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LOWER BOUNDS FOR THE LARGEST EIGENVALUE
OF THE GCD MATRIX ON \{1, 2, \ldots, n\}

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Dedicated to the memory of Miroslav Fiedler

Abstract. Consider the \(n \times n\) matrix with \((i, j)\)’th entry \(\text{gcd}(i, j)\). Its largest eigenvalue \(\lambda_n\) and sum of entries \(s_n\) satisfy \(\lambda_n > s_n / n\). Because \(s_n\) cannot be expressed algebraically as a function of \(n\), we underestimate it in several ways. In examples, we compare the bounds so obtained with one another and with a bound from S. Hong, R. Loewy (2004). We also conjecture that \(\lambda_n > 6\pi^{-2} n \log n\) for all \(n\). If \(n\) is large enough, this follows from F. Balatoni (1969).

Keywords: eigenvalue bounds; greatest common divisor matrix

MSC 2010: 15A42, 15B36, 11A05

1. Introduction

Given \(n > 1\), let \(A_n = (a_{ij})\) be the greatest common divisor (gcd) matrix on \(\{1, 2, \ldots, n\}\), that is, \(a_{ij} = \text{gcd}(i, j)\), \(i, j = 1, 2, \ldots, n\). Let \(\lambda_n\) be its largest eigenvalue and \(s_n\) the sum of its entries. Denote by \(e_n\) the \(n\)-vector with each entry one. Applying the Rayleigh quotient and noting that \(e_n\) is not an eigenvector corresponding to \(\lambda_n\), we have

\[
\lambda_n > \frac{e_n^T A_n e_n}{e_n^T e_n} = \frac{s_n}{n} =: l_n,
\]

see [5], Theorem 4.2.2. The lower bound for the largest eigenvalue of a Hermitian matrix, obtained in this way, is often quite good if the matrix is positive definite and (entrywise) positive. Because \(A_n\) is positive definite, see [3], Theorem 2, we are motivated to a closer look at \(l_n\).
The study of gcd matrices traces back to Smith in [7] but did not attract much attention until recent decades. Hong and Loewy in [4] may be regarded as initiators of studying eigenstructures of gcd and related matrices. For a brief historical survey on this topic with references, see Altınışık et al. [1].

Because \( s_n \) cannot be expressed algebraically as a function of \( n \), we underestimate it; then we are actually studying lower bounds for \( l_n \). The simplest way is to replace all off-diagonal entries of \( A_n \) by 1; let \( B_n = (b_{ij}) \) be the matrix so obtained. Since the sum of its entries is

\[
\frac{n(n+1)}{2} + n(n-1) = \frac{3n^2 - n}{2} =: t_n,
\]

we have

\[
\lambda_n > \frac{t_n}{n} = \frac{3n - 1}{2} =: u_n.
\]

Our task is to find for \( \lambda_n \) better bounds than \( u_n \). Because we are interested also in asymptotic bounds, we will first (Section 2) take a look at the asymptotics of \( \lambda_n \) and \( l_n \). Thereafter (Sections 3–7) we will improve \( u_n \). We will take a suitable nonzero and (entrywise) nonnegative matrix \( E_n = (e_{ij}) \) with the following properties:

(i) Its all diagonal entries are zero.
(ii) Its all off-diagonal entries satisfy \( b_{ij} + e_{ij} \leq a_{ij} \).
(iii) The sum of its entries, denoted by \( \tau_n \), is easy to calculate.

Then

\[
s_n \geq t_n + \tau_n > t_n,
\]

which implies, by (1),

\[
\lambda_n > u_n + \frac{\tau_n}{n} > u_n.
\]

Different choices of \( E_n \) give different improvements. We will finally in examples compare our bounds with one another (Section 8) and with a bound of Hong and Loewy in [4] (Section 9). Concluding remarks (Section 10) complete our paper.

### 2. Asymptotics of \( \lambda_n \) and \( l_n \)

It is well-known, see [8], equation (25), that

\[
s_n = \frac{6}{\pi^2} n^2 \log n + O(n^2),
\]

so

\[
l_n = \frac{6}{\pi^2} n \log n + O(n).
\]
Experiments make us conjecture that

\[ \lambda_n > \frac{6}{\pi^2} n \log n =: v_n. \]  

It is also well-known, see [2], Theorem, that

\[ \lambda_n = O(n^{1+\varepsilon}) \]  

for all \( \varepsilon > 0 \) but

\[ \lambda_n \neq O(n(\log n)^k) \]  

for all \( k \geq 1 \). Therefore (3) is true if \( n \) is large enough. In fact, \( v_n \) is then a very poor bound, because

\[ \lim_{n \to \infty} \frac{v_n}{\lambda_n} = 0 \]

by (4) and (5).

3. First attempt: \( e_{ij} = 1 \) if \( i \neq j \) and \( a_{ij} \geq 2 \)

We obtained the bound \( u_n \) by replacing all off-diagonal entries of \( A_n \) by one. To improve it, we replace by two all of them that are at least two. In other words, we define \( E_n \) by setting \( e_{ij} = 1 \) if \( i \neq j \) and \( a_{ij} \geq 2 \), and \( e_{ij} = 0 \) otherwise. The number of ones before the diagonal is \( i - 1 - \varphi(i) \), where \( i > 1 \) and \( \varphi \) is the Euler totient function. Hence

\[ \tau_n = 2 \sum_{i=2}^{n} (i - 1 - \varphi(i)) = n^2 - n + 2(1 - \Phi(n)), \]

where

\[ \Phi(n) = \sum_{i=1}^{n} \varphi(i). \]

By (2),

\[ \lambda_n > \frac{3n - 1}{2} + n - 1 + 2 \frac{1 - \Phi(n)}{n} = \frac{5n - 3}{2} + 2 \frac{1 - \Phi(n)}{n} =: w_n. \]

Asymptotically, see [6], Section I.21,

\[ \Phi(n) = \frac{3}{n^2} n^2 + O(n^\delta) \]  

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for some $\delta$ with $1 < \delta < 2$; hence

$$w_n = \left(\frac{5}{2} - \frac{6}{\pi^2}\right)n + O(n^\delta)$$

for some $\delta$ with $0 < \delta < 1$.

4. Second attempt: Restrict $i$ and $j$ even

To find a (weaker) bound without $\Phi(n)$, we restrict $i$ and $j$ to be even. So we set $e_{ij} = 1$ if $i$ and $j$ are different and even, and $e_{ij} = 0$ otherwise. Then

$$\tau_n = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right).$$

By (2),

$$\lambda_n > \frac{3n-1}{2} + \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) =: x_n.$$ 

If $n$ is even, then

$$x_n = \frac{3n-1}{2} + \frac{1}{2} \left(\frac{n}{2} - 1\right) = \frac{7n}{4} - 1.$$ 

If $n$ is odd, then

$$x_n = \frac{3n-1}{2} + \frac{n-1}{2n} \left(\frac{n-1}{2} - 1\right) = \frac{7n}{4} - \frac{3}{2} + \frac{3}{4n}.$$ 

Asymptotically

$$x_n = \frac{7n}{4} + O(1).$$

5. Third attempt: Change $e_{ij} = 2$ if $i \neq j$ and $3 \mid i, j$

If $i$ and $j$ are multiples of three and $i \neq j$, then $a_{ij} \geq 3$ but $b_{ij} = 1$. The number of such pairs $(i, j)$ is

$$\left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1\right) =: \alpha_n.$$ 

“Old $E_n$” (i.e., $E_n$ constructed in the previous section) has then either $e_{ij} = 0$ or $e_{ij} = 1$. We change all these entries into two. Call “new $E_n$” the matrix effecting so.

If $i \neq j$ and $6 \mid i, j$, then old $e_{ij} = 1$. The number of such pairs $(i, j)$ is

$$\left\lfloor \frac{n}{6} \right\rfloor \left(\left\lfloor \frac{n}{6} \right\rfloor - 1\right) =: \beta_n.$$ 

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If \( i \neq j \) and \( 3 \mid i, j \) but not \( 6 \mid i, j \), then old \( e_{ij} = 0 \). The number of such pairs is \( \alpha_n - \beta_n \). Therefore we obtain “new \( \tau_n \)” by adding 

\[
2(\alpha_n - \beta_n) + \beta_n = 2\alpha_n - \beta_n
\]

to “old \( \tau_n \)”. Hence, by (2),

\[
\lambda_n > \frac{3n-1}{2} + \frac{1}{n} \left[ \left\lfloor \frac{n}{2} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 2 \left\lfloor \frac{n}{3} \right\rfloor \left( \left\lfloor \frac{n}{3} \right\rfloor - 1 \right) - \left\lfloor \frac{n}{6} \right\rfloor \left( \left\lfloor \frac{n}{6} \right\rfloor - 1 \right) \right] =: x'_n.
\]

The polynomial expression of \( x'_n \) depends on the remainder \( r = n - 6 \left\lfloor \frac{n}{6} \right\rfloor \).

If \( r = 0 \), then \( \left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2}n, \left\lfloor \frac{n}{3} \right\rfloor = \frac{1}{3}n, \left\lfloor \frac{n}{6} \right\rfloor = \frac{1}{6}n \); so

\[
x'_n = \frac{3n-1}{2} + \frac{1}{n} \left[ \frac{n}{2} \left( \frac{n}{2} - 1 \right) + 2 \frac{n}{3} \left( \frac{n}{3} - 1 \right) - \frac{n}{6} \left( \frac{n}{6} - 1 \right) \right] = \frac{35n}{18} - \frac{3}{2}.
\]

If \( r = 1 \), then \( \left\lfloor \frac{n}{2} \right\rfloor = \frac{1}{2}(n-1), \left\lfloor \frac{n}{3} \right\rfloor = \frac{1}{3}(n-1), \left\lfloor \frac{n}{6} \right\rfloor = \frac{1}{6}(n-1) \); so

\[
x'_n = \frac{3n-1}{2} + \frac{1}{n} \left[ \frac{n-1}{2} \left( \frac{n-1}{2} - 1 \right) + 2 \frac{n-1}{3} \left( \frac{n-1}{3} - 1 \right) - \frac{n-1}{6} \left( \frac{n-1}{6} - 1 \right) \right]
= \frac{35n}{18} - \frac{43}{18} + \frac{13}{9n}.
\]

We continue similarly. If \( r = 2 \), then

\[
x'_n = \frac{35n}{18} - \frac{41}{18} + \frac{16}{9n}.
\]

If \( r = 3 \), then

\[
x'_n = \frac{35n}{18} - \frac{11}{6}.
\]

If \( r = 4 \), then

\[
x'_n = \frac{35n}{18} - \frac{31}{18} - \frac{2}{9n}.
\]

If \( r = 5 \), then

\[
x'_n = \frac{35n}{18} - \frac{47}{18} + \frac{13}{9n}.
\]

This procedure can be pursued further. The next step is to change \( e_{ij} = 3 \) if \( i \) and \( j \) are multiples of four and \( i \neq j \). But we stop here, because the calculations become complicated.
6. Fourth attempt: $e_{i,ki} = e_{ki,i} = i - 1$

Denote $n_i = \lfloor n/i \rfloor$. The entries

$$a_{i,2i} = a_{i,3i} = \ldots = a_{i,n_i} = i,$$

$$a_{2i,i} = a_{3i,i} = \ldots = a_{n_i,i} = i, \quad i = 2, 3, \ldots, n_2,$$

are greater than one, but the corresponding entries are $b_{ij} = 1$. In order to give them their original values, we define $E_n$ by

$$e_{i,2i} = e_{i,3i} = \ldots = e_{i,n_i} = i - 1,$$

$$e_{2i,i} = e_{3i,i} = \ldots = e_{n_i,i} = i - 1, \quad i = 2, 3, \ldots, n_2,$$

and $e_{ij} = 0$ otherwise. Then

$$\tau_n = \sum_{i=2}^{n_2} 2 \sum_{k=2}^{n_i} e_{i,ki} = \sum_{i=2}^{n_2} 2(n_i - 1)(i - 1)$$

$$= 2[(n_2 - 1) + (n_3 - 1) \cdot 2 + (n_4 - 1) \cdot 3 + \ldots + (n_{n_2 - 1} - 1)(n_2 - 2) + 1 \cdot (n_2 - 1)]$$

$$= 2\{[1 + \ldots + (n_2 - 1)] + [1 + \ldots + (n_3 - 1)] + \ldots + [1 + \ldots + (n_{n_2 - 1} - 1)] + 1\}$$

$$= 2 \sum_{k=2}^{n_2} [1 + 2 + \ldots + (n_k - 1)] = \sum_{k=2}^{n_2} n_k(n_k - 1),$$

which is tedious to compute. So we underestimate it.

Because

$$n_k > \frac{n}{k} - 1,$$

we have

$$\tau_n > \sum_{k=2}^{n_2} \left(\frac{n}{k} - 1\right)\left(\frac{n}{k} - 2\right) = \sum_{k=2}^{n_2} \left(\frac{n^2}{k^2} - 3\frac{n}{k} + 2\right)$$

$$= n^2 \sum_{k=2}^{n_2} \frac{1}{k^2} - 3n \sum_{k=2}^{n_2} \frac{1}{k} + 2(n_2 - 1).$$

Hence, by (2),

$$\lambda_n > \frac{3n - 1}{2} + n \sum_{k=2}^{n_2} \frac{1}{k^2} - 3 \sum_{k=2}^{n_2} \frac{1}{k} + \frac{2(n_2 - 1)}{n} =: y_n.$$
If $n$ is even, then

$$y_n = \frac{3n - 1}{2} + n \sum_{k=2}^{n/2} \frac{1}{k^2} - 3 \sum_{k=2}^{n/2} \frac{1}{k} + \frac{2(\frac{1}{2}n - 1)}{n}$$

$$= \frac{3n}{2} + n \sum_{k=2}^{n/2} \frac{1}{k^2} - 3 \sum_{k=2}^{n/2} \frac{1}{k} + \frac{1}{2} - \frac{2}{n}.$$ 

If $n$ is odd, then

$$y_n = \frac{3n - 1}{2} + n \sum_{k=2}^{(n-1)/2} \frac{1}{k^2} - 3 \sum_{k=2}^{(n-1)/2} \frac{1}{k} + \frac{2(\frac{1}{2}(n - 1) - 1)}{n}$$

$$= \frac{3n}{2} + n \sum_{k=2}^{(n-1)/2} \frac{1}{k^2} - 3 \sum_{k=2}^{(n-1)/2} \frac{1}{k} + \frac{1}{2} - \frac{3}{n}.$$ 

Since

$$\sum_{k=1}^{n} \frac{1}{k} = O(\log n)$$

and

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} + O\left(\frac{1}{n}\right),$$

we have asymptotically

$$y_n = \frac{3n}{2} + n\left(\frac{\pi^2}{6} - 1 + O\left(\frac{1}{n}\right)\right) + O(\log n) = \left(\frac{\pi^2}{6} + \frac{1}{2}\right)n + O(\log n).$$

### 7. Fifth Attempt: Underestimate $y_n$

We underestimate $y_n$ in order to find a polynomial expression. We apply the inequalities

$$\sum_{k=1}^{n} \frac{1}{k} < \log n, \quad \sum_{k=1}^{n} \frac{1}{k^2} > \frac{2n(2n - 1) \pi^2}{(2n + 1)^2 6}.$$ 

The first inequality is easy to show. The second is from Wikipedia, where it is shown in order to prove (7). A reference to Yaglom and Yaglom [9] is given there. Now

$$n \sum_{k=2}^{n} \frac{1}{k^2} - 3 \sum_{k=2}^{n} \frac{1}{k} > n\left[\frac{2n(2n - 1) \pi^2}{(2n + 1)^2 6} - 1\right] - 3 \log n.$$
which implies, by (6),
\[
\lambda_n > \frac{3n - 1}{2} + \left[ \frac{2n_2(2n_2 - 1) \pi^2}{6} - 1 \right] n - 3 \log n_2 + \frac{2(n_2 - 1)}{n} =: y'_n.
\]

If \( n \) is even, then
\[
y'_n = \left[ \frac{2 \cdot \frac{1}{2} n(2 \cdot \frac{1}{2} n - 1) \pi^2}{6} + \frac{1}{2} \right] n - 3 \log \frac{n}{2} + \frac{2(\frac{1}{2} n - 1)}{n}
\]
\[
= \frac{n^2(n - 1) \pi^2}{6} + \frac{n + 1}{2} - 3 \log \frac{n}{2} - \frac{2}{n}.
\]

If \( n \) is odd, then
\[
y'_n = \left[ \frac{2 \cdot \frac{1}{2} (n - 1)(2 \cdot \frac{1}{2} (n - 1) - 1) \pi^2}{6} + \frac{1}{2} \right] n - 3 \log \frac{n - 1}{2} + \frac{2(\frac{1}{2} (n - 1) - 1)}{n}
\]
\[
= \frac{(n - 1)(n - 2) \pi^2}{6} + \frac{n + 1}{2} - 3 \log \frac{n - 1}{2} - \frac{3}{n}.
\]

Asymptotically
\[
y'_n = \left( \frac{\pi^2}{6} + \frac{1}{2} \right) n + O(\log n).
\]

8. Examples

In the asymptotic expression of all our bounds (excluding the conjectured bound \( v_n \)), the main term is of the form \( cn \). The coefficient \( c \) (with four digits precision) is
for \( u_n \): \( c = \frac{3}{2} = 1.5 \),
for \( x_n \): \( c = \frac{4}{3} = 1.75 \),
for \( w_n \): \( c = \frac{5}{3} - 6/\pi^2 = 1.892 \),
for \( x'_n \): \( c = \frac{35}{18} = 1.944 \),
for \( y'_n, y_n \): \( c = \frac{1}{6} \pi^2 + \frac{1}{2} = 2.145 \).

Therefore, and since \( v_n = O(n \log n) \) by definition, we have
\[
(8) \quad u_n < x_n < w_n < x'_n < y'_n < y_n < v_n
\]
when \( n \) is large.
Example 1. \( n = 3, \lambda_3 = 4.214, l_3 = u_3 = 4 \). Since \( B_3 = A_3 \), there is nothing to be improved.

Example 2. \( n = 4, \lambda_4 = 6.421, l_4 = 6, u_4 = 5.5 \). In all our procedures, \( B_4 + E_4 = A_4 \). So \( w_4 = x_4 = x'_4 = 6 = l_4 \), but \( y_4 = 5.5 = u_4 \). The benefit obtained in changing \( B_4 \) is then lost in computing \( y_4 \). The bound \( y'_4 = 3.079 \). The conjectured bound \( v_4 = 3.371 \).

Example 3. \( n = 5, \lambda_5 = 7.770, l_5 = 7.4, u_5 = 7 \). Again all procedures work completely; so \( w_5 = x_5 = x'_5 = 7.4 = l_5 \). The bound \( y_5 = 7.15 \) is better than \( u_5 \). The gain in changing \( B_5 \) is thus larger than the loss in computing \( y_5 \). The bound \( y'_5 = 4.268 \). The conjectured bound \( v_4 = 4.892 \).

Example 4. \( n = 6, \lambda_6 = 11.05, l_6 = 10.17, u_6 = 8.5 \). The bound \( w_6 = 9.833 \). The procedure of Section 5 yields \( B_6 + E_6 = A_6 \), but that in Section 4 does not. We have \( x_6 = 9.5 \) and \( x'_6 = 10.17 = l_6 \). The bound \( y_6 = 8.833 \) is better than \( u_6 \). The bound \( y'_6 = 5.913 \). The conjectured bound \( v_6 = 6.536 \).

Example 5. \( n = 20, \lambda_20 = 49.62, l_20 = 44, u_20 = 29.5 \). In the previous examples, the bound \( y'_n \) and the conjectured bound \( v_n \) are the poorest, but they improve when \( n \) increases. The bound \( y_{20} = 35.61 \) is better than \( x_{20} = 34 \) but worse than \( x'_{20} = 36.71 \). The bound \( y'_{20} = 31.84 \) is better than \( u_{20} \) but worse than \( x_{20} \). The bound \( w_{20} = 35.8 \). The conjectured bound \( v_{20} = 36.42 \).

Example 6. \( n = 50, \lambda_{50} = 156.73, l_{50} = 134.5, u_{50} = 74.5 \). We have \( x_{50} = 86.5 \) and \( x'_{50} = 94.98 \). The bound \( y_{50} = 97.30 \) is better than \( x'_{50} \). The bound \( y'_{50} = 93.28 \) is better than \( x_{50} \) but worse than \( x'_{50} \). The bound \( w_{50} = 92.58 \). The conjectured bound \( v_{50} = 118.91 \). The ordering

\[ u_{50} < x_{50} < w_{50} < y'_5 < x'_{50} < y_{50} < v_{50} \]

is almost the same as the asymptotic ordering (8). Only \( y'_{50} \) and \( x'_{50} \) are reversed.

Example 7. \( n = 150, \lambda_{150} = 617.0, l_{150} = 498.3, u_{150} = 224.5 \). Now \( x_{150} = 261.5, w_{150} = 282.1, x'_{150} = 290.2, y'_{150} = 304.4, y_{150} = 308.5, v_{150} = 456.9 \) are in the asymptotic ordering.
9. Comparison with a bound of Hong and Loewy

Hong and Loewy proved as a special case of [4], Theorem 4.7 (ii), that

\[ \lambda_n \geq n e^{-\gamma} \left( 1 - \frac{c}{\log n} \right), \]

where \( \gamma \) is Euler’s constant and \( c \) is a certain positive number. Since \( c \) is unknown and cannot easily be overestimated, this bound is useless in comparison.

These authors actually studied power gcd matrices. So let \( A_n^{(p)} \) denote the entrywise \( p \)’th power of \( A_n \) with largest eigenvalue \( \mu_n \). A special case of [4], Theorem 4.7 (i), states that if \( p > 1 \), then

\[ \mu_n \geq \frac{n^p}{\zeta(p)} =: h_n, \]

where \( \zeta \) is the Riemann zeta function. We use this bound in comparison in two ways.

First, because \( A_n^{(p)} \geq A_n \) (entrywise), we have

\[ \mu_n \geq \lambda_n, \]

see [5], Theorem 8.1.18. Hence our bounds apply also to \( \mu_n \) but are poor unless \( p \) is near to one. On the other hand, if \( p \to 1 \), then \( \zeta(p) \to \infty \) and so \( h_n \to 0 \). Therefore \( h_n \) is poor if \( p \) is near to 1, which favors our bounds unless \( n \) is very large.

Second, applying to \( A^{(p)} \) the procedures described in Sections 1 and 3, we obtain

\[ \mu_n > \frac{1}{n} \sum_{k=1}^{n} k^p + n - 1 =: \tilde{u}_n, \]

\[ \mu_n > \frac{1}{n} \sum_{k=1}^{n} k^p + n - 1 + (2^p - 1) \left( n - 1 + 2 \frac{1 - \Phi(n)}{n} \right) =: \tilde{w}_n. \]

If \( p \) is an integer, the power sum can be expressed polynomially by using Faulhaber’s formula in [10].

We compare our bounds with \( h_n \) for \( p = 2, 1.5, 1.1 \). If \( p \) is not an integer and \( n \) is not small, the bounds \( \tilde{u}_n \) and \( \tilde{w}_n \) are tedious to compute with a non-programmable calculator. Therefore we consider these bounds only in case of \( p = 2 \). We denote by \( f_n \) and \( g_n \) the best and, respectively, the worst of the bounds presented in Sections 1 and 3–7.

Example 8. \( p = 2, \mu_4 = 17.514, \mu_5 = 25.37, \mu_6 = 40.30 \). The bound \( \tilde{u}_4 = 10.5 \) is better than \( h_4 = 9.727 \), but \( h_5 = 15.20 \) is better than \( \tilde{u}_5 = 15 \). The bound \( \tilde{w}_5 = 15.4 \) is better than \( h_5 \), but \( h_6 = 21.89 \) is better than \( \tilde{w}_6 = 21.50 \). The bound \( h_n \) is better than our bounds if \( n \geq 6 \), and remarkably better if \( n \) is large.
Example 9. $p = 1.5$, $\mu_6 = 19.36$, $\mu_{20} = 125.65$, $\mu_{150} = 3050.2$. Again our bounds are better for small $n$. For example, $g_6 = y_6' = 5.913$ is better than $h_6 = 5.626$. As $n$ increases, $h_n$ begins to do better, but the range of $n$ where our bounds succeed is wider than in Example 8. The bound $h_{20} = 34.24$, for example, beats $g_{20} = u_{20} = 29.50$ but loses to $f_{20} = x_{20}' = 36.71$. Again $h_n$ is remarkably better if $n$ is large.

Example 10. $p = 1.1$, $\mu_4 = 6.918$, $\mu_{20} = 58.09$, $\mu_{150} = 810.63$. Now our bounds are better for all matrices of reasonable size. For example, $g_4 = y_4' = 3.079$, $h_4 = 0.434$, $g_{150} = u_{150} = 224.5$, $h_{150} = 23.39$. Even for $n = 1.01 \cdot 10^{12}$ the bound $g_n = u_n = 1.5150 \cdot 10^{12}$ is better than $h_n = 1.5139 \cdot 10^{12}$, but for $n = 1.02 \cdot 10^{12}$ the ordering changes: $g_n = u_n = 1.5300 \cdot 10^{12}$, $h_n = 1.5304 \cdot 10^{12}$.

10. Conclusions and remarks

We expected that $l_n = s_n/n$ is a quite good lower bound for $\lambda_n$. By underestimating $s_n$, we found several easily computable bounds. We compared them with one another and studied their asymptotical behavior. We also noted that $\lambda_n > v_n$ if $n$ is large, and conjectured this for all $n$. The examples suggest a stronger conjecture that actually $l_n > v_n$. We also compared our bounds with a bound of Hong and Loewy. For this purpose, we extended $u_n$ and $w_n$ to concern the largest eigenvalue of $A_n(p)$, $p > 1$.

By using the vector $A_n e_n$ instead of $e_n$ in the Rayleigh quotient, we obtain

$$\lambda_n > \frac{(A_n e_n)^T A_n (A_n e_n)}{e_n^T e_n} = \frac{su A_n^3}{su A_n^2},$$

where $su$ denotes the sum of entries. This bound is better than $l_n$ but seems difficult to be underestimated for our purpose.

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References


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