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ON COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF AANA RANDOM VARIABLES

Haiwu Huang, Hanjun Zhang and Qingxia Zhang

In this work, a complete moment convergence theorem is obtained for weighted sums of asymptotically almost negatively associated (AANA) random variables without assumption of identical distribution under some mild moment conditions. As an application, the complete convergence theorems for weighted sums of negatively associated (NA) and AANA random variables are obtained. The result not only generalizes the corresponding ones of Sung [13] and Huang et al. [8], but also improves them.

Keywords: AANA random variables, complete moment convergence, complete convergence, weighted sums

Classification: 60F15

1. INTRODUCTION

Firstly, let us restate the definition of AANA random variables.

Definition 1.1. A sequence of random variables $\{X_n, n \geq 1\}$ is called AANA if there exists a nonnegative sequence $\mu(n) \to 0$ as $n \to \infty$ such that

$$\mathrm{Cov}\left(f_1\left(X_n\right), f_2\left(X_{n+1}, \ldots, X_{n+k}\right)\right) \leq \mu(n)\left(\mathrm{Var}\left(f_1(X_n)\right) \mathrm{Var}\left(f_2\left(X_{n+1}, \ldots, X_{n+k}\right)\right)\right)^{1/2},$$

for all $n, k \geq 1$ and all coordinatewise non-decreasing continuous functions f_1 and f_2 whenever the variances exist. $\{\mu(n), n \geq 1\}$ is so-called the mixing coefficient sequence.

The concept of AANA was introduced by Chandra and Ghosal [3]. It is obviously seen that the family of AANA random variables contains NA random variables (with $\mu(n) = 0, n \geq 1$) and some more sequences of random variables which are not much deviated from being NA random variables. As pointed out and proved by Joag-Dev and Proschan [10], a number of well known multivariate distributions possess the NA property. Hence, extending and improving the limit properties of NA random variables to the case of AANA random variables has very important significance in the theory and application.

Since the concept of AANA was introduced by Chandra and Ghosal [3], many applications have been established. For more details, we can refer to [3, 4, 7, 9, 11, 12, 14, 15, 16, 17, 19, 20, 21], and so forth.

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Recently, Sung [13] obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

Theorem 1.2. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n), \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ for $1 < \alpha \leq 2$. If

$$\begin{array}{ll} E|X|^{\alpha} < \infty & \text{for} \quad \alpha > \gamma, \\ E|X|^{\alpha} \log(1 + |X|) < \infty & \text{for} \quad \alpha = \gamma, \\ E|X|^{\gamma} < \infty & \text{for} \quad \alpha < \gamma, \end{array} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{1.3}$$

Huang et al. [8] extended Theorem 1.2 for NA random variable to AANA random variables by using a method different from Sung [13], and obtained the following results.

Theorem 1.3. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} = O(n), \tag{1.4}$$

for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and (1.2) holds for $\alpha \neq \gamma$, then (1.3) holds.

Theorem 1.4. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.4) for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha = \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and (1.2) holds for $\alpha = \gamma$, then (1.3) holds.

Inspired by the above theorems obtained by Sung [13] and Huang et al. [8], in this work, we will further study the strong convergence for weighted sums of AANA random variables without assumption of identical distribution, and obtain an improved result (i. e., so-called complete moment convergence, which will be introduced in Definition 1.6 under some mild moment conditions. As applications, the complete convergence theorems for weighted sums of NA and AANA random variables are obtained. The main result of this paper not only generalizes the corresponding ones of Sung [13] and Huang et al. [8], but also improves them, respectively.

We will give some definitions used in this paper as follows.

Definition 1.5. A sequence of random variables $\{X_n, n \geq 1\}$ converges completely to a constant λ if for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|X_n - \lambda| > \varepsilon\right) < \infty.$$

This notion was firstly given by Hsu and Robbins [6].

Definition 1.6. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If for all $\varepsilon \geq 0$,

$$\sum_{n=1}^{\infty} a_n E\left(b_n^{-1}|X_n| - \varepsilon\right)_+^q < \infty.$$

Then the above result was called the complete moment convergence by Chow [5].

Definition 1.7. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P\left(|X_n| \geq x\right) \leq CP\left(|X| \geq x\right),$$

for all $x \geq 0$ and $n \geq 1$.

2. MAIN RESULTS AND PROOFS

To prove the main results, we need the following lemmas.

Lemma 2.1. (Yuan and An [20]) Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$, let $\{f_n, n \geq 1\}$ be a sequence of all non-decreasing (or all non-increasing) continuous functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$.

Lemma 2.2. (Yuan and An [20]) Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$ such that $\sum_{i=1}^{\infty} \mu^{1/(p-1)}(i) < \infty$ for some $p \in \left(3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}\right]$, where the integer number $k \geq 1$, $EX_n = 0$. Then there exists a positive constant $C = C(p)$ depending only on p such that for all $n \geq 1$,

$$E\left(\max_{1 \leq j \leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C\left(\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right). \tag{2.1}$$

Lemma 2.3. (Adler and Rosalsky [1]; Adler, Rosalsky and Taylor [2]) Suppose that $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For all $q > 0$ and $x > 0$, the following two statements hold:

$$E|X_n|^q I(|X_n| \leq x) \leq C_1 (E|X|^q I(|X| \leq x) + x^q P(|X| > x)), \tag{2.2}$$

$$E|X_n|^q I(|X_n| > x) \leq C_2 E|X|^q I(|X| > x), \tag{2.3}$$

where C_1 and C_2 are positive constants.

Lemma 2.4. (Wu, Sung and Volodin [18]) Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $p > \max\{\alpha, \gamma\}$, then

$$\sum_{n=1}^{\infty} \frac{1}{nb_n^p} \sum_{i=1}^{n} E|a_{ni}X|^p I(|a_{ni}X| \leq b_n) \leq \begin{cases} CE|X|^\alpha & \text{for} \quad \alpha > \gamma, \\ CE|X|^\alpha \log(1 + |X|) & \text{for} \quad \alpha = \gamma, \\ CE|X|^\gamma & \text{for} \quad \alpha < \gamma. \end{cases} \tag{2.4}$$

Throughout this paper, the symbol C represents a positive constant whose value may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$. $I(A)$ be the indicator function of the set A. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants.

Now, we state and prove the main result as follows.

Theorem 2.5. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, assume that $EX_n = 0$ for $1 < \alpha \leq 2$. If $0 < q < \alpha$, the following statements hold:

(i) for $\alpha > \gamma$, then $E|X|^\alpha < \infty$ implies

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon\right)_+^q < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{2.5}$$

(ii) for $\alpha = \gamma$, then $E|X|^\alpha \log(1 + |X|) < \infty$ implies (2.5).

(iii) for $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies (2.5).

Proof. For $\forall\varepsilon > 0$, since

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon\right)_+^q \\
= &\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}P\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon>t^{1/q}\right)\mathrm{d}t \\
= &\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{1}P\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon+t^{1/q}\right)\mathrm{d}t \\
+ &\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon+t^{1/q}\right)\mathrm{d}t \qquad (2.6)\\
\le &\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon b_n\right) \\
+ &\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>b_n t^{1/q}\right)\mathrm{d}t \\
\triangleq\ & I+J.
\end{aligned}$$

To prove (2.5), it suffices to show that $I < \infty$ and $J < \infty$. By the above listed Theorems 1.3 and 1.4 of Huang et al. [8]], we can easily obtain that $I < \infty$.

Without loss of generality, assume that $a_{ni} \ge 0$. For all $t \ge 1$ and $i \ge 1$, define

$$\begin{aligned}
Y_i &= -b_n t^{1/q} I\left(a_{ni}X_i < -b_n t^{1/q}\right) + a_{ni}X_i I\left(|a_{ni}X_i| \le b_n t^{1/q}\right) \\
&\quad + b_n t^{1/q} I\left(a_{ni}X_i > b_n t^{1/q}\right), \\
Z_i &= a_{ni}X_i - Y_i \\
&= \left(a_{ni}X_i + b_n t^{1/q}\right) I\left(a_{ni}X_i < -b_n t^{1/q}\right) + \left(a_{ni}X_i - b_n t^{1/q}\right) I\left(a_{ni}X_i > b_n t^{1/q}\right), \\
A &= \bigcap_{i=1}^{n}(Y_i = a_{ni}X_i), \quad B = \bar{A} = \bigcup_{i=1}^{n}(Y_i \ne a_{ni}X_i) = \bigcup_{i=1}^{n}\left(|a_{ni}X_i| > b_n t^{1/q}\right), \\
E_n &= \left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > b_n t^{1/q}\right).
\end{aligned}$$

It is easy to check that for $\forall\varepsilon > 0$,

$$E_n = E_nA \cup E_nB \subset \left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}Y_i\right| > b_n t^{1/q}\right) \cup \left(\bigcup_{i=1}^{n}\left(|a_{ni}X_i| > b_n t^{1/q}\right)\right),$$

which implies that

$$\begin{aligned} P\left(E_n\right) &\leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right| > b_n t^{1/q}\right) + P\left(\bigcup_{i=1}^{n}\left(|a_{ni}X_i| > b_n t^{1/q}\right)\right) \\ &\leq P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(Y_i - EY_i\right)\right| > b_n t^{1/q} - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_i\right|\right) \\ &+ \sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n t^{1/q}\right). \end{aligned} \tag{2.7}$$

Firstly, we will show that

$$\max_{t\geq 1}\frac{1}{b_n t^{1/q}}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_i\right| \to 0 \quad \text{as} \quad n\to\infty. \tag{2.8}$$

For $0 < \alpha \leq 1$, it follows from (2.2) of Lemma 2.3, the c_r inequality, the Markov inequality and $E|X|^\alpha < \infty$ that

$$\begin{aligned} &\max_{t\geq 1}\frac{1}{b_n}t^{1/q}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_i\right| \\ &\leq C\max_{t\geq 1}\frac{1}{b_n}t^{1/q}\sum_{i=1}^{n}|EY_i| \\ &\leq C\max_{t\geq 1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n} E\left|a_{ni}X_i\right| I\left(|a_{ni}X_i| \leq b_n t^{1/q}\right) + C\max_{t\geq 1}\sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n t^{1/q}\right) \\ &\leq C\max_{t\geq 1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n}\left(E\left|a_{ni}X\right| I\left(|a_{ni}X| \leq b_n t^{1/q}\right) + b_n t^{1/q} P\left(|a_{ni}X| > b_n t^{1/q}\right)\right) \\ &\quad + C\max_{t\geq 1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/q}\right) \\ &\leq C\max_{t\geq 1}\frac{1}{b_n^\alpha t^{\alpha/q}}\sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\left(|a_{ni}X| \leq b_n t^{1/q}\right) + C\max_{t\geq 1}\frac{1}{b_n^\alpha t^{\alpha/q}}\sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha \\ &\leq 2C\frac{1}{b_n^\alpha}\sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha \\ &\leq C(\log n)^{-\alpha/\gamma}E|X|^\alpha \to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{2.9}$$

Note that when $a_{ni}X_i > b_n t^{1/q}$, $0 < Z_i = a_{ni}X_i - b_n t^{1/q} < a_{ni}X_i$; when $a_{ni}X_i < -b_n t^{1/q}$, $a_{ni}X_i < Z_i = a_{ni}X_i + b_n t^{1/q} < 0$. Hence, $|Z_i| < |a_{ni}X_i|\, I\left(|a_{ni}X_i| > b_n t^{1/q}\right)$.

For $1 < \alpha \le 2$, it follows from $EX_n = 0$, (2.3) of Lemma 2.3, the c_r inequality and $E|X|^\alpha < \infty$ again that

$$\begin{aligned}
\max_{t\ge 1} \frac{1}{b_n t^{1/q}} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EY_i\right| &= \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EZ_i\right| \\
&\le C \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \sum_{i=1}^{n} E\left|Z_i\right| \\
&\le C \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \sum_{i=1}^{n} E\left|a_{ni}X_i\right| I\left(\left|a_{ni}X_i\right| > b_n t^{1/q}\right) \\
&\le C \max_{t\ge 1} \frac{1}{b_n t^{1/q}} \sum_{i=1}^{n} E\left|a_{ni}X\right| I\left(\left|a_{ni}X\right| > b_n t^{1/q}\right) \\
&\le C \max_{t\ge 1} \frac{1}{b_n^\alpha t^{\alpha/q}} \sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\left(\left|a_{ni}X\right| > b_n t^{1/q}\right) \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^\alpha \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{2.10}$$

By (2.9) and (2.10), we can obtain (2.8) immediately. Hence, for n large enough and all $t \ge 1$,

$$P\left(E_n\right) \le P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} (Y_i - EY_i)\right| > \frac{b_n t^{1/q}}{2}\right) + \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right| > b_n t^{1/q}\right). \tag{2.11}$$

To prove $J < \infty$, it suffices to show that

$$J_1 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} (Y_i - EY_i)\right| > \frac{b_n t^{1/q}}{2}\right) \mathrm{d}t < \infty, \tag{2.12}$$

$$J_2 \triangleq \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \sum_{i=1}^{n} P\left(\left|a_{ni}X_i\right| > b_n t^{1/q}\right) \mathrm{d}t < \infty. \tag{2.13}$$

By Lemma 2.1, it obviously follows that $\{Y_n - EY_n, n \ge 1\}$ is still a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \ge 1\}$. Hence, it follows from the Markov inequality and Lemma 2.2 that

$$\begin{aligned}
J_1 &\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \frac{1}{b_n^p t^{p/q}} E\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} (Y_i - EY_i)\right|^p\right) \mathrm{d}t \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \frac{1}{b_n^p t^{p/q}} \left(\sum_{i=1}^{n} E|Y_i - EY_i|^p + \left(\sum_{i=1}^{n} E|Y_i - EY_i|^2\right)^{p/2}\right) \mathrm{d}t \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \int_1^\infty \frac{1}{t^{p/q}} \sum_{i=1}^{n} E|Y_i|^p \mathrm{d}t + C \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \int_1^\infty \frac{1}{t^{p/q}} \left(\sum_{i=1}^{n} E|Y_i|^2\right)^{p/2} \mathrm{d}t \\
&\triangleq J_{11} + J_{12}.
\end{aligned} \tag{2.14}$$

Hence, it follows from (2.2) of Lemma 2.3 that

$$\begin{aligned}
J_{11} \leq\ & C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_n t^{1/q}\right)\mathrm{d}t \\
&+C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_{1}^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X_i|^p I\left(|a_{ni}X_i|\leq b_n t^{1/q}\right)\mathrm{d}t \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/q}\right)\mathrm{d}t \\
&+C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_{1}^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X|^p I\left(|a_{ni}X|\leq b_n t^{1/q}\right)\mathrm{d}t \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/q}\right)\mathrm{d}t \\
&+\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_{1}^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X|^p I\left(|a_{ni}X|\leq b_n\right)\mathrm{d}t \\
&+C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_{1}^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X|^p I\left(b_n<|a_{ni}X|\leq b_n t^{1/q}\right)\mathrm{d}t \\
\triangleq\ & J_2^{'}+J_{111}+J_{112}.
\end{aligned} \tag{2.15}$$

From $0<q<\alpha$ and $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$, we can obtain that

$$\begin{aligned}
J_2 \leq\ & \sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/q}\right)\mathrm{d}t \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{n}\int_{0}^{\infty}\sum_{i=1}^{n}P\left(\frac{|a_{ni}X|^q}{b_n^q}>t\right)\mathrm{d}t \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E|a_{ni}X|^q}{b_n^q}I\left(|a_{ni}X|>b_n\right) \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|>b_n\right) \\
=\ & C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|^{\alpha}>n(\log n)^{\alpha/\gamma}\right) \\
\leq\ & C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|X|^{\alpha}>\frac{n(\log n)^{\alpha/\gamma}}{\sum_{i=1}^{n}|a_{ni}|^{\alpha}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|X|>(\log n)^{1/\gamma}\right)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\alpha/\gamma}E|X|^{\alpha}I\left(|X|>(\log n)^{1/\gamma}\right)\\
&= C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\alpha/\gamma}\sum_{k=n}^{\infty}E|X|^{\alpha}I\left((\log k)^{1/\gamma}<|X|<(\log (k+1))^{1/\gamma}\right)\\
&= C\sum_{k=1}^{\infty}E|X|^{\alpha}I\left((\log k)^{1/\gamma}<|X|<(\log (k+1))^{1/\gamma}\right)\sum_{n=1}^{k}\frac{1}{n}(\log n)^{-\alpha/\gamma}.
\end{aligned}
$$

Note that

$$
\sum_{n=1}^{k}\frac{1}{n}(\log n)^{-\alpha/\gamma}=\begin{cases} C & \text{for}\quad \alpha>\gamma,\\ C\log\log k & \text{for}\quad \alpha=\gamma,\\ C(\log k)^{1-\alpha/\gamma} & \text{for}\quad \alpha<\gamma.\end{cases}
$$

Hence,

$$
J_2\leq\begin{cases} CE|X|^{\alpha} & \text{for}\quad \alpha>\gamma,\\ CE|X|^{\alpha}\log 1+|X| & \text{for}\quad \alpha=\gamma,\\ CE|X|^{\gamma} & \text{for}\quad \alpha<\gamma.\end{cases}
$$

Under the conditions of Theorem 2.5, it follows that $J_2<\infty$ and $J_2^{'}<\infty$.

For $0<q<\alpha\leq 2$, $p\in\left(3\cdot 2^{k-1},4\cdot 2^{k-1}\right]$, where the integer number $k\geq 1$, it follows from Lemma 2.4 and conditions of Theorem 2.5 that

$$
\begin{aligned}
J_{111} &= C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{E|a_{ni}X|^{p}}{b_n^p}I\left(|a_{ni}X|\leq b_n\right)\int_1^{\infty}\frac{1}{t^{p/q}}\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|a_{ni}X|^{p}I\left(|a_{ni}X|\leq b_n\right)\\
&< \infty. \qquad (2.16)
\end{aligned}
$$

Taking $t=x^q$, it follows from (2.2) of Lemma 2.3 and the Markov inequality that

$$
\begin{aligned}
J_{112} &\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}x^{q-p-1}\sum_{i=1}^{n}E|a_{ni}X|^{p}I\left(b_n<|a_{ni}X|\leq b_nx\right)\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{m=1}^{\infty}\int_m^{m+1}x^{q-p-1}\sum_{i=1}^{n}E|a_{ni}X|^{p}I\left(b_n<|a_{ni}X|\leq b_nx\right)\mathrm{d}x\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{m=1}^{\infty}m^{q-p-1}\sum_{i=1}^{n}E|a_{ni}X|^{p}I\left(b_n<|a_{ni}X|\leq b_n(m+1)\right)\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{s=1}^{m}m^{q-p-1}E|a_{ni}X|^{p}I\left(b_ns<|a_{ni}X|\leq b_n(s+1)\right)\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^{p}I\left(b_ns<|a_{ni}X|\leq b_n(s+1)\right)\sum_{m=s}^{\infty}m^{q-p-1}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^pI\left(b_ns<|a_{ni}X|\le b_n\left(s+1\right)\right)s^{q-p}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{s=1}^{\infty}\frac{b_n^p(s+1)^p}{b_n^q(s+1)^q}E|a_{ni}X|^qI\left(b_ns<|a_{ni}X|\le b_n\left(s+1\right)\right)s^{q-p}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}E|a_{ni}X|^qI\left(|a_{ni}X|>b_n\right)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(|a_{ni}X|>b_n\right)\\
&< \infty \quad \text{(see the proof of } \mathrm{J}_2<\infty). \qquad (2.17)
\end{aligned}
$$

It follows from the c_r inequality and Lemma 2.3 that

$$
\begin{aligned}
&\left(\sum_{i=1}^{n}E|Y_i|^2\right)^{p/2}\\
&\le C\left(\sum_{i=1}^{n}\left(b_n^2t^{2/q}P\left(|a_{ni}X_i|>b_nt^{1/q}\right)+E|a_{ni}X_i|^2I\left(|a_{ni}X_i|\le b_nt^{1/q}\right)\right)\right)^{p/2}\\
&\le C\left(\sum_{i=1}^{n}b_n^2t^{2/q}P\left(|a_{ni}X_i|>b_nt^{1/q}\right)\right)^{p/2}+C\left(\sum_{i=1}^{n}E|a_{ni}X_i|^2I\left(|a_{ni}X_i|\le b_nt^{1/q}\right)\right)^{p/2}\\
&\le C\sum_{i=1}^{n}b_n^pt^{p/q}P\left(|a_{ni}X_i|>b_nt^{1/q}\right)+C\sum_{i=1}^{n}\left(E|a_{ni}X_i|^2\right)^{p/2}I\left(|a_{ni}X_i|\le b_nt^{1/q}\right)\\
&\le C\sum_{i=1}^{n}b_n^pt^{p/q}P\left(|a_{ni}X|>b_nt^{1/q}\right)+C\sum_{i=1}^{n}\left(E|a_{ni}X|^2\right)^{p/2}I\left(|a_{ni}X|\le b_nt^{1/q}\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
J_{12} &\le C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_nt^{1/q}\right)\mathrm{d}t\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}\left(E|a_{ni}X_i|^2\right)^{p/2}I\left(|a_{ni}X_i|\le b_nt^{1/q}\right)\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_nt^{1/q}\right)\mathrm{d}t\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}\left(\frac{E|a_{ni}X|^2}{b_n^2t^{2/q}}\right)^{p/2}I\left(|a_{ni}X|\le b_nt^{1/q}\right)\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_nt^{1/q}\right)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}\frac{E|a_{ni}X|^{\alpha}}{b_n^{\alpha}t^{\alpha/q}}I\left(|a_{ni}X|\le b_nt^{1/q}\right)\mathrm{d}t\\
= \;& C\sum_{n=1}^{\infty}\frac{1}{n}\int_{1}^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_nt^{1/q}\right)\mathrm{d}t\\
&+C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\int_{1}^{\infty}\frac{1}{t^{\alpha/q}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(|a_{ni}X|\le b_nt^{1/q}\right)\mathrm{d}t\\
&+C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\int_{1}^{\infty}\frac{1}{t^{\alpha/q}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I\left(b_n<|a_{ni}X|\le b_nt^{1/q}\right)\mathrm{d}t\\
< \;& \infty \quad \text{(see the proof of } J_{11}<\infty \text{ by replacing } p \text{ with } \alpha).
\end{aligned}
\tag{2.18}
$$

The proof of Theorem 2.5 is completed. □

Remark 2.6. Under the conditions of Theorem 2.5, we can obtain that

$$
\begin{aligned}
\infty \;&>\; \sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon\right)_+^q\\
&=\; \sum_{n=1}^{\infty}\frac{1}{n}\int_{0}^{\infty}P\left(\frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon>t^{1/q}\right)\mathrm{d}t\\
&\ge\; C\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>b_n\varepsilon+t^{1/q}\right)\mathrm{d}t\\
&\ge\; C\int_{0}^{1}\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>b_n\varepsilon\right)\mathrm{d}t\\
&=\; C\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>b_n\varepsilon\right)\quad \text{for}\quad \forall\varepsilon>0.
\end{aligned}
\tag{2.19}
$$

Hence, from (2.19), we can know that the complete moment convergence implies the complete convergence. Compared with the results of Huang et al.[8], it is worth pointing out that our main result is much stronger under the same conditions. So, Theorem 2.5 is an extension and improvement of the corresponding ones of Sung [13] and Huang et al. [8].

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