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On the Example of Almost Pseudo-Z-symmetric Manifolds*

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Abstract

In the present paper we have obtained a new example of non-Ricci-flat almost pseudo-Z-symmetric manifolds in the class of equidistant spaces, which admit non-trivial geodesic mappings.

Key words: (pseudo-) Riemannian manifold, almost pseudo-Z-symmetric spaces, equidistant spaces.

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1 Introduction

In [4] was introduced an almost pseudo-Z-symmetric space, which is an \(n\)-dimension (pseudo-) Riemannian space \(V_n\) where the special tensor

\[
Z_{ij} = R_{ij} + \varphi g_{ij},
\]

satisfied the recurrent condition

\[
Z_{ij,k} = (a_k + b_k)Z_{ij} + a_j Z_{ik} + a_i Z_{jk}
\]  

(1)

\(R_{ij}\), \(g_{ij}\) and \(\varphi\) being Ricci tensor, metric tensor and scalar function.

These manifolds are generalization of symmetric and recurrent spaces which were introduced by É. Cartan [2], and A. G. Walker [19], respectively.

These manifolds were generalized in many directions, see, for example [13, pp. 292–295, 335, 338], [18]. Geodesic and holomorphically projective mappings

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of mentioned manifolds were studied in many papers too, see [6, 8, 11, 12, 13, 15, 17]. Among others, J. Mikeš [9] proved that non-Einstein Ricci-symmetric (pseudo-) Riemannian spaces \((R_{ij,k} = 0)\) do not admit non-trivial geodesic mappings. In paper [10] were constructed projective symmetric space which is not symmetric. For example, generalized recurrent spaces were studied in [5, 7, 14, 16].

In the paper [4] was studied almost pseudo-Z-symmetric space. As we can see, the Example 8, on p. 39–40, is false for explicit calculation. In this paper, we construct new example of these manifolds.

2 Equidistant manifolds

Having found the example of almost pseudo-Z-symmetric manifolds faulty [4], the present authors have constructed an example in the class of special equidistant space.

In an equidistant space with non isotropic concircular vector field there exists canonical coordinate system, where the metric tensor has the following form [17, pp. 92–95], [13, p. 150]:

\[
ds^2 = e \, dx^1^2 + f(x^1) \, d\tilde{s}^2, \tag{2}\]

where \(e = \pm 1\), \(f\) is a differentiable function and

\[
d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n) \, dx^a dx^b
\]

is a metric of \((n-1)\)-dimensional (pseudo-) Riemannian manifold \(\tilde{V}_{n-1}\).

Here and after indices \(a, b, \ldots = 2, 3, \ldots, n\).

In 1954 N. S. Sinyukov (see [17], [13, pp. 140-155]), thanks to their geometrical properties, gave them the name equidistant space. Around the year 1920 the H. W. Brinkmann [1] started studying these space and in the 1940 K. Yano [20] studied concircular vector fields. Many newly obtained results are possible to see in [3].

We denote that if \(f' \neq 0\), then this manifold admits non-trivial geodesic mappings, see [17, 11, 13]. In the coordinate system (2) the components of metric and inverse metric tensors have the following form:

\[
g_{11} = e; \quad g_{1a} = 0; \quad g_{ab} = f(x^1) \tilde{g}_{ab}
\]

\[
g^{11} = e; \quad g^{1a} = 0; \quad g^{ab} = f(x^1)^{-1} \tilde{g}^{ab}, \tag{3}\]

where \(f (\neq 0)\) is a function of variable \(x^1\) and \(\tilde{g}_{ab}\) and \(\tilde{g}^{ab}\) are components of metric and inverse metric tensors of \((n-1)\)-dimension on (pseudo-) Riemannian space \(\tilde{V}_{n-1}\), their component are functions of variables \(x^2, x^3, \ldots, x^n\).

Now, non-zero components of Christofell symbols:

\[
\Gamma^h_{ij} = \Gamma_{ijk} g^{kh} \quad \text{and} \quad \Gamma_{ijk} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})
\]
On the example of almost pseudo-Z-symmetric manifolds

where $\partial_i \equiv \partial/\partial x^i$, have the following form:

$$
\Gamma_{1ab} \equiv \Gamma_{a1b} = \frac{1}{2} f' \tilde{g}_{ab}; \quad \Gamma_{ab1} = -\frac{1}{2} f' \tilde{g}_{ab}; \quad \Gamma_{abc} = f \tilde{\Gamma}_{abc}
$$

and non-zero components of Christoffel symbols of second kind:

$$
\Gamma_{ab}^1 = -\frac{e}{2} f' \tilde{g}_{ab}; \quad \Gamma_{1b}^c \equiv \Gamma_{b1}^c = \frac{1}{2} \frac{f'}{f} \delta^c_b; \quad \Gamma_{ab}^c = \tilde{\Gamma}_{ab}^c
$$

Following computation of non-zero components of the Riemannian tensor

$$
R_{hijk} = \partial_j \Gamma_{hi}^k - \partial_k \Gamma_{hi}^j + \Gamma_{hi}^\alpha \Gamma_{\alpha j}^k - \Gamma_{\alpha i}^j \Gamma_{\alpha k}^\alpha
$$

$$
R_{1ab} = -\frac{e}{2} (f'' - \frac{f'^2}{2f}) \tilde{g}_{ab},
$$

$$
R_{1b1} = -\frac{1}{2f} (f'' - \frac{f'^2}{2f}) \delta^c_b \tilde{g}_{ab},
$$

$$
R_{abc} = \tilde{R}_{abc} - \frac{e}{4} \frac{f''}{f} (\tilde{g}_{ac} \delta^d_b - \tilde{g}_{ab} \delta^d_c).
$$

Contracting Riemannian tensor by metric tensor, we lower indices and obtain Riemannian tensor of type $(0,4)$

$$
R_{hijk} = g_{h\alpha} R_{\alpha ijk}.
$$

After computation, we get the following non-zero components:

$$
R_{11a1b} = -R_{a11b} = R_{a1b1} = R_{a11b} = -\frac{1}{2} (f'' - \frac{f'^2}{2f}) \tilde{g}_{ab},
$$

$$
R_{abcd} = f \tilde{R}_{abcd} - \frac{e}{4} f'' (\tilde{g}_{ac} \delta^d_b - \tilde{g}_{ab} \delta^d_c).
$$

The Ricci tensor $R_{ij} = R_{i\alpha j}$ has these non-zero components:

$$
R_{11} = R_{1a1} = -\frac{1}{2f} (n - 1)(f'' - \frac{f'^2}{2f})
$$

$$
R_{ab} = \tilde{R}_{ab} - \frac{e}{4} (f'' - \frac{f'^2}{2f}) \tilde{g}_{ab}.
$$

3 Special equidistant almost pseudo-Z-symmetric spaces

The above mentioned almost pseudo-Z-symmetric spaces are defined in formula (1). Next, we shall study these spaces supposing that this space $V_n$ is equidistant, and moreover $\tilde{V}_{n-1}$ is Ricci flat space and component $Z_{11}$ of tensor $Z$ is equal to zero.

Firstly, we compute non-zero components of tensor $Z_{ij} = R_{ij} + \varphi g_{ij}$:

$$
Z_{11} = R_{11} + \varphi (x^1) \tilde{g}_{11} = -\frac{1}{2f} (n - 1)(f'' - \frac{f'^2}{2f}) + e \varphi;
$$

$$
Z_{ab} = -\left( \frac{e}{2} (f'' - \frac{f'^2}{2f}) - \varphi f \right) \tilde{g}_{ab}.
$$
From our proposition \((Z_{11} = 0)\) it follows that the function \(\varphi\) has the following form:

\[
\varphi = \frac{e}{2}(n - 1) \left( f'' - \frac{f'^2}{2f} \right),
\]

and thus

\[
Z = -\frac{en}{2} \left( f'' - \frac{f'^2}{2f} \right).
\]

Secondly, we remember that covariant derivations of \(Z_{ij}\) have the following definition

\[
Z_{ij,k} = \partial_k Z_{ij} - Z_{\alpha j} \Gamma^\alpha_{ik} - Z_{i\alpha} \Gamma^{\alpha}_{jk},
\]

and equation (1):

\[
Z_{ij,k} = (a_k + b_k) Z_{ij} + a_j Z_{ik} + a_i Z_{jk}
\]

will have the form

\[
Z_{11,1} \equiv \partial_1 Z_{11} = (3a_1 + b_1)Z_{11};
\]

\[
Z_{11,c} \equiv 0 = (a_c + b_c)Z_{11};
\]

\[
Z_{1b,1} \equiv 0 = a_b Z_{11};
\]

\[
Z_{1b,c} \equiv -\frac{f'}{2f} Z_{bc} + \frac{e}{2} f' Z_{11} \tilde{g}_{bc} = a_1 Z_{bc};
\]

\[
Z_{ab,1} \equiv \partial_1 Z_{ab} - \frac{f'}{f} Z_{ab} = (a_1 + b_1)Z_{ab};
\]

\[
Z_{ab,c} \equiv 0 = (a_c + b_c)Z_{ab} + a_a Z_{bc} + a_b Z_{ac}.
\]

Because \(Z_{11} = 0\), the above equations simplify to the following form:

\[
-\frac{f'}{2f} Z_{bc} = a_1 Z_{bc};
\]

\[
\partial_1 Z_{ab} - \frac{f'}{f} Z_{ab} = (a_1 + b_1)Z_{ab};
\]

\[
(a_c + b_c)Z_{ab} + a_a Z_{bc} + a_b Z_{ac} = 0.
\]

Naturally \(Z_{ij} \neq 0\), then \(Z\) must not be equal to zero. Then for \(n \geq 4\) and from (11) it implies \(a_a = b_a = 0\). From (9) and (10) follows:

\[
a_1 = -\frac{1}{2} \frac{f'}{f}, \quad \text{and} \quad b_1 = -a_1 - \frac{f'}{f} \partial_1 \ln |Z|.
\]

On the base of above discussion, we can formulate this theorem:

**Theorem 1** The equidistant space with metric (2) where metric \(d\tilde{s}^2\) defined Ricci-flat space is almost pseudo-Z-symmetric space for any non-zero function \(f(x^1) \in C^3, f'' - \frac{f'^2}{2f} \neq 0\).
In this space we have tensor $Z_{ij} = R_{ij} - \varphi g_{ij}$, where

$$\varphi = \frac{e(n - 1)}{2} \left( f'' - \frac{f'^2}{2f} \right),$$

and

$$a_i = -\delta^1_i \left( \frac{f'}{2f} \right) \quad \text{and} \quad b_i = -\delta^1_i \frac{f'}{2f} \left( 1 - 2 \left( \ln \left| f'' - \frac{f'^2}{2f} \right| \right) \right).$$

References


