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The Killing Tensors on an $n$-dimensional Manifold with $SL(n,\mathbb{R})$-structure

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Abstract

In this paper we solve the problem of finding integrals of equations determining the Killing tensors on an $n$-dimensional differentiable manifold $M$ endowed with an equiaffine $SL(n,\mathbb{R})$-structure and discuss possible applications of obtained results in Riemannian geometry.

Key words: Differentiable manifold, $SL(n,\mathbb{R})$-structure, Killing tensors.

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1 Introduction

1.1. The “structural point of view” of affine differential geometry was introduced by K. Nomizu in 1982 in a lecture at Münster University with the title “What is Affine Differential Geometry?” (see [12]). In the opinion of K. Nomizu, the geometry of a manifold $M$ endowed with an equiaffine structure is called affine differential geometry.

In recent years, there has been a new ware of papers devoted to affine differential geometry. Today the number of publications (including monographs) on affine differential geometry reached a considerable level. The main part of these publications is devoted to geometry of hypersurfaces (see [15, 16] for the history and references).
1.2. In the present paper we solve the problem of finding integrals of equations determining the Killing tensors (see [8] for the definitions, properties and applications) on an $n$-dimensional differentiable manifold $M$ endowed with an equiaffine structure. The paper is a direct continuation of [18]. The same notations are used here.

The first of two present theorems proved in our paper is an affine analog of the statement published in the paper [17], which appeared in the process of solving problems in General relativity.

2 Definitions and results

2.1. In order to clarify the approach to problem of finding integrals of equations determining the Killing tensors on an $n$-dimensional differentiable manifold $M$ we shall start with a brief introduction to the subject which emphasizes the notion of an equiaffine $SL(n, \mathbb{R})$-structure.

Let $M$ be a connected differentiable manifold of dimension $n$ $(n > 2)$, and let $L(M)$ be the corresponding bundle of linear frames with structural group $GL(n, \mathbb{R})$. We define $SL(n, \mathbb{R})$-structure on $M$ as a principal $SL(n, \mathbb{R})$-sub-bundle of $L(M)$. It is well known that an $SL(n, \mathbb{R})$-structure is simply a volume element on $M$, i.e. an $n$-form $\eta$ that does not vanishing anywhere (see [6, Chapter I, §2]).

We recall the famous problem of the existence of a uniquely determined linear connection $\nabla$ reducible to $G$ for each given $G$-structure on $M$ (see [1, p. 213]). For example, if $M$ is a manifold with a pseudo-Riemannian metric $g$ of an arbitrary index $k$, then the bundle $L(M)$ admits a unique linear connection $\nabla$ without torsion that is reducible to $O(m, k)$-structure. Such a connection is called the Levi-Civita connection. It is characterized by the following condition $\nabla g = 0$.

A linear connection $\nabla$ having zero torsion and reducible to $SL(n, \mathbb{R})$ is said to be equiaffine and can be characterized by the following equivalent conditions (see [15, p. 99], [16, pp. 57–58]):

1. $\nabla \eta = 0$;
2. the Ricci tensor $\text{Ric}$ of $\nabla$ is symmetric; that means $\text{Ric}(X, Y) = \text{Ric}(Y, X)$ for any vector fields $X, Y \in C^\infty TM$.

An equiaffine $SL(n, \mathbb{R})$-structure or an equiaffine structure on an $n$-dimensional differentiable manifold $M$ is a pair $(\eta, \nabla)$, where $\nabla$ is a linear connection with zero torsion and $\eta$ is a volume element which is parallel relative to $\nabla$ (see [13, p. 43]).

The curvature tensor $R$ of an equiaffine connection $\nabla$ admits a point-wise $SL(n, \mathbb{R})$-invariant decomposition of the form

$$R = (n - 1)^{-1}[\text{id}_M \otimes \text{Ric} - \text{Ric} \otimes \text{id}_M] + \mathcal{W}$$

where $\mathcal{W}$ is the Weyl projective curvature tensor (see [16, p. 73–74], [2, §40]). Then two classes of equiaffine structures can be distinguished in accordance
with this decomposition: the Ricci-flat equiaffine $SL(n, \mathbb{R})$-structures for which $\text{Ric} = 0$, and the equiprojective $SL(n, \mathbb{R})$-structures for which
\[
R = (n - 1)^{-1}[\text{id}_M \otimes \text{Ric} - \text{Ric} \otimes \text{id}_M].
\]

**Remark 1** Recall that a linear connection $\nabla$ with zero torsion is called Ricci-flat if the Ricci tensor $\text{Ric} = 0$ (see [9]). On the other hand, a connection $\nabla$ is called equiprojective if the Weyl projective curvature tensor $W = 0$ (see [15, §18]). In the literature equiprojective connections sometimes are called projectively flat (see, for example, [16, p. 73]).

An autodiffeomorphism of the manifold $M$ is an automorphism of $SL(n, \mathbb{R})$-structure if and only if it preserves the volume element $\eta$. Let $X$ be a vector field on $M$. The function $\text{div} X$ defined by the formula $(\text{div} X)\eta = L_X \eta$ where $L_X$ is the Lie differentiation in the direction of the vector field $X$ is called the divergence of $X$ with respect to the $n$-form $\eta$ (see [7, Appendix no. 6]). Obviously, $X$ is an infinitesimal automorphism of an $SL(n, \mathbb{R})$-structure if and only if $\text{div} X = 0$. Such a vector field $X$ is said to be solenoidal.

For an arbitrary vector field $X$ on $M$ with a linear connection $\nabla$ we can introduce the tensor field $A_X = L_X - \nabla_X$ regarded as a field of linear endomorphisms of the tangent bundle $TM$. If $M$ is an $n$-dimensional with an equiaffine $SL(n, \mathbb{R})$-structure then the formula $\text{trace} A_X = -\text{div} X$ can be verified directly (see [7, Appendix no. 6]).

We have the $SL(n, \mathbb{R})$-invariant decomposition
\[
A_X = -n^{-1}(\text{div} X) \text{id}_M + \hat{A}_X
\]
at every point $x \in M$.

Two classes of vector fields on $M$ endowed with an equiaffine $SL(n, \mathbb{R})$-structure can be distinguished in accordance with this decomposition: the solenoidal vector fields and the concircular vector fields for which, by definition (see [14, p. 322], [9]), we have $A_X = -n^{-1}(\text{div} X) \text{id}_M$.

The integrability conditions of the structure equation $A_X = -n^{-1}(\text{div} X) \text{id}_M$ of the concircular vector field $X$ is the Ricci’s identity
\[
Y(\text{div} X)Z - Z(\text{div} X)Y = nR(Y, Z)X
\]
for any vector fields $Y, Z \in C^\infty TM$ (see [2, §11]). This identity are equivalent to the condition $W(Y, Z)X = 0$ for any vector fields $Y, Z \in C^\infty TM$. It follows that an equiaffine $SL(n, \mathbb{R})$-structure on an $n$-dimensional manifold $M$ is equiprojective if and only if there exist $n$ linearly independent concircular vector fields $X_1, X_2, \ldots, X_p$ on $M$ (see also [24]). This statement is an affine analog of the well known fact for the Riemannian manifold $M$ of constant sectional curvature (see [3]).

**Remark 2** A pseudo-Riemannian manifold $(M, g)$ with a projectively flat Levi-Civita connection $\nabla$ is a manifold of constant section curvature (see [15, §18]). Therefore a manifold $M$ endowed with an equiprojective $SL(n, \mathbb{R})$-structure is an affine analog of a pseudo-Riemannian manifold of constant sectional curvature.
2.2. We consider an $n$-dimensional manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure and denote by $\Lambda^p M$ ($1 \leq p \leq n - 1$) the $p^{th}$ exterior power $\Lambda^p (T^* M)$ of the cotangent bundle $T^* M$ of $M$. Hence $C^\infty \Lambda^p M$, the space of all $C^\infty$-sections of $\Lambda^p M$, is the space of skew-symmetric covariant tensor fields of degree $p$ ($1 \leq p \leq n - 1$).

Let $\gamma: J \subset \mathbb{R} \to M$ be an arbitrary geodesic on $M$ with affine parameter $t \in J$. In this case, we have $\nabla_{\dot{\gamma}} \frac{d\gamma}{dt} = 0$ for the tangent vector $\frac{d\gamma}{dt}$ of $\gamma$.

**Definition 1** (see [18]). A skew-symmetric tensor field $\omega \in C^\infty \Lambda^p M$ ($1 \leq p \leq n - 1$) on an $n$-dimensional manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure is called Killing-Yano tensor of degree $p$ if the tensor

$$i_\frac{d\gamma}{dt} \omega := \text{trace} \left( \frac{d\gamma}{dt} \otimes \omega \right)$$

is parallel along an arbitrary geodesic $\gamma$ on $M$.

From this definition we conclude that

$$\left( \nabla_{\frac{d\gamma}{dt}} \omega \right) \left( \frac{d\gamma}{dt}, X_2, \ldots, X_p \right) = 0$$

for any vector fields $X_2, \ldots, X_p \in C^\infty TM$. Since the geodesic $\gamma$ may be chosen arbitrary, the above relation is possible if and only if $\nabla \omega \in C^\infty \Lambda^{p+1} M$, which is equivalent to $d\omega = (n+1)^{-1} \nabla \omega$ for the exterior differential operator $d: C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M$.

Obviously, the set of Killing-Yano tensors of degree $p$ ($1 \leq p \leq n - 1$) constitutes an $\mathbb{R}$-module of tensor fields on $M$, denoted by $K^p (M, \mathbb{R})$.

Let $X_1, \ldots, X_p$ be $p$ linearly independent concircular vector fields on $M$ ($1 \leq p \leq n - 1$). Then direct inspection shows that the tensor field $\omega$ of degree $n - p$ dual to the tensor field $\omega = alt \{X_1 \otimes \cdots \otimes X_p\}$ relative to the $n$-form $\eta$ is a Killing-Yano tensor (see also [18]). Therefore on any $n$-manifold $M$ with equiprojective $SL(n, \mathbb{R})$-structure, there exist at least $n! [p! (n - p)!]^{-1}$ linearly independent Killing-Yano tensors (see [18]). Moreover the following theorem is true.

**Theorem 1** On an $n$-dimensional manifold $M$ endowed with an equiprojective $SL(n, \mathbb{R})$-structure $(\eta, \nabla)$, there exist a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary Killing-Yano tensor $\omega$ of degree $p$ ($1 \leq p \leq n - 1$) has the components

$$\omega_{i_1 \ldots i_p} = e^{(p+1)\psi} (A_{i_0 i_1 \ldots i_p} x^{i_0} + B_{i_1 \ldots i_p})$$

(2.1)

where $A_{i_0 i_1 \ldots i_p}$ and $B_{i_1 \ldots i_p}$ are arbitrary constants skew-symmetric w.r.t. all their indices and $\psi = (n + 1)^{-1} \ln(\eta)$.

From the theorem we conclude that the maximum of linearly independent the Killing–Yano tensors is by calculating the number $K^p_n$ of independent $A_{i_0 i_1 \ldots i_p}$ and $B_{i_1 \ldots i_p}$ which exist after accounting for the symmetries on the indices. It follows that $K^p_n = \frac{(n+1)!}{(p+1)! (n-p)!}$ is the maximum number linearly independent the Killing–Yano tensors.
Corollary 1 Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$-structure then
\[
\dim K^p(M, \mathbb{R}) = \frac{(n + 1)!}{(p + 1)!(n - p)!}.
\]

On our fixed manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure, we denote by $S^p M$ the bundle of symmetric covariant tensor fields of degree $p$ on $M$. Hence $C^\infty S^p M$, the space of all $C^\infty$-sections of $S^p M$, is the space symmetric covariant tensor fields of degree $p$.

Definition 2 (see [18]). A symmetric tensor field $\varphi \in C^\infty S^p M$ on an $n$-dimensional manifold $M$ with an equiaffine $SL(n, \mathbb{R})$-structure is called Killing tensor of degree $p$ if
\[
\varphi \left( \frac{d\gamma}{dt}, \ldots, \frac{d\gamma}{dt} \right) = \text{const.}
\]
along an arbitrary geodesic $\gamma$ on $M$.

Let $\varphi \left( \frac{d\gamma}{dt}, \ldots, \frac{d\gamma}{dt} \right) = \text{const.}$ along an arbitrary geodesic $\gamma$ on $M$ and hence $\varphi$ is a Killing tensor. Then the above relation is possible if and only if
\[
\delta^* \varphi := \sum_{cicl} \{ \nabla \varphi \} = 0
\]
where for the local components $\nabla_{i_0} \varphi_{i_1 \ldots i_p}$ of $\nabla \varphi$ we define the sum
\[
\sum_{cicl} \{ \nabla_{i_0} \varphi_{i_1 \ldots i_p} \}
\]
as the sum of the terms obtained by a cyclic permutation of indices $i_0, i_1, \ldots, i_p$.

Obviously, the set of Killing tensors constitutes an $\mathbb{R}$-module of tensor fields on $M$, denoted by $T^p(M, \mathbb{R})$.

Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$-structure $(\eta, \nabla)$, and $\omega_1, \ldots, \omega_p$ be $p$ linearly independent Killing-Yano tensors of degree 1 on $M$. Then direct inspection shows that the tensor field $\varphi := \text{sym}\{\omega_1 \otimes \cdots \otimes \omega_p\}$ is a Killing tensor of degree $p$. Therefore on any $n$-manifold $M$ with equiprojective $SL(n, \mathbb{R})$-structure, there exist at least $(n + p - 1)!p!(n - 1)!-1$ linearly independent Killing tensors (see also [23]). Moreover the following theorem is true.

Theorem 2 On an $n$-dimensional manifold $M$ endowed with an equiprojective $SL(n, \mathbb{R})$-structure $(\eta, \nabla)$, there exist a local coordinate system $x^1, \ldots, x^n$ in which the components $\varphi_{i_1 \ldots i_p}$ of an arbitrary Killing tensor $\varphi$ of degree $p$ can be expressed in the form of an $p^{th}$ degree polynomial in the $x^i$’s
\[
\varphi_{i_1 \ldots i_p} = e^{2p \psi} \sum_{q=0}^{p} A_{i_1 \ldots i_{p-j} j_1 \ldots j_q} x^{j_1} \ldots x^{j_q}
\]
where the coefficients $A_{i_1 \ldots i_p j_1 \ldots j_q}$ are constant and symmetric in the set of indices $i_1, \ldots, i_p$ and the set of indices $j_1, \ldots, j_q$. In addition to these properties the coefficients $A_{i_1 \ldots i_p j_1 \ldots j_q}$ have the following symmetries

\[
\sum_{c i d l} \{A_{i_1 \ldots i_p j_1 \ldots j_{p-s}} \} j_{p-s+1} = 0
\]  
(2.3)

for $s = 1, \ldots, p - 1$ and

\[
\sum_{c i d l} \{A_{i_1 \ldots i_p j_1} \} = 0.
\]  
(2.4)

From the theorem we conclude that the maximum number of linearly independent the Killing tensors is obtained by calculating the number $T_{p}^n$ of independent $A_{i_1 \ldots i_p j_1 \ldots j_q}$ ($q = 0, 1, \ldots, n$) which exist after accounting for the symmetries on the indices the dependence relations (2.3) and (2.4). By [4] it follows that

\[
T_{p}^n = \frac{p(p+1)^2(p+2)^2 \ldots (m+p-1)^2(m+p)}{(p+1)!p!}
\]

is the maximum number linearly independent the Killing–Yano tensors. Then we have the following proposition.

**Corollary 2** Let $M$ be an $n$-dimensional manifold endowed with an equiprojective $SL(n, \mathbb{R})$-structure then

\[
\dim T_{p}^n(M, \mathbb{R}) = \frac{p(p+1)^2(p+2)^2 \ldots (m+p-1)^2(m+p)}{p!(p+1)!}.
\]

### 3 Proofs of theorems

**3.1.** We let $f: \bar{M} \rightarrow M$ denote the mapping of an $\bar{n}$-dimensional manifold $\bar{M}$ endowed with an equiaffine $SL(\bar{n}, \mathbb{R})$-structure onto another an $n$-dimensional manifold $M$ endowed with an equiaffine $SL(n, \mathbb{R})$-structure, and let $f^*$ be the differential of this mapping. For any covariant tensor field $\omega$ on $M$, we can then define the covariant tensor field $f^*\omega$ on $\bar{M}$, where $f^*$ is the transformation transposed to the transformation $f$.

If $\dim \bar{M} = \dim M = n$ and $f: \bar{M} \rightarrow M$ is a projective diffeomorphism, i.e., a mapping that transforms an arbitrary geodesic in $\bar{M}$ into a geodesic in $M$, then we have the following lemma.

**Lemma 1** Let $f: \bar{M} \rightarrow M$ be a projective diffeomorphism of $n$-dimensional manifolds endowed with the equiaffine $SL(n, \mathbb{R})$-structures $(\bar{\eta}, \bar{\nabla})$ and $(\eta, \nabla)$ respectively. Then for an arbitrary Killing-Yano tensor $\omega$ of degree $p$ ($1 \leq p \leq n - 1$) on the manifold $M$ the tensor field $\bar{\omega} = e^{-(p+1)\psi}(f^*\omega)$ with $\psi = (n+1)^{-1} \ln(\eta/\bar{\eta})$ will be the Killing-Yano tensor of degree $p$ on the manifold $\bar{M}$.
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Proof. It is known that the diffeomorphism $f: \bar{M} \to M$ can be realized following the principle of equality of the local coordinates $\bar{x}^1 = x^1, \ldots, \bar{x}^n = x^n$ at the corresponding points $\bar{x}$ and $x = f(\bar{x})$ of these manifolds. In this case, we have the equalities (see [15, §18], [9, 10, 26])

$$\Gamma^k_{ij} = \Gamma^k_{ij} + \psi_i \delta^k_j + \psi_j \delta^k_i$$

(3.1)

for the objects $\Gamma^k_{ij}$ and $\Gamma^k_{ij}$ of the a equiaffine connections $\nabla$ and $\bar{\nabla}$ in the coordinate system $x^1, \ldots, x^n$ that is common w.r.t. the mapping $f: \bar{M} \to M$, and for the gradient $\psi_j = (n+1)^{-1} \partial_j \ln[\eta/\bar{\eta}]$.

Equalities (3.1) imply that the mapping $f^{-1}$, which is inverse to the projective diffeomorphism $f: \bar{M} \to M$, is a projective mapping [10, p. 262].

We set $\omega_{i_1 \ldots i_p}$ be the local components of a Killing-Yano tensor $\omega$ of degree $p$ $(1 \leq p \leq n-1)$ arbitrary defined on the manifold $M$; by definition, these components satisfy the equations

$$\nabla_{i_0} \omega_{i_1 \ldots i_p} + \nabla_{i_1} \omega_{i_0 \ldots i_p} = 0.$$  

(3.2)

From equalities (3.2) we find directly that the components

$$\bar{\omega}_{i_1 \ldots i_p} = e^{-(p+1)\psi} \omega_{i_1 \ldots i_p}$$

(3.3)

of the tensor field $\bar{\omega} = e^{-(p+1)\psi} (f^* \omega)$ satisfy the equations

$$\nabla_{i_0} \bar{\omega}_{i_1 \ldots i_p} + \nabla_{i_1} \bar{\omega}_{i_0 \ldots i_p} = 0.$$  

(3.4)

Hence, the tensor field $\bar{\omega}$ is a Killing-Yano tensor of degree $p$ $(1 \leq p \leq n-1)$ on the manifold $\bar{M}$.  

3.2. Let $A^n$ be an $n$-dimensional affine space with a volume element given by the determinant: $\det(e_1, \ldots, e_n) = 1$, where $\{e_1, \ldots, e_n\}$ is the standard basis of the underlying vector space for $A^n$. We denote by $\nabla$ the standard linear connection in $A^n$ relative to which the volume element “det” is parallel (see [13], [16, p. 10]).

Let $f: \bar{M} \to A^n$ be a projective diffeomorphism from a manifold $\bar{M}$ endowed with equiaffine $SL(n, \mathbb{R})$-structure onto an affine space $A^n$ endowed with standard equiaffine $SL(n, \mathbb{R})$-structure. It is well known that manifolds endowed with equiprojective $SL(n, \mathbb{R})$-structures and only these manifolds are projectively diffeomorphic to an affine space $A^n$ (see [15, §18], [9]) therefore in our case the $SL(n, \mathbb{R})$-structure of the manifold $\bar{M}$ must be an equiprojective structure.

If $A^n$ is an affine space with the Cartesian system of coordinates $\bar{x}_1, \ldots, \bar{x}^n$ then the components $\bar{\omega}_{i_1 \ldots i_p}$ of the Killing-Yano tensor $\bar{\omega}$ of degree $p$ $(1 \leq p \leq n-1)$ in equation (3.4) must now satisfy

$$\partial_j \bar{\omega}_{i_1 \ldots i_p} + \partial_i \bar{\omega}_{j_1 \ldots i_p} = 0$$

(3.5)

where $\partial_j = \frac{\partial}{\partial x^j}$. From (3.5) we conclude the following equations

$$\partial_k \partial_j \bar{\omega}_{i_1 \ldots i_p} + \partial_k \partial_i \bar{\omega}_{j_1 \ldots i_p} = 0;$$

(3.6)
\[
\begin{align*}
\partial_j \partial_i \bar{\omega}_{ki1 \ldots i_p} + \partial_j \partial_k \bar{\omega}_{i1 \ldots i_p} &= 0; \\
\partial_i \partial_k \bar{\omega}_{ji1 \ldots i_p} + \partial_i \partial_j \bar{\omega}_{k11 \ldots i_p} &= 0.
\end{align*}
\] (3.7) (3.8)

From (3.6), (3.7), (3.8) we find
\[
\partial_k \partial_j \bar{\omega}_{i1 \ldots i_p} = 0,
\] (3.9)

by using identities \(\frac{\partial^2 h}{\partial x^i \partial x^j} = \frac{\partial^2 h}{\partial x^j \partial x^i}\) which are carried out for an arbitrary smooth function \(h: \mathbb{A}^n \to \mathbb{R}\). The integrals of equations (3.9) take the form
\[
\bar{\omega}_{i1 \ldots i_p} = A_{i_0 i_1 \ldots i_p} \bar{x}^{i_0} + B_{i_1 \ldots i_p},
\] (3.10)

for any skew-symmetric constants \(A_{i_0 i_1 \ldots i_p}\) and \(B_{i_1 \ldots i_p}\) (see also [23, 19]). Taking the components (3.10) of the Killing-Yano tensor \(\bar{\omega}\) in \(\mathbb{A}^n\) and using Lemma 1, we can formulate Theorem 1.

3.3. Let \(\bar{M}\) be a manifold of dimension \(n\) endowed with the equiaffine \(SL(n, \mathbb{R})\)-structure \((\bar{\eta}, \bar{\nabla})\) and \(M\) be a manifold of some dimension endowed with the equiaffine \(SL(n, \mathbb{R})\)-structure \((\eta, \nabla)\). Let there be given a projective diffeomorphism \(f: \bar{M} \to M\), then we have the following lemma.

**Lemma 2** Let \(f: \bar{M} \to M\) be a projective diffeomorphism of \(n\)-dimensional manifolds endowed with the equiaffine \(SL(n, \mathbb{R})\)-structures \((\bar{\eta}, \bar{\nabla})\) and \((\eta, \nabla)\) respectively. Then for an arbitrary Killing tensor \(\varphi\) of degree \(p\) on the manifold \(M\) the tensor field \(\bar{\varphi} = e^{-2p\psi} (f^* \varphi)\) with \(\psi = (n+1)^{-1} \ln(\eta/\bar{\eta})\) will be the Killing tensor of degree \(p\) on the manifold \(\bar{M}\).

**Proof** We set \(\varphi_{i_1 \ldots i_p}\) to be components of the Killing tensor \(\varphi\) arbitrary defined on the manifold \(\bar{M}\); by definition, these components satisfy the following equations \(\sum_{cicl} (\nabla_{i_0} \varphi_{i_1 \ldots i_p}) = 0\). Then we find directly that the components \(\bar{\varphi}_{i_1 \ldots i_p} = e^{-2p\psi} \varphi_{i_1 \ldots i_p}\) of the tensor \(\bar{\varphi} = e^{-2p\psi} \varphi\) satisfy the equations
\[
\sum_{cicl} (\bar{\nabla}_{i_0} \bar{\varphi}_{i_1 \ldots i_p}) = e^{-2p\psi} \sum_{cicl} (\nabla_{i_0} \varphi_{i_1 \ldots i_p}) = 0.
\] (3.11)

From (3.11) we conclude that the tensor field \(\bar{\varphi}\) is a Killing tensor of degree \(p\) on the manifold \(\bar{M}\). \(\Box\)

3.4. It follows from Nijenhuis (see [11]) that in an \(n\)-dimensional affine space \(\mathbb{A}^n\) the components \(\bar{\varphi}_{i_1 \ldots i_p}\) of the Killing tensor \(\bar{\varphi}\) of degree \(p\) can be expressed in the form of an \(p^{th}\) degree polynomial in the \(\bar{x}\)'s
\[
\varphi_{i_1 \ldots i_p} = e^{-2p\psi} \sum_{q=0}^{p} A_{i_1 \ldots i_p j_1 \ldots j_q} \bar{x}^{j_1} \ldots \bar{x}^{j_q}.
\] (3.12)

The coefficients \(A_{i_1 \ldots i_p j_1 \ldots j_q}\) are constant and symmetric in the set of indices \(i_1, \ldots, i_p\) and the set of indices \(j_1, \ldots, j_q\). In addition to these properties the coefficients \(A_{i_1 \ldots i_p j_1 \ldots j_q}\) have the following symmetries
\[
\sum_{cicl} \{A_{i_1 \ldots i_p j_1 \ldots j_{p-s}}\} j_{p-s+1} = 0
\]
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for $s = 1, \ldots, p - 1$ and

$$\sum_{ciel} \{A_{i_1 \ldots i_p j_1}\} = 0.$$  

Taking the components (3.12) of the Killing tensor $\tilde{\varphi}$ in $A^n$ and using Lemma 2, we can formulate Theorem 2.

4 Applications to Riemannian geometry

4.1. Let $(M, g)$ be a pseudo-Riemannian manifold of dimensional $n$. Then from the present theorems 1 and 2 we conclude that an arbitrary Killing vector $\omega$ has the following local covariant components

$$\omega_i = e^{2\psi}(A_{ik}x^k + B_i)$$

where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $A$'s and $B$'s are constants and $A_{ik} + A_{ki} = 0$ (see also [17]). It follows that the group of infinitesimal isometric transformations has $\frac{1}{2}n(n + 1)$ parameters (see also [2, §71]).

4.2. Following [25, 5], a skew-symmetric covariant tensor field $\vartheta$ of degree $p$ ($1 \leq p \leq n - 1$) is called a conformal Killing tensor if $\vartheta \in \ker D$ for

$$D = \nabla - \frac{1}{p + 1}d - \frac{1}{n - p + 1}g \wedge d^*$$

where $d^*$ is the codifferential operator $d^*: C^\infty \Lambda^{p+1}M \to C^\infty \Lambda^p M$ and

$$(g \wedge d^* \vartheta)_{i_0 i_1 \ldots i_p} = \sum_{a=1}^{p} (-1)^{a+1} g_{i_0 i_a} (d^* \vartheta)_{i_1 \ldots \hat{i_a} \ldots i_p}.$$  

Obviously, the set of conformal Killing tensors constitutes an vector space of tensor fields on $(M, g)$, denoted by $C^p(M, \mathbb{R})$ (see [21]). If a conformal Killing tensor $\vartheta$ belongs to $\ker d^*$, then it is a Killing-Yano tensor. On the other hand, if a conformal Killing tensor $\vartheta$ belongs to $\ker d$, it is called a closed conformal Killing tensor or a planar tensor (see [20, 21, 22]). We denote the vector space of these tensors by $P^p(M, \mathbb{R})$.

By [5] on an arbitrary $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) the vector space $C^p(M, \mathbb{R})$ of conformal Killing tensors is decomposed uniquely in the form

$$C^p(M, \mathbb{R}) = K^p(M, \mathbb{R}) \oplus P^p(M, \mathbb{R}).$$  

From (4.1) we conclude that any conformal Killing tensor $\vartheta$ of degree $p$ is decomposed uniquely in the form $\vartheta = \omega + \theta$ where $\omega$ is a Killing-Yano tensor of degree $p$ and $\theta$ is a closed conformal Killing tensor of degree $p$.

Following theorem 1, on an $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) there is a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary Killing-Yano tensor $\omega$ of degree $p$ ($2 \leq p \leq n - 1$) has the components

$$\omega_{i_1 \ldots i_p} = e^{(p+1)\psi}(A_{i_0 i_1 \ldots i_p} x^{i_0} + B_{i_1 \ldots i_p})$$  

(4.2)
where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0i_1\ldots i_p}$, $B_{i_1\ldots i_p}$ are arbitrary skew-symmetric constants. On the other hand, by [19] on a pseudo-Riemannian manifold $(M, g)$ of constant nonzero curvature $C$ ($C \neq 0$) the components $\theta_{i_1\ldots i_p}$ of a closed conformal Killing tensor $\theta$ of degree $p$ ($1 \leq p \leq n - 1$) can be found from the equations

$$\theta_{i_1i_2\ldots i_p} = -\frac{1}{pC} \nabla_i \omega_{i_2\ldots i_p}$$

(4.3)

where $\nabla_i \omega_{i_2\ldots i_p} = \partial_i \omega_{i_2\ldots i_p} - \omega_{k\ldots i_p} \Gamma^k_{i_2i_1} - \cdots - \omega_{i_2\ldots k} \Gamma^k_{i_pi_1}$ is the expression for the covariant derivative $\nabla \omega$ of the Killing-Yano tensor of degree $p - 1$. Moreover, by virtue of (3.1) on a pseudo-Riemannian manifold $(M, g)$ of constant curvature $C$ ($C \neq 0$) the Christoffel symbols $\Gamma^k_{ij}$ have the following form $\Gamma^k_{ij} = \psi_i \delta^k_j + \psi_j \delta^k_i$ (see also [17]). Therefore, we can deduce from (4.2) and (4.3) that

$$\theta_{i_1\ldots i_p} = -\frac{1}{C} e^{p\psi} (\psi_{[i_1} A_{k]i_2\ldots i_p]} x^k + \psi_{[i_1} B_{i_2\ldots i_p]} + \frac{1}{p} A_{i_1i_2\ldots i_p}).$$

Consequently we have

**Theorem 3** On an n-dimensional pseudo-Riemannian manifold $(M, g)$ of constant nonzero sectional curvature $C$ ($C \neq 0$) there is a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary conformal Killing tensor $\vartheta$ of degree $p$ ($2 \leq p \leq n - 1$) has the components

$$\vartheta_{i_1\ldots i_p} = e^{(p+1)\psi} (A_{k_1\ldots i_p} x^k + B_{i_1\ldots i_p} + \frac{1}{p} C_{i_1i_2\ldots i_p})$$

where $\psi = [2(n + 1)]^{-1} \ln |\det g|$, $\psi_k = \frac{\partial \psi}{\partial x^k}$ and $A_{i_0i_1\ldots i_p}$, $B_{i_1\ldots i_p}$, $C_{i_1\ldots i_p}$ and $D_{i_1\ldots i_p}$ are arbitrary skew-symmetric constants.

**Remark 3** For a conformal Killing vector field, see K. Yano and T. Nagano [27].

**References**

The Killing tensors on an $n$-dimensional manifold...


