.7.** Let $X$ be a strongly zero-dimensional space. Then $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $X$ is functionally countable.

As an application of Corollary 2.7, consider the space of irrational numbers, denoted by $\mathbb{P}$. The ring $C_c(\mathbb{P})$ is not isomorphic to any ring of continuous functions. Notice that for showing this fact, all ring theoretic methods which were chosen in [9] are not applicable here.

**Remark 2.8.** In [13], it is shown that a $P$-space is functionally countable if and only if $X$ is pseudo-$\mathcal{R}_1$-compact. Since every $P$-space is strongly zero-dimensional, Corollary 2.7 shows that for an arbitrary $P$-space $X$, the ring $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $X$ is pseudo-$\mathcal{R}_1$-compact.

**Remark 2.9.** With regard to Theorem 2.4 and Corollary 2.7, for a strongly zero-dimensional space $X$, $\nu_0X$ is functionally countable if and only if $X$ is functionally countable. The interested reader is encouraged to find an example of a zero-dimensional non functionally countable space $X$, for which $\nu_0X$ is functionally countable.

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