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On Uniqueness Theorems for Ricci Tensor^{*}

Marina B. KHRIPUNOVA^{1a}, Sergey E. STEPANOV^{1b},
Irina I. TSYGANOK^{1c}, Josef MIKEŠ²

¹*Department of Mathematics, Finance University under the Government
of Russian Federation, Leningradsky Prospect, 49-55,
125468 Moscow, Russian Federation*

^a*e-mail: MBKHripunova@fa.ru*

^b*e-mail: s.e.stepanov@mail.ru*

^c*e-mail: i.i.tsyganok@mail.ru*

²*Department of Algebra and Geometry, Faculty of Science, Palacký
University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

e-mail: josef.mikes@upol.cz

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Abstract

In Riemannian geometry the prescribed Ricci curvature problem is as follows: given a smooth manifold M and a symmetric 2-tensor r , construct a metric on M whose Ricci tensor equals r . In particular, DeTurck and Koiso proved the following celebrated result: the Ricci curvature uniquely determines the Levi-Civita connection on any compact Einstein manifold with non-negative section curvature. In the present paper we generalize the result of DeTurck and Koiso for a Riemannian manifold with non-negative section curvature. In addition, we extended our result to complete non-compact Riemannian manifolds with nonnegative sectional curvature and with finite total scalar curvature.

Key words: Uniqueness theorem for Ricci tensor, compact and complete Riemannian manifolds, vanishing theorem.

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1 Introduction

The main point of the papers [1, 2] and the monograph [3, pp. 140–153] is that in certain circumstances the metric (or at last the connection) is uniquely

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determined by the Ricci tensor. In particular, in [1, Corollary 3.3] and [3, Theorem 5.42] anyone can read the following: Let (M, \bar{g}) be a compact *Einstein manifold* with non-negative sectional curvature and with the Ricci tensor $\text{Ric}(\bar{g}) = \bar{g}$, then another Riemannian metric g on M with $\text{Ric}(g) = \bar{g}$ has the same Levi-Civita connection as \bar{g} . We note that this proposition is a corollary of the Eells and Sampson *vanishing theorem* for harmonic maps of compact Riemannian manifolds (see [4, p. 124]).

In the present paper we consider a compact *Riemannian manifold* (M, \bar{g}) with non-negative sectional curvature and with $\text{Ric}(\bar{g}) \leq \bar{g}$. Under these conditions, we prove that if g is another Riemannian metric on M with the Ricci tensor $\text{Ric}(g) = \bar{g}$, then g and \bar{g} have the same Levi-Civita connection. Furthermore, if the full holonomy group $\text{Hol}(\bar{g})$ is irreducible then the metric $g = C\bar{g}$ for some constant $C > 0$. In turn, it is well known that $\text{Ric}(\bar{g}) = \text{Ric}(C\bar{g})$. This proposition was announced in report [5] at the 12th International Conference on Geometry and Applications (September 1–5, 2015, Varna, Bulgaria). We extend the above scheme to show that if (M, \bar{g}) is a non-compact manifold (M, \bar{g}) with non-negative sectional curvature and with the Ricci tensor $\text{Ric}(\bar{g}) \leq \bar{g}$ then there is no complete Riemannian metric g such that its Ricci tensor $\text{Ric}(g) = \bar{g}$ and its total scalar curvature $s_g(M)$ is finite. This proposition is a corollary of the Schoen and Yau *vanishing theorem* for harmonic maps of complete non-compact Riemannian manifolds (see [8]).

Our statements generalize and complement the results of the papers [1] and [2], and the monograph [3].

2 Harmonic maps

For the discussion of harmonic maps we will follow Eells and Sampson [4]. Let (M, g) and (\bar{M}, \bar{g}) be two Riemannian manifolds with the Levi-Civita connections $\nabla := \nabla(g)$ and $\bar{\nabla} := \nabla(\bar{g})$, and $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ be a smooth map. The *energy density* of f is defined as the scalar function

$$e(f) = 2^{-1} \|df\|^2 \quad (1)$$

where $\|df\|^2$ is the squared norm of the differential of f with respect to metric on the bundle $T^*M \otimes f^*T\bar{M}$. Then the *total energy* of f is obtained by integrating the energy density $e(f)$ over M

$$E(f) = \int_M e(f) \, dVol_g \quad (2)$$

where dV_g denotes the measure on (M, g) induced by the metric g . If f is of class C^2 and $E(f) < +\infty$, and f is an extremum of the *Dirichlet energy functional* $E(f)$, then f is called a *harmonic map* and satisfies the *Euler–Lagrange equation*

$$\text{trace}_g D \, df = 0 \quad (3)$$

where D is the connection in the bundle $T^*M \otimes f^*T\bar{M}$ induced from the Levi-Civita connections ∇ and $\bar{\nabla}$ of (M, g) and (\bar{M}, \bar{g}) , respectively.

For any harmonic $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ we have the *Weitzenböck formula* (see [4])

$$\Delta e(f) = Q(f) + \|Ddf\|^2 \quad (4)$$

where Δ is the Laplace–Beltrami operator $\Delta = \operatorname{div} \nabla$ and

$$Q(f) = g(\operatorname{Ric}, f^* \bar{g}) - \operatorname{trace}_g(\operatorname{trace}_g(f^* \overline{\operatorname{Riem}})) \quad (5)$$

where $\operatorname{Ric} = \operatorname{Ric}(g)$ is the Ricci tensor of (M, g) and $\overline{\operatorname{Riem}}$ is the Riemannian curvature tensor of (\bar{M}, \bar{g}) . Let the inequality $\overline{\operatorname{sec}} \leq 0$ be satisfied anywhere on (\bar{M}, \bar{g}) and the inequality $\operatorname{Ric} \geq 0$ be satisfied anywhere on compact (M, g) , then $Q(f)$ is non-negative everywhere on M . Since our hypothesis implies that the left hand side of (3) is non-negative, then using the *Hopf's lemma* (see [6, pp. 30–31]), one can verify that $e(f)$ is constant. In this case, from (4) we obtain $Ddf = 0$. In this case, f is totally *geodesic map* (see [7]). Now we can formulate the following *vanishing theorem* on harmonic maps. Namely, if $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ is any harmonic mapping between a compact Riemannian manifold (M, g) with the Ricci tensor $\operatorname{Ric} \geq 0$ and a Riemannian manifold (\bar{M}, \bar{g}) with the sectional curvature $\overline{\operatorname{sec}} \leq 0$ then f is *totally geodesic* and has constant *energy density* $e(f)$. Furthermore, if there is at least one point of M at which its Ricci curvature $\operatorname{Ric} > 0$, then every harmonic map $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ is constant (see [4, p. 124]).

In turn, Schoen and Yau have showed in [8] that $\sqrt{e(f)}$ is subharmonic function on (M, g) if $Q(f) \geq 0$. On other hand, Yau has proved in other his paper [9] that every non-negative L^2 -integrable subharmonic function on a complete Riemannian manifold must be constant. Applying this to $\sqrt{e(f)}$, we conclude that $\sqrt{e(f)}$ is a constant if the total energy $E(f) < +\infty$ (see also [8]). On the other hand, every complete non-compact Riemannian manifold with nonnegative Ricci curvature has infinite volume (see [9]). In our case, we have $\operatorname{Ric} \geq 0$ then the volume of (M, g) is infinite. This forces the constant $e(f)$ to be zero and f to be a constant map (see also [8]). Now we can formulate another celebrated *vanishing theorem* on harmonic maps: If the sectional curvature of (\bar{M}, \bar{g}) is non-positive and (M, g) is a complete non-compact manifold with $\operatorname{Ric} \geq 0$, then any harmonic map $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ with the finite energy $E(f)$ is a constant map (see [8], [10, p. 116]). We remark that in the original paper [8] the manifold (\bar{M}, \bar{g}) was assumed to be compact. However, this assumption is superfluous (see [10, p. 116]).

3 The main theorem

If we consider the manifold M with two Riemannian metrics g and \bar{g} then the identity mapping $\operatorname{Id}: (M, g) \rightarrow (M, \bar{g})$ is harmonic if and only if the deformation tensor $T = \bar{\nabla} - \nabla$ is a section of the tensor bundle $TM \otimes S_0^2 M$, because in this case the Euler–Lagrange equation (3) has the form $\operatorname{trace}_g T = 0$ (see [1, 3]). In particular, if (M, g) is a manifold of strictly positive Ricci Ric curvature, then $\operatorname{Id}: (M, g) \rightarrow (M, \operatorname{Ric})$ is a harmonic map (see [1]). Next we can formulate and prove the following

Theorem 1 *Let (M, \bar{g}) be a compact Riemannian manifold with the sectional curvature $\overline{\text{sec}} \geq 0$ and with the Ricci tensor $\overline{\text{Ric}} \leq \bar{g}$. If g is another Riemannian metric on M with the Ricci tensor $\text{Ric} = \bar{g}$, then g and \bar{g} have the same Levi-Civita connection. Furthermore, if the full holonomy group $\text{Hol}(\bar{g})$ of (M, \bar{g}) is irreducible then $\text{Ric} = \overline{\text{Ric}}$.*

Proof With the above assumptions, we have $\text{Ric} = \bar{g} > 0$, then the identity map $\text{Id}: (M, g) \rightarrow (M, \bar{g})$ is harmonic. In this case, we have $e(f) = \frac{1}{2}s$ for the energy density $e(f)$ of the harmonic identity map $\text{Id}: (M, g) \rightarrow (M, \bar{g})$ and the scalar curvature $s = \text{trace}_g \text{Ric}$ of the Riemannian manifold (M, g) (see [1], [3, p. 152]). Therefore, s satisfies the Weitzenböck formula (4) which has the following form (see [1]):

$$\frac{1}{2}\Delta s = Q(f) + \|D \text{d}f\|^2 \quad (6)$$

where $Q(f) = g^{ik}g^{jl}(\bar{g}_{ij}\bar{g}_{kl} - \bar{R}_{ijkl})$ and $\|D \text{d}f\|^2 = g^{ij}g^{kl}\bar{g}_{pq}T_{ik}^p T_{jl}^q \geq 0$ for local components g_{ij} , \bar{g}_{kl} , \bar{R}_{ijkl} and T_{kl}^i of metric tensors g and \bar{g} , the Riemannian curvature tensor $\overline{\text{Riem}}$ and the deformation tensor T , respectively. On the other hand, we have the identity (see [3, p. 436], [11])

$$(\bar{g}_{ij}\bar{R}_{kl} - \bar{R}_{ijkl})\varphi^{ik}\varphi^{jl} = \sum_{i<j} \overline{\text{sec}}(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 \quad (7)$$

where φ is any smooth symmetric tensor field such that $\varphi(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}$ for the Kronecker delta δ_{ij} and some orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ at any point $x \in M$. Then equation (6) can be rewritten in the form

$$\frac{1}{2}\Delta s = \sum_{i<j} \overline{\text{sec}}(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 + g^{ik}g^{jl}\bar{g}_{ij}(\bar{g}_{kl} - \bar{R}_{kl}) + \|T\|^2 \quad (8)$$

where $g(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}$. We remark that under the stated assumptions the right side of (8) is non-negative, since then $\Delta s \geq 0$. Therefore, the scalar curvature s is a positive subharmonic function on (M, g) . If (M, \bar{g}) is a compact Riemannian manifold, then using the Hopf's lemma (see [6, pp. 30–31]), one can verify that $s = \text{const}$. In this case, from (8) we obtain $T = 0$. Then g and \bar{g} have the same Levi-Civita connection, i.e. $\bar{\nabla}g = 0$. Furthermore, if the full holonomy group $\text{Hol}(\bar{g})$ of (M, \bar{g}) is irreducible then the metric $g = C\bar{g}$ for some constant $C > 0$ (see [3, pp. 282, 285–287]). In this case, we have the identity $\text{Ric} = \overline{\text{Ric}}$ because $\text{Ric}(\bar{g}) = \text{Ric}(C\bar{g})$ for some positive constant C (see [3, pp. 44, 152]). \square

4 Two vanishing theorems

In [13] the following non-existence theorem was proved: Let (M, \bar{g}) be a compact Riemannian manifold with all sectional curvature less than $(\bar{n} - 1)^{-1}$. Then there is no Riemannian metric g on M such that its Ricci tensor $\text{Ric} = \bar{g}$. In its turn, in [2] the following vanishing theorem was proved: Let \bar{g} be a metric

on a compact manifold M with the sectional curvature $\overline{\text{sec}} < +1$, then any metric g does not exist on M such that its Ricci tensor $\text{Ric} = \bar{g}$. We also get a non-existence result which complements the above propositions. In turn, we can formulate and prove an analogue of these propositions in the following form.

Theorem 2 *Let (M, \bar{g}) be a compact Riemannian manifold with nonnegative section curvatures and with the Ricci tensor $\overline{\text{Ric}} \leq \bar{g}$. If in addition there is at least one point of M at which the Ricci tensor $\overline{\text{Ric}} < \bar{g}$, then there is no Riemannian metric g on M such that its Ricci tensor $\text{Ric} = \bar{g}$.*

Proof Let M be a compact manifold. We may assume that M is oriented by taking the twofold covering of M if necessary. Then by Green's theorem (see [6, pp. 31–33]) we obtain from (6) the following identity

$$\int_M Q(f) dVol_g + \int_M \|T\|^2 dVol_g = 0. \quad (9)$$

If the inequalities $\overline{\text{sec}} \geq 0$ and $\overline{\text{Ric}} \leq \bar{g}$ are satisfied and there is a one point x of M in which $\overline{\text{Ric}} < \bar{g}$ then the inequality $\int_M Q(f) dVol_g > 0$ holds. This inequality contradicts the equation (9). In this case, the harmonic mapping f must be constant. \square

Theorem 3 *Let (M, \bar{g}) be a non-compact Riemannian manifold with the section curvature $\overline{\text{sec}} \geq 0$ and with the Ricci tensor $\overline{\text{Ric}} < \bar{g}$. Then there is no complete Riemannian metric g on (M, \bar{g}) such that its Ricci tensor $\text{Ric} = \bar{g}$ and its total scalar curvature $s(M)$ is finite.*

Proof Let (M, \bar{g}) be a non-compact Riemannian manifold with the section curvature $\overline{\text{sec}} \geq 0$ and with the Ricci tensor $\overline{\text{Ric}} < \bar{g}$, then $Q(f)$ is non-negative everywhere on M . If we assume that there is complete Riemannian metric g on (M, \bar{g}) such that its Ricci tensor $\text{Ric} = \bar{g} > 0$, then the volume of (M, g) is infinite (see [9]). Moreover, we have $e(f) = \frac{1}{2}s$ for the energy density $e(f)$ of the harmonic identity map $\text{Id}: (M, g) \rightarrow (M, \bar{g})$ and the scalar curvature $s = \text{trace}_g \text{Ric}$ of the Riemannian manifold (M, g) . In this case, \sqrt{s} is a strictly positive subharmonic function on a complete Riemannian manifold (M, g) of infinite volume (see [8]). In addition, if we suppose that the total scalar curvature $\int_M s dVol_g < +\infty$, then s must be zero (see [8], [12, p. 262]). On the other hand, according to the condition of our theorem the scalar curvature $s = \text{trace}_g \bar{g} > 0$ and hence there is no complete Riemannian metric g on non-compact (M, \bar{g}) such that its Ricci tensor $\text{Ric} = \bar{g}$. \square

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