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On Metrizable Locally Homogeneous Connections in Dimension Two*

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Abstract

We discuss metrizability of locally homogeneous affine connections on affine 2-manifolds and give some partial answers, using the results from [1, 5, 13, 12].

Key words: Manifold, affine connection, Riemannian connection, Lorentzian connection, Killing vector field, locally homogeneous space.

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1 Introduction

In [1], locally homogeneous affine connections (with arbitrary torsion) on two-dimensional manifolds were completely classified. As a basic tool, Lie algebras of affine Killing vector fields were used. The paper [1] brings answers in the case of torsion-free connections, generalizing previous results reached in [5] by group-theoretical methods, and in [9] by a direct method.

In [11, 12] we examined metrizability of linear (= affine) connections [6, 7] in 2-manifolds, in [13] we found necessary and sufficient conditions for metrizability of the so-called connections of type A (with constant Christoffels in a fixed coordinate system), in [14] we have given necessary conditions and also a sufficient condition for metrizability of connection of the so-called type B, [1]. Here we would like to relate metrizability of locally homogeneous affine connections to Lie algebras of vector fields. We give at least a partial answer here.

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2 Locally homogeneous affine connections

Let (M, ∇) be a smooth connected n-manifold together with an affine connection ∇. The curvature and Ricci tensor are denoted by R and Ric, respectively. An affine connection ∇ is *homogeneous* if there is a group of global affine transformations acting transitively in the manifold M. A connection is (affine) *locally homogeneous* if for each pair of points p, q in M there are neighborhoods U of p, V of q and an affine transformation $f\colon (U, \nabla|_U) \to (V, \nabla|_V)$ such that $f(p) = q$ [1]. A torsion-free connection is locally symmetric if and only if its curvature is covariantly constant, $\nabla R = 0$ [2]. Any locally symmetric connection is locally homogeneous (but not vice versa).

2.1 Dimension two

In what follows we examine manifolds M_2 of dimension two. Two-dimensional locally homogeneous Riemannian manifolds are just those with constant curvature. On the other hand there exist many different locally homogeneous affine structures on 2-manifolds.

An affine connection ∇ in a domain U of $\mathbb{R}^2[u, v]$ is given uniquely by a family of components (Christoffel symbols) which are eight functions $a, b, c, \tilde{c}, d, \tilde{d}, e, f$ in two variables u, v (local coordinates in U) such that

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= a\partial_u + b\partial_v, & \nabla_{\partial_u}\partial_v &= c\partial_u + d\partial_v, \\ \nabla_{\partial_u}\partial_v &= \tilde{c}\partial_u + \tilde{d}\partial_v, & \nabla_{\partial_v}\partial_v &= e\partial_u + f\partial_v \end{aligned} \tag{1}$$

where $\partial_u = \frac{\partial}{\partial u}$, $\partial_v = \frac{\partial}{\partial v}$. In the usual notation $\Gamma^1_{11} = a(u, v)$, $\Gamma^2_{11} = b(u, v)$, $\Gamma^1_{12} = c(u, v)$, $\Gamma^2_{12} = d(u, v)$, $\Gamma^1_{21} = \tilde{c}(u, v)$, $\Gamma^2_{21} = \tilde{d}(u, v)$, $\Gamma^1_{22} = e(u, v)$, $\Gamma^2_{22} = f(u, v)$. For torsion-free ($T = 0$) connections, $\tilde{c} = c$ and $\tilde{d} = d$.

Example 1 If for a point $p \in M_2$ there is a neighorhood U and system of local coordinates (u, v) in U such that the components, from (1), of a connection ∇ in U are *constants* $a, b, c, \tilde{c}, d, \tilde{d}, e, f$ (∇ has constant Christoffels in U) we say that a connection ∇ is *of type* A in U.

Lemma 1 *Any connection of type A in $\mathbb{R}^2$ is homogeneous, consequently locally homogeneous.*

Example 2 Let for a point $p \in M_2$ there is a neighorhood U and a system of local coordinates (u, v) in U and there exist real constants $A, B, C, \tilde{C}, D, \tilde{D}, E, F$, not all of them are zero, such that the following formulae hold in U:

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= u^{-1}(A\partial_u + B\partial_v), & \nabla_{\partial_u}\partial_v &= u^{-1}(C\partial_u + D\partial_v), \\ \nabla_{\partial_v}\partial_u &= u^{-1}(\tilde{C}\partial_u + \tilde{D}\partial_v), & \nabla_{\partial_v}\partial_v &= u^{-1}(E\partial_u + F\partial_v), \end{aligned} \tag{2}$$

i.e. $a = u^{-1}A$, $b = u^{-1}B$ etc. In accordance with [5, 1] we say that a connection $\nabla|_U$ is a connection *of type* B.

Lemma 2 *Each locally flat connection (i.e. with $T = 0$ and $R = 0$) in (an open domain of) M_2 is of both types A and B.*

Indeed, if all components of the connections vanish in some coordinate neighborhood U_p of an arbitrary point p, $\Gamma^i_{jk} = 0$, then the connection is obviously of type A. If we use the transformation $u' = e^u$, $v' = v$ the only non-zero component of the same connection is coordinates (u', v') is $a' = -1/u'$, therefore the connection is of type B as well ($A = -1$ while other constants equal zero).

It appears that connections of type A and B play an important role in classification: in fact, besides Riemannian connections of spheres, (all) connections of type A and some of the connections of type B, there are no other locally homogeneous affine connections defined in plain domains.

2.2 Classification theorems

First attempts to classify systematically torsion-free locally homogeneous linear connections in plain domains appeared relatively recently. Some partial results can be found in [3, 4]. In [9], B. Opozda (after a lot of tedious calculations) has found and proved in a direct way a general formula which describes all torsion-free locally homogeneous linear connections on 2-manifolds in suitable local coordinates:

Theorem 1 *Assume a torsion-free locally homogeneous affine connection ∇ in a 2-dimensional manifold M_2. Then, either the connection is a natural (Levi-Civita) connection of a space of constant curvature or, in a neighborhood U_p of each point $p \in M_2$, there is a system (u, v) of local coordinates in which either ∇ is of type A, i.e. has constant Christoffels, or ∇ is of type B, i.e. has components satisfying* (2)*.*

Formulation of the Theorem 1 might avoke a (wrong) impression that it somehow separates metric connections from connections of type A and type B; the classes of connections mentioned in the theorem are *not* disjoint; among connections of type B, Levi-Civita connections of Riemannian as well as Lorentzian metrics of constant curvature can be found. Note that the 2-dimensional sphere has a special position, it is neither of type A, nor of type B [1].

The result of Theorem 1 was later on reached by O. Kowalski and others in [5] by means of a more elegant and sophisticated group-theoretical method based on Killing vector fields and the Olver's list of Lie algebras of vector fields in $\mathbb{R}^2$ (Table 1 – Transitive, Imprimitive, Table 6 – Primitive, from the book [8, pp. 472 and 475] by P. J. Olver). Let us point out that the classification procedure in [5] uses "canonical" local coordinate systems quite different from thouse used originally in [9].

The mentioned list of algebras was refined by O. Kowalski in [1] and step by step, the following was proved.

Theorem 2 (Classification Theorem) [1] *Assume a locally homogeneous affine connection with arbitrary torsion on a 2-manifold M_2. Then, in a neighborhood*

U_p of each point $p \in M_2$, either the connection is the Levi-Civita connection of the unit sphere or, there is a system (u, v) of local coordinates and constants $A, B, C, \tilde{C}, D, \tilde{D}, E, F$, not all equal zero, such that either the connection is of type A and has constant Christoffels $a = A, \ldots, f = F$, or the connection is in U_p of type B, expressed by the formula (2).

The proof makes use of group-theoretical view-point and is based on affine Killing vector fields of the linear connection and the Kowalski's refinement of Olver's list of Lie algebras of vector fields, which we call *Kowalski's list* here. In the proof of Theorem 2, the starting point is to follow the extended table of transitive Lie algebras of vector fields [1, pp. 3, 4, 5] which is derived from the Olver list [8, pp. 472, 475]. For any particular item $\underline{g}$, it is examined for which linear connections (in the same 2-dimensional domain and with respect to the same local coordinates) the given algebra $\underline{g}$ is the full algebra of affine Killing vector fields; such connections are called *corresponding* to $\underline{g}$. When some connection has more different Killing algebras the maximal one is found. It appears that there are Lie algebras of vector fields for which no such connection exists.

Remark that a similar classification in dimension three is an open and seemingly hard problem.

2.3 Lie algebras of vector fields

A vector field X on a manifold (M, ∇) is an affine *Killing vector field* if it satisfies $[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0$ for all $Y, Z \in \mathcal{X}(M)$, [2, Chapter VI]. All Killing vector fields in a domain of (M, ∇) form a *full affine Killing (Lie) algebra.* In local coordinates, the defining condition for Killing fields is represented by a system of partial differential equations, and it is sufficient to check the condition for coordinate vector fields instead of arbitrary fields Y, Z. A Killing vector field in M_2 has coordinate expression $X = p(u, v)\partial_u + q(u, v)\partial_v$ where the functions p, q satisfy a system of eight basic partial differential equations given by (3) in [1, p. 6]; for symmetric connections, the corresponding system reduces to six linear PDEs presented by the formula (6) in [5, p. 90]. The following holds:

Proposition 1 [5] *A smooth connection ∇ on a smooth manifold M is locally homogeneous if and only if for each point there exist at least two linearly independent affine Killing vector fields defined in a neighborhood.*

If a connection has constant Christoffels in some local coordinates (u, v) in a U (is of type A) then the coordinate fields ∂_v, ∂_u are obviously affine Killing vector fields. If a connection is of type B then it has (at least) Killing vector fields ∂_v and $u\partial_u + v\partial_v$. It holds [1]:

Lemma 3 *The following equivalences hold for any Lie algebra $\underline{g}$ of vector fields (given n a simply connected domain $U(u, v)$ of the plane) from the Kowalski's list:*

(i) All connections corresponding to $\underline{g}$ are of type A if and only if in $\underline{g}$, there exist two linearly independent vector fields X, Y such that $[X, Y] = 0$.

(ii) All connections corresponding to $\underline{g}$ are of type B if and only if in $\underline{g}$, there exist two linearly independent vector fields X, Y such that $[X, Y] = X$.

2.4 Metrizability of affine 2-manifolds

In dimension two, the curvature tensor R can be completely recovered from the Ricci tensor, $R(X,Y)Z = \mathrm{Ric}\,(Y,Z)X - \mathrm{Ric}\,(X,Z)Y$ for $X, Y, Z \in \mathcal{X}(M)$. In local coordinates, the components of tensors are related by $R^i_{hjk} = \delta^i_j R_{kh} - \delta^i_k R_{jh}$. If $n = 2$ the curvature R of (M_2, ∇) is recurrent if and only if the Ricci tensor Ric is recurrent, and $R = 0$ if and only if Ric $= 0$ (which can be used e.g. by checking local symmetry).

A linear connection ∇ on M and the metric g of a (pseudo-) Riemannian manifold (M, g) are *compatible* if $\nabla g = 0$ holds. Any (pseudo-) Riemannian manifold admits a unique linear connection fully characterized by vanishing of the torsion tensor T and the condition $\nabla g = 0$, called the *Levi-Civita connection* of (M, g). On the other hand if (M, ∇) is given we might be interested in all compatible metrics. If such a metric exists (or exists locally) the connection is *metrizable* (or *locally metrizable*, respectively).

The Ricci tensor of a (pseudo-) Riemannian manifold (M_2, g) is proportional to its metric tensor, $\mathrm{Ric} = Kg$ where K is the Gauss curvature; $R^i_{hjk} = K(\delta^i_k g_{hj} - \delta^i_j g_{hk})$ holds. Hence the Ricci tensor of (M_2, g) must be symmetric. Moreover, for each nowhere flat (M_2, g), the Ricci tensor is recurrent and the corresponding 1-form is exact (= gradient) [12] (if $R = 0$, Ric is also recurrent with $\omega = 0$, hence recurrency is one of necessary conditions for metrizability of a two-manifold). Moreover, for a nowhere flat (M_2, g), i.e. with Ric $\neq 0$, the Ricci tensor is non-degenerate, $\det(R_{ij})$ has rank 2.

Lemma 4 *A nowhere flat torsion-free linear connection on M_2 is metrizable if and only if its Ricci tensor is symmetric, regular and recurrent with the corresponding one-form being exact, and the compatible (homothetic) metrics can be given explicitly: if $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$ and $df = \omega$ then $g_k = \exp(k - f) \cdot \mathrm{Ric}$ where $k \in \mathbb{R}$ is a constant.*

2.5 Metrizable connections of type A and B

Now let us pay attention to metrizable locally homogeneous linear connections in open domains of 2-manifolds. According to the Classification Theorems, we are left with connections of type A and B.

Type A. A connection with constant Christoffels defined in $U \subset \mathbb{R}^2[u, v]$ has the curvature tensor R as well as the Ricci tensor constant, moreover Ric is always symmetric, with components

$$R_{11} = b(f - c) + d(a - d), \quad R_{12} = R_{21} = cd - be, \quad R_{22} = e(a - d) + c(f - c),$$

a torsion-free connection with constant Christoffels on M_2 is recurrent if and only if there exist constants χ_1, χ_2 such that the following system holds:

$$\begin{gathered}(\chi_1+2a)R_{11}+2bR_{12}=0, \quad bR_{11}+(\chi_1+a+d)R_{12}+bR_{22}=0,\\ 2cR_{12}+(\chi_1+2d)R_{22}=0, \quad (\chi_2+2c)R_{11}+2dR_{12}=0,\\ eR_{11}+(\chi_2+c+f)R_{12}+dR_{22}=0, \quad 2eR_{12}+(\chi_2+2f)R_{22}=0.\end{gathered} \tag{3}$$

If we account also non-singularity and symmetry of Ric we find [13, 14]:

Lemma 5 *Exactly for the following choices of constants we get a metrizable connection ∇ of type A in M_2*

(i) *$a, c, e, f \in \mathbb{R}$ are constants satisfying $ae - c^2 + cf = 0$, $b = d = 0$;*

(ii) *$a, b, e, f \in \mathbb{R}$ are constants, $d \neq 0$, $c = be/d$ and the equality $d^2a - eb^2 + dbf - d^3 = 0$ is satisfied.*

In all cases, Ric $= 0$. Hence besides locally flat connections, there are no other metrizable torsion-free type A connections (with constant Christoffels) on 2-manifolds.

Type B. The situation is quite different. There exist classes of metrizable non-flat connections of type B, corresponding to both Riemannian and Lorentzian 2-spaces of constant curvature.

Example 3 (Lorentzian metric of constant curvature 1) In $\mathbb{R}^2[u, v]$ assume the metric $g = u^{-2}(du^2 - dv^2)$ of constant positive curvature $K = 1$. The corresponding natural (Levi-Civita) connection has the form (2) with non-zero coefficients $A = D = \tilde{D} = E = 1$, hence is of type B, and Ric $= g$ holds.

Example 4 (The Poincaré model of hyperbolic plane) Assume the manifold $\mathbb{H} = \{(u, v) \in \mathbb{R}^2; u > 0\}$ endowed with the (positive definite) Riemannian metric $g = u^{-2}(du^2 + dv^2)$. The Levi-Civita connection $\hat{\nabla}$ of $(\mathbb{H}, g)$ is of the form (2) with the only non-zero coefficients $A = D = \tilde{D} = -1$, $E = 1$. That is, ∇ is a connection of type B, with the Ricci tensor Ric $= -g$; g is the metric of constant negative Gauss curvature $K = -1$.

Lemma 6 ([13]) *Each choice of real constants $A = D \neq 0$, $E \neq 0$, $B = C = F = 0$ determines by formula (2), in (a neighborhood U of) $M_2 = \mathbb{R}^2[u, v] \setminus \{(0, v); v \in \mathbb{R}\}$, a non-flat locally metrizable linear connection of type B*

$$\nabla_{\partial_u}\partial_u = u^{-1}A\partial_u, \quad \nabla_{\partial_u}\partial_v = u^{-1}A\partial_v, \quad \nabla_{\partial_v}\partial_v = u^{-1}E\partial_u \tag{4}$$

with the symmetric recurrent Ricci tensor

$$R_{11} = u^{-2}A, \quad R_{12} = 0, \quad R_{22} = -u^{-2}E$$

satisfying

$$\nabla \operatorname{Ric} = df \otimes \operatorname{Ric}, \qquad f(u, v) = -2(A+1)\ln|u|.$$

The corresponding family of compatible (homothetic) metrics is

$$g_k = e^{k-f(u)}Au^{-2}du^2 - e^{k-f(u)}Eu^{-2}dv^2, \qquad k \in R. \tag{5}$$

Examples 3, 4 belong just to this class of connections.

3 Properties of some torsion-free connections associated with the Kowalski's list

Now we would like to relate our previous results on metrizability to Lie algebras of affine Killing fields from the Kowalski's list [1, p. 3]. Recall that in the cases of Lie algebras 1.4, 1.5 h, 1.6 h', 1.7 for $k > 2$, 1.8 for $k \geq 2$, 1.9 for $k > 2$, 1.10, 1.11, 2.7 (6.7 of Olver's list) and 2.8 (6.8 of Olver's list) there exists no corresponding invariant affine connection.

Lemma 7 ([1],[5, Lemma 3.1, p. 91]) *The following algebras from the Kowalski's list characterize, as affine Killing algebras, (torsion-free) metrizable connections with vanishing Christoffel symbols:*
the case 1.7 for $\alpha \neq 0, 2, \frac{1}{2}$, *and* $k = 1, 2$;
the case 1.7 for $\alpha = 0$, *or* $\alpha = \frac{1}{2}$, *and* $k = 2$;
the case 1.8 for $k = 1$;
the case 1.9 for $k = 1, 2$;
the case 2.1, 2.4, 2.5 and 2.6.

Now let us recall at least some algebras from the list [1, p. 3] and discuss properties, particularly metrizability, of the corresponding connections.

Case 1.1 The Lie algebra $\underline{g} = \text{span}(\partial_v, -u\partial_u + v\partial_v, -2uv\partial_u + v^2\partial_v) \simeq \underline{sl}(2)$. In general, the corresponding connection is not torsion-free, [1, p. 15]; if we assume $T = 0$ the connection is

$$\nabla_{\partial_u}\partial_u = -\frac{1}{2}u^{-1}\partial_u, \quad \nabla_{\partial_u}\partial_v = \gamma u\partial_u - \frac{1}{2}u^{-1}\partial_v, \quad \nabla_{\partial_v}\partial_v = \epsilon u^3\partial_u + 2\gamma u\partial_v$$

where $\gamma, \epsilon \in \mathbb{R}$ are arbitrary constants, with Christoffel symbols

$$a = -d = -\frac{1}{2}u^{-1}, \quad b = 0, \quad c = \gamma u, \quad e = \epsilon u^3, \quad f = 2\gamma u.$$

As $\underline{g}$ is isomorphic with $\underline{sl}(2)$, and $\underline{sl}(2)$ is a simple Lie algebra, $\underline{g}$ does not admit any subalgebra $\text{span}(X, Y)$ such that $[X, Y] = 0$. Consequently the corresponding connection ∇ is *not* of type A. On the other hand the connection is of type B due to Lemma 5: $X = \partial_v$ and $Y = -u\partial_u + v\partial_v$ satisfy $[X, Y] = X$. If we use the coordinate transformation $(u, v) \mapsto (\tilde{u}, \tilde{v})$ where $\tilde{u} = \frac{1}{u}$, $\tilde{v} = v$ we transform the connection to the form (2) explicitly, with constants

$$A = -\frac{3}{2}, \quad B = 0, \quad C = \gamma \in \mathbb{R}, \quad D = -\frac{1}{2}, \quad E = -\varepsilon \in \mathbb{R}, \ F = 2C = 2\gamma.$$

The Ricci tensor is symmetric if and only if $F = -C$, that is, $C = 0$. If this is the case then the only component of Ric which might be non-zero in new coordinates is $R_{22} = 2\varepsilon\tilde{u}^{-2}$. Hence $\text{Ric} = 2\varepsilon\tilde{u}^{-2}d\tilde{v}^2$ is singular, we get no compatible metric in this case.

Case 1.2 The Lie algebra $\underline{g} = \text{span}(\partial_v, -u\partial_u + v\partial_v, -(2uv+1)\partial_u + v^2\partial_v) \simeq \underline{sl}(2)$. The corresponding (locally) homogeneous torsion-free connection is

$$\nabla_{\partial_u}\partial_u = 0, \quad \nabla_{\partial_u}\partial_v = -2u\partial_u, \quad \nabla_{\partial_v}\partial_v = 4u^3\partial_u + 2u\partial_v,$$

with Christoffels $a = b = d = 0$, $c = -2u$, $e = 4u^3$, $f = 2u$. According to Lemma 5, this connection is *not* of type A, but it is of type B. To see it explicitly we can use again the coordinate transformation $\tilde{u} = \frac{1}{u}$, $\tilde{v} = v$. The Ricci tensor, with respect to (u, v), is $\mathrm{Ric} = -4\,du dv + 4u^2 dv^2$, and ∇ is the Levi-Civita connection of the Lorentzian metric $g = \mathrm{Ric}$ of constant positive curvature $K = 1$, or equivalently of the Lorentzian metric $g = -\mathrm{Ric}$ of constant negative curvature $K = -1$.

Case 1.3 The Lie algebra $\underline{g} = \mathrm{span}(\partial_v, v\partial_v, u\partial_u, -uv\partial_u + v^2\partial_v)$. The corresponding torsion-free connection is given by

$$\nabla_{\partial_u}\partial_u = 0, \quad \nabla_{\partial_u}\partial_v = \frac{1}{u}\partial_v, \quad \nabla_{\partial_v}\partial_v = 0,$$

has Christoffels $A = B = C = E = F = 0$, $D = 1/u$, $T = 0$, $\mathrm{Ric} = 0$, so ∇ is flat, hence it is of both the types A and B, and is metrizable.

Case 1.5 a) Locally homogeneous connections whose corresponding algebra of Killing vector fields is $\underline{g} = \mathrm{span}(\partial_u, \partial_v)$ has arbitrary constants as its Christoffel symbols, are of type A and are not of type B, [1, Prop. 5, p. 14]. Such a connection is metrizable if and only if $c = \tilde{c}$, $d = \tilde{d}$ (then $T = 0$) and one of the conditions (i) or (ii) from Lemma 5 is satisfied.

Case 1.5 b) Locally homogeneous connections whose corresponding algebra of Killing vector fields is $\underline{g} = \mathrm{span}(\partial_u, e^v\partial_v)$ are not of type A, but are of type B, [1, Prop. 6, p. 14]. Christoffels of a torsion-free connection of this class depend on six parameters and read

$$\begin{aligned} &a(u) = C_1 u + C_2, \quad b(u) = C + 1,\\ &c(u) = -C_1 u^2 + (C_3 - C_2)u + C_4, \quad d(u) = -C_1 u + C_3,\\ &e(u) = C_1 u^3 + (C_2 - C_3)u^2 + (C_5 - 2C_4 - 1)u + C_6,\\ &f(u) = C_1 u^2 - 2C_3 u + C_5, \quad C_1, \dots, C_6 \in \mathbb{R} \quad C_1 \neq 0. \end{aligned} \tag{6}$$

If moreover the conditions

$$C_2 + C_3 = 0, \quad C_4 = -\frac{C_2^2}{C_1}, \quad C_5 = \frac{C_2^2}{C_1}, \quad C_6 = -\frac{C_2(C_1 - C_2^2)}{C_2^2} \tag{7}$$

hold then the connection is locally symmetric (and not locally flat in general). Components of the Ricci tensor are

$$R_{11} = C_1, \quad R_{12} = R_{21} = -C_1 u - C_2, \quad R_{22} = C_1 u^2 + 2C_2 u + \frac{C^2 + 2}{C_1} - 1,$$

Ric is symmetric, recurrent ($\nabla\,\mathrm{Ric} = 0$) and non-degenerate since $\det(R_{ij}) = C_1 \neq 0$.

$$\mathrm{Ric} = C_1\,du^2 - (C_1 u + C_2)\,du dv + (C_1 u^2 + 2C_2 u + C^2 + 2/C_1 - 1)\,dv^2$$

is a Lorentzian metric of constant curvature. The coordinate transformation $u' = e^{-v}$, $v' = ue^{-v}$ turns the Christoffel symbols to the shape (2.2) explicitly, with constants (in new coordinates)

$$A' = -C_5 - 1, \quad B' = C_6, \quad C' = C_3, \quad D' = -C_4 - 1, \quad E' = -C_1, \quad F' = C_2.$$

Case 2.2, or (6.2) from [5, 8], concerns the algebra

$$\underline{g} = \operatorname{span}(\partial_v, u\partial_u + v\partial_v, 2uv\partial_u + (v^2 - u^2)\partial_v) \simeq \underline{sl}(2).$$

The corresponding torsion-free connection ∇ with Christoffels $a = d = -e = u^{-1}$,

$$\nabla_{\partial_u}\partial_u = -(1/u)\partial_u, \quad \nabla_{\partial_u}\partial_v = -(1/u)\partial_v, \quad \nabla_{\partial_v}\partial_v = (1/u)\partial_v$$

and is obviously of type B from (2), with constants $B = C = F = 0$, $A = D = -1$, $E = 1$, i.e. of the shape (4), and is not of type A, from similar reasons as above. The Ricci tensor $\operatorname{Ric} = -u^{-2}(du^2 - dv^2)$ provides the Riemannian metric $g = -\operatorname{Ric} = u^{-2}(du^2 + dv^2)$ of constant negative curvature $K = -1$. Locally, ∇ is the Levi-Civita connection of the (standard) hyperbolic plane (Example 4).

Case 2.3, case 6.3. in Olver's list is the Lie algebra

$$\underline{g} = \operatorname{span}(v\partial_u - u\partial_v, (1 + v^2 - u^2)\partial_v + 2uv\partial_u, 2uv\partial_v + (1 - v^2 + u^2)\partial_u) \simeq so(3).$$

The generators give a system of 18 partial differential equations for Christoffels (difficult to solve) [1, (24)–(26)]. If we denote $\varrho = \ln(1 + v^2 + u^2)$ the unique solution of the system reads $a = d = -e = -\varrho u$, $b = -c = -f = \varrho v$. Then $T = 0$, and $\operatorname{Ric} = 4(1 + v^2 + u^2)^{-2}du^2 + 4(1 + v^2 + u^2)^{-2}dv^2$ is a Riemannian metric of constant positive curvature $K = 1$, and it is locally the Levi-Civita connection of the unit sphere.

Theorem 3 ([5, Theorem 6.4.]) *Let ∇ be a torsion-free affine connection in (an open domain of) of a 2-manifold M_2. The corresponding Ricci tensor* Ric *is a pseudo-Riemannian metric, and ∇ is the Levi-Civita connection of $g =$* Ric*, just in the following cases:*

(i) *The connection of constant positive curvature with Christoffel symbols $a = d = -e = -\varrho u$, $b = -c = -f = \varrho v$, $\varrho = \ln(1 + v^2 + u^2)$, corresponding to the case 2.3, with $g = \operatorname{Ric} = 4(1 + v^2 + u^2)^{-2}du^2 + 4(1 + v^2 + u^2)^{-2}dv^2$ as a compatible metric; ∇ is locally the Levi-Civita connection of the unit sphere; neither of type A nor of type B.*

(ii) *The connection of constant negative curvature $K = -1$ with Christoffels $a = d = -e = \frac{1}{u}$, $b = c = f = 0$, corresponding to the case 2.2, is compatible to the Riemannian metric $g = -\operatorname{Ric} = u^{-2}(du^2 + dv^2)$; ∇ is locally the Levi-Civita connection of the standard hyperbolic plane. It is of type B, not of type A.*

(iii) *The connection of constant positive curvature, with Christoffels $a = b = d = 0$, $c = -2u$, $e = 4u^3$, $f = 2u$, corresponding to case 1.2; $g = \operatorname{Ric} = -4dudv + 4u^2dv^2$ is a compatible Lorentzian metric metric; ∇ is of type B but not of type A.*

(iv) *∇ is given by* (6) *and the condition* (7) *is satisfied. The Ricci tensor is a Lorentzian metric of constant positive curvature.*

In all cases, the connection ∇ is locally symmetric ($\nabla \operatorname{Ric} = 0$).

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