Nguyen Van Quang; Pham Tri Nguyen
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*Applications of Mathematics*, Vol. 61 (2016), No. 6, 669–684

Persistent URL: [http://dml.cz/dmlcz/145915](http://dml.cz/dmlcz/145915)
**STRONG LAWS OF LARGE NUMBERS FOR SEQUENCES OF BLOCKWISE AND PAIRWISE $m$-DEPENDENT RANDOM VARIABLES IN METRIS SPACES**

Nguyen Van Quang, Vinh, Pham Tri Nguyen, Hanoi

(Received April 3, 2016)

*Abstract.* The aim of the paper is to establish strong laws of large numbers for sequences of blockwise and pairwise $m$-dependent random variables in a convex combination space with or without compactly uniformly integrable condition. Some of our results are even new in the case of real random variables.

*Keywords:* strong law of large numbers; convex combination space; pairwise $m$-dependent; blockwise $m$-dependent; compactly uniformly integrable

*MSC 2010:* 60F15, 60B05, 60G50

1. Introduction


Terán and Molchanov [7] introduced the concept of a convex combination space, which is a metric space endowed with a convex combination operation. The class of these metric spaces is not only larger than the class of Banach spaces but also larger than the class of hyperspaces of compact subsets, as well as the class of upper

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This research was supported in part by Foundation for Science and Technology Development of Vietnam's Ministry of Education and Training. No. B2016.DNA.26.
semicontinuous functions (fuzzy sets) with compact support in a Banach space. Some very basic sets, such as singletons and balls, may fail to be convex in the convex combination space. Terán and Molchanov [7] also stated various basic properties of the convex combination operation and used these properties to derive the SLLN for pairwise i.i.d. random variables, which extended Theorem 1 of Etemadi [3]. Some more limit theorems for random variables taking values in a convex combination space can be found in Quang and Thuan [6], Thuan et al. [9].

Continuing in this direction, in this study we establish some results on SLLN for sequences of blockwise and pairwise $m$-dependent random variables in a convex combination space with respect to the blocks $[2^k, 2^{k+1})$ and with or without compactly uniformly integrable condition. Our results are more general than some previously reported ones. The paper is organized as follows. In Section 2, we state and summarize basic results in a convex combination space $\mathcal{X}$ and some related concepts. In Section 3, some results on SLLN for sequences of blockwise and pairwise $m$-dependent, blockwise $m$-dependent, pairwise $m$-dependent independent identically distributed random variables are established.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{A}, P)$ is a complete probability space. For $A \in \mathcal{A}$, the notation $I\{A\}$ (or $I_A$) is the indicator function of $A$. At first, we present a short introduction to the approach given by Terán and Molchanov [7]. Let $(\mathcal{X}, d)$ be a metric space. Based on $\mathcal{X}$, we introduce a convex combination operation which for all $n \geq 2$, numbers $\lambda_1, \ldots, \lambda_n > 0$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$, and all $u_1, \ldots, u_n \in \mathcal{X}$, produces an element of $\mathcal{X}$, which is denoted by $[\lambda_1, u_1; \ldots; \lambda_n, u_n]$. Assume that $[1, u] = u$ for every $u \in \mathcal{X}$ and that the following axioms are satisfied:

(i) (Commutativity)

$$[\lambda_i, u_i]_{i=1}^{n} = [\lambda_{\sigma(i)}, u_{\sigma(i)}]_{i=1}^{n} \text{ for every permutation } \sigma \text{ of } \{1, \ldots, n\}.$$ 

(ii) (Associativity)

$$[\lambda_i, u_i]_{i=1}^{n+2} = [\lambda_1, u_1; \ldots; \lambda_n, u_n; \lambda_{n+1} + \lambda_{n+2}, \left(\frac{\lambda_{n+j}}{\lambda_{n+1} + \lambda_{n+2}}, u_{n+j}\right)_{j=1}^{2}] \text{.}$$

(iii) (Continuity) If $u, v \in \mathcal{X}$ and $\lambda^{(k)} \to \lambda \in (0, 1)$ as $k \to \infty$, then

$$[\lambda^{(k)}, u; 1 - \lambda^{(k)}, v] \to [\lambda, u; 1 - \lambda, v].$$
Based on the inductive method and (ii), this axiom can be extended to convex combinations of $n$ element as follows: if $u_i, v_i \in \mathcal{X}$, $\lambda_i \in (0; 1)$, $\sum_{i=1}^{n} \lambda_i = 1$, then

$$d([\lambda, u_1; 1 - \lambda, u_2], [\lambda, v_1; 1 - \lambda, v_2]) \leq \lambda d(u_1, v_1) + (1 - \lambda) d(u_2, v_2).$$

(v) (Convexification) For each $u \in \mathcal{X}$, there exists $\lim_{n \to \infty} [n^{-1}, u]_{i=1}^{n}$, which will be denoted by $K_Xu$ (or $Ku$ if no confusion can arise), and $K$ is called the convexification operator.

Then the metric space $\mathcal{X}$ endowed with a convex combination operation is referred to as the convex combination space. The above axioms imply the following properties:

(2.1) For every $u_{11}, \ldots, u_{mn} \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n > 0$ with $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = 1$, we have

$$[\alpha_i, [\beta_j, u_{ij}]_{i=1}^{m}]_{j=1}^{n} = [\alpha_i \beta_j, u_{ij}]_{i=1}^{m}, j=1.$$

(2.2) The convex combination operation is jointly continuous in its $2n$ arguments.

(2.3) The convexification operator $K$ is linear, i.e. $K([\alpha_j, u_{ij}]_{j=1}^{n}) = [\alpha_j, K u_{ij}]_{j=1}^{n}.$

(2.4) If $u \in \mathcal{X}$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{j=1}^{n} \lambda_j = 1$, then $K([\lambda_j, u]_{j=1}^{n}) = Ku = [\lambda_j, Ku]_{j=1}^{n}$. Hence, $K$ is an idempotent operator in $\mathcal{X}$.

(2.5) For every $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $u, v \in \mathcal{X}$,

$$[\lambda_1, u; \lambda_2, Kv; \lambda_3, K\lambda] = [\lambda_1 u; (\lambda_2 + \lambda_3), Kv].$$

(2.6) The mapping $K$ is non-expansive with respect to the metric $d$, which means that $d(Ku, Kv) \leq d(u, v)$.

Let $\{\lambda_k: k \geq 1\} \subset (0; 1), \lambda_k \to 0$ as $k \to \infty$, and $u, v \in \mathcal{X}$. By (iv) and property (2.4), we have

$$d([\lambda_k, Ku; 1 - \lambda_k, Kv], Kv) = d([\lambda_k, Ku; 1 - \lambda_k, Kv], [\lambda_k, Kv; 1 - \lambda_k, K\lambda])$$

$$\leq \lambda_k d(Ku, Kv) \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that $[\lambda_k, Ku; 1 - \lambda_k, Kv] \to K\lambda$ as $k \to \infty$, and this remark makes it possible to extend weights $\lambda_i$ from $(0; 1)$ to $[0; 1]$ for elements in $K(\mathcal{X})$, which means that we can define $[\lambda_i, u_i]_{i \in I} = [\lambda_i, u_i]_{i \in J}$ with $J = \{i \in I: \lambda_i > 0\}$, where $\lambda_i \in [0, 1], u_i \in K(\mathcal{X})$ and $\sum_{i \in I} \lambda_i = \sum_{i \in I} \lambda_i = 1$. 671
A mapping $X: \Omega \to \mathcal{X}$ is called an $\mathcal{X}$-valued random variable if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathcal{X})$, where $\mathcal{B}(\mathcal{X})$ is the Borel $\sigma$-algebra on $\mathcal{X}$. The distribution $P_X$ of an $\mathcal{X}$-valued random variable $X$ is defined by $P_X(B) = P\{X^{-1}(B)\}$ for all $B \in \mathcal{B}(\mathcal{X})$, and two $\mathcal{X}$-valued random variables $X,Y$ are said to be identically distributed if $P_X = P_Y$.

The collection of $\mathcal{X}$-valued random variables $\{X_i: i \in I\}$ is said to be independent (pairwise independent) if the collection of $\sigma$-algebras $\{\sigma(X_i): i \in I\}$ is independent (pairwise independent, respectively), where $\sigma(X) = \{X^{-1}(B): B \in \mathcal{B}(\mathcal{X})\}$.

Let $m$ be a fixed nonnegative integer. We say that a finite collection $\{X_i: 1 \leq i \leq n\}$ of $\mathcal{X}$-valued random variables is $m$-dependent if either $n \leq m + 1$, or $n > m + 1$ and the random variables $\{X_1, \ldots, X_i\}$ are independent of the random variables $\{X_j, \ldots, X_n\}$ whenever $j - i > m$. A finite collection of $\mathcal{X}$-valued random variables $\{X_i: 1 \leq i \leq n\}$ is said to be pairwise $m$-dependent if either $n \leq m + 1$, or $n > m + 1$ and $X_i$ and $X_j$ are independent whenever $j - i > m$. A sequence $\{X_n: n \geq 1\}$ of $\mathcal{X}$-valued random variables is said to be pairwise $m$-dependent if $X_i$ and $X_j$ are independent whenever $j - i > m$. A sequence $\{X_n: n \geq 1\}$ of $\mathcal{X}$-valued random variables is said to be blockwise $m$-dependent (blockwise and pairwise $m$-dependent) if for each $k \in \mathbb{N}_0$, the collection $\{X_n: 2^k \leq n < 2^{k+1}\}$ is $m$-dependent (pairwise $m$-dependent, respectively).

From now on, we assume that $(\mathcal{X}, d)$ is a separable and complete metric space. We fix $u_0 \in K(\mathcal{X})$ and consider $u_0$ to be a special element of $\mathcal{X}$. A random variable $X: \Omega \to \mathcal{X}$ is said to be integrable if $Ed(u_0, X) < \infty$. Note that this definition does not depend on the selection of the element $u_0$. The space of all integrable $\mathcal{X}$-valued random variables will be denoted by $L^1_\mathcal{X}$, and the metric on $L^1_\mathcal{X}$ is defined by $\Delta(X,Y) = Ed(X,Y)$.

If $X$ is a simple function that takes a distinct value $x_i \in \mathcal{X}$ for each non-null set $\Omega_i$, $i = 1, \ldots, n$, the expectation of $X$ is defined by $EX = [P(\Omega_i), Kx_i]_{i=1}^n$. By continuity of all Borel functions $X \in L^1_\mathcal{X}$, then for $X \in L^1_\mathcal{X}$, the expectation of $X$ is defined as the limit of the expectations sequence of simple random variables. Note that if $X,Y \in L^1_\mathcal{X}$ then $d(EX, EY) \leq Ed(X,Y)$.

An example for a convex combination space is the space of nonempty compact subsets. Let $\mathcal{X}$ be a convex combination space and $k(\mathcal{X})$ the set of nonempty compact subsets of $\mathcal{X}$. A set $A \subset \mathcal{X}$ is called convex if $[\lambda_i, u_i]_{i=1}^n \in A$ for all $u_i \in A$ and $\sum_{i=1}^n \lambda_i = 1$. We denote as $\text{co}A$ the convex hull of $A \subset \mathcal{X}$, and $\overline{\text{co}}A$ is the closed convex hull of $A$. Let $d_H$ be the Hausdorff metric on $k(\mathcal{X})$, that is $d_H(A,B) = \max\left\{\sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(b,a)\right\}$ for $A, B \in k(\mathcal{X})$. It follows from Theorem 6.2 [7] that the space $k(\mathcal{X})$ with the convex combination $[\lambda_i, A_i]_{i=1}^n = \{[\lambda_i, u_i]_{i=1}^n: u_i \in A_i$ for all $i\}$ and the metric $d_H$ is a separable complete convex combination space. We
also have $K_{k(\mathcal{X})}A = \overline{\bigcup K_X A} = \overline{\{K_X u: u \in A\}}$. Further examples for convex combination spaces can be found in [7].

Now we introduce the concept of compact uniform integrability in Cesàro sense for a sequence of random variables taking values in a metric space, which is naturally extended from a Banach space to a metric space. A sequence $\{X_n: n \geq 1\}$ of $X$-valued ($k(\mathcal{X})$-valued) random variables is said to be compactly uniformly integrable in Cesàro sense (Cesàro CUI for short) if for every $\varepsilon > 0$, there exists a compact subset $K_\varepsilon$ of $\mathcal{X}$ ($k(\mathcal{X})$, respectively) such that

$$\sup_{n \geq 1} n^{-1} \sum_{i=1}^{n} E(d(u_0, X_i)I\{X_i \notin K_\varepsilon\}) \leq \varepsilon.$$  

(resp. $\sup_{n \geq 1} n^{-1} \sum_{i=1}^{n} E(d_H(\{u_0\}, X_i)I\{X_i \notin K_\varepsilon\}) \leq \varepsilon$).

By using a method similar to that used in the proof of Proposition 2.1 of [9], it is easy to show that the concept of Cesàro CUI does not depend on the selected element $u_0$. A sequence $\{X_n: n \geq 1\}$ of real-valued random variables is said to be stochastically dominated by a real-valued random variable $X$ if there exists a constant $C$ ($0 < C < \infty$) such that

$$P(|X_n| > t) \leq CP(|X| > t), \quad n \geq 1, \quad t \geq 0.$$  

We complete this summary by a lemma which will be used in the proof of Proposition 3.1 in the next section.

**Lemma 2.1** ([6], Lemma 3.3). Let $\{a_i, b_i: 1 \leq i \leq n\} \subset [0,1]$ be a collection of nonnegative constants with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$. Then

$$d([a_i, Kx_i]_{i=1}^{n}, [b_i, Kx_i]_{i=1}^{n}) \leq \sum_{i=1}^{n} |a_i - b_i|d(u_0, x_i),$$

where $x_1, \ldots, x_n \in \mathcal{X}$ are arbitrary.
3. SLLN for Sequences of Blockwise and Pairwise $m$-Dependent $\mathcal{X}$-Valued Random Variables

Let $u_0$ be a fixed element of $K(\mathcal{X})$, which is mentioned in Section 2. We denote $\|x\|_{u_0} := d(x, u_0)$ for all $x \in \mathcal{X}$. In the first theorem, we will establish a result on the SLLN for sequences of blockwise $m$-dependent and Cesàro CUI $\mathcal{X}$-valued random variables in a convex combination space. To do that, we need the following proposition.

**Proposition 3.1.** Let $K$ be a compact subset of $\mathcal{X}$. If $\{X_n : n \geq 1\}$ is a sequence of blockwise and pairwise $m$-dependent $\mathcal{X}$-valued random variables satisfying $P(X_n \in K) = 1$ for all $n$, then

$$d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \to 0 \quad \text{a.s. as } n \to \infty.$$  

**Proof.** For $\varepsilon > 0$, by the compactness of $K$, there exists $\{c_1, c_2, \ldots, c_p\} \subset K$ such that

$$K \subset \bigcup_{t=1}^p B(c_t, \varepsilon), \quad \text{where } B(c_t, \varepsilon) = \{x \in \mathcal{X} : d(x, c_t) < \varepsilon\}.$$

For $n \geq 1$, we define the $\mathcal{X}$-valued random variables as follows:

$$Y_n(\omega) = \begin{cases} 
c_0 := u_0 & \text{if } X_n(\omega) \notin K, \\
c_1 & \text{if } X_n(\omega) \in B(c_1, \varepsilon) \cap K, \\
c_t & \text{if } X_n(\omega) \in B(c_t, \varepsilon) \cap \left( \bigcup_{k=1}^{t-1} B(c_k, \varepsilon) \right)^c \cap K, \ t = 2, \ldots, p.
\end{cases}$$

It is obvious that the sequence $\{Y_n : n \geq 1\}$ is also blockwise and pairwise $m$-dependent. By the triangular inequality, we have

$$d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \leq d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Y_i]_{i=1}^n) + d([n^{-1}, Y_i]_{i=1}^n, [n^{-1}, KY_i]_{i=1}^n) + d([n^{-1}, KY_i]_{i=1}^n, [n^{-1}, EY_i]_{i=1}^n) + d([n^{-1}, EY_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) := (A_1) + (A_2) + (A_3) + (A_4).$$

For $(A_1)$, by the construction of $Y_n$, we have

$$(A_1) = d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Y_i]_{i=1}^n) \leq \frac{1}{n} \sum_{i=1}^n d(X_i, Y_i) \leq \varepsilon \quad \text{a.s.}$$

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For $(A_2)$, we put

\[ Q_t^i(\omega) = \text{card}\{i: 1 \leq i \leq n, Y_i(\omega) = c_t\} = \sum_{i=1}^n I\{Y_i = c_t\}(\omega), \]

\[ T_n(\omega) = \{t: 0 \leq t \leq p, Q_t^i(\omega) > 0\}, \quad n \geq 1. \]

By properties (2.1) and (2.4), we have

\[ [n^{-1}, Y_i]_{i=1}^n = [n^{-1}Q_t^i, [(Q_t^i)^{-1}, c_t]_{i=1}^Q]_{t \in T_n} \]

and

\[ [n^{-1}, KY_i]_{i=1}^n = [n^{-1}Q_t^i, [(Q_t^i)^{-1}, Kc_t]_{i=1}^Q]_{t \in T_n} = [n^{-1}Q_t^i, Kc_t]_{t \in T_n}. \]

Therefore,

\[ (A_2) = d([n^{-1}, Y_i]_{i=1}^n, [n^{-1}, KY_i]_{i=1}^n) \leq \sum_{t \in T_n} \frac{Q_t^i}{n} d([(Q_t^i)^{-1}, c_t]_{i=1}^Q, Kc_t). \]

We will prove that $(A_2) \leq \varepsilon$ for all $\omega \in \Omega$ when $n$ is sufficiently large.

Indeed, we consider each $t = 0, 1, \ldots, p$. By the definition of $K$, we have

\[ \lim_{n \to \infty} d([n^{-1}, c_t]_{i=1}^n, Kc_t) = 0. \]

Thus, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that

\[ d([n^{-1}, c_t]_{i=1}^n, Kc_t) < \frac{\varepsilon}{p+1} \quad \text{for all } n \geq n_1(\varepsilon) \text{ and for all } t = 0, 1, \ldots, p. \]

We put

\[ M_t(\varepsilon) = \max_{1 \leq k < n_1(\varepsilon)} d([k^{-1}, c_t]_{i=1}^k, Kc_t), \quad M(\varepsilon) = \max_{0 \leq t \leq p} M_t(\varepsilon) \]

and choose the smallest integer number $n(\varepsilon)$ such that $n(\varepsilon) \geq \varepsilon^{-1}(p+1)M(\varepsilon)n_1(\varepsilon)$.

Now, for all $n \geq n(\varepsilon)$:

If $Q_t^i(\omega) \geq n_1(\varepsilon)$, then

\[ \frac{Q_t^i(\omega)}{n} d([(Q_t^i(\omega))^{-1}, c_t]_{i=1}^Q(\omega), Kc_t) < \frac{\varepsilon}{p+1} \quad \text{(since } n^{-1}Q_t^i(\omega) \leq 1). \]

If $0 < Q_t^i(\omega) < n_1(\varepsilon)$, then

\[ \frac{Q_t^i(\omega)}{n} d([(Q_t^i(\omega))^{-1}, c_t]_{i=1}^Q(\omega), Kc_t) \leq \frac{n_1(\varepsilon)}{n(\varepsilon)} M(\varepsilon) \leq \frac{\varepsilon}{p+1}. \]
Hence, for \( n \geq n(\varepsilon) \) and for all \( \omega \in \Omega \),
\[
\frac{Q_n^t(\omega)}{n} d([Q_n^t(\omega)]^{-1}, c_t)^{Q_n^t(\omega)}/Kc_t) \leq \frac{\varepsilon}{p+1}.
\]
This implies that
\[
(A_2) \leq \sum_{t \in T_n} \frac{Q_n^t}{n} d([Q_n^t]^{-1}, c_t)^{Q_n^t/Kc_t} \leq \sum_{t=0}^{p} \frac{\varepsilon}{p+1} = \varepsilon
\]
for values of \( n \) that are sufficiently large.

For \((A_3)\), we have \( KY_i = [I\{Y_i = c_t\}, Kc_t]_{t=0}^n, EY_i = [P\{Y_i = c_t\}, Kc_t]_{t=0}^p \). Then, by properties \((2.1)\) and \((2.5)\), and Lemma \(2.1\)
\[
(A_3) = d([n^{-1}, I\{Y_i = c_t\}, Kc_t]_{t=0}^n, [n^{-1}, P\{Y_i = c_t\}, Kc_t]_{t=0}^n) = d([n^{-1}I\{Y_i = c_t\}, Kc_t]_{t=0}^n, [n^{-1}P\{Y_i = c_t\}, Kc_t]_{t=0}^n)
\]
\[
= d\left(\left[\frac{1}{n} \sum_{i=1}^{n} I\{Y_i = c_t\}, Kc_t\right]_{t=0}^p, \left[\frac{1}{n} \sum_{i=1}^{n} P\{Y_i = c_t\}, Kc_t\right]_{t=0}^p\right)
\]
\[
\leq \sum_{t=0}^{p} \frac{1}{n} \sum_{i=1}^{n} (I\{Y_i = c_t\} - P\{Y_i = c_t\}) \|c_t\|_{u_0}
\]
\[
= \sum_{i=1}^{n} \frac{1}{n} \sum_{i=1}^{n} (I\{Y_i = c_t\} - P\{Y_i = c_t\}) \|c_t\|_{u_0}.
\]

Using the fact that \(\{I\{Y_i = c_t\} - P\{Y_i = c_t\} : n \geq 1\}\) is the sequence of blockwise and pairwise \(m\)-dependent real-valued random variables that is stochastically dominated by random variable \(X = 2\) and applying Theorem \(1\) of \([8]\), we have
\[
\frac{1}{n} \sum_{i=1}^{n} (I\{Y_i = c_t\} - P\{Y_i = c_t\}) \to 0 \quad \text{a.s. as } n \to \infty.
\]
Thus \((A_3)\) \(\to 0\) a.s. as \(n \to \infty\).

For \((A_4)\), by the definition of \(Y_n\), we have
\[
(A_4) = d([n^{-1}, EY_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} d(EY_i, EX_i) \leq \frac{1}{n} \sum_{i=1}^{n} Ed(Y_i, X_i) < \varepsilon.
\]
Combining the above parts, we obtain
\[
\limsup_{n \to \infty} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \leq 3\varepsilon \quad \text{a.s.}
\]
By the arbitrariness of \(\varepsilon > 0\), we derive conclusion \((1)\). \(\square\)
Applying the above proposition, we obtain the SLLN for sequences of blockwise \(m\)-dependent and Cesàro CUI \(X\)-valued random variables.

**Theorem 3.2.** Let \( \{X_n: n \geq 1\} \) be a sequence of blockwise \(m\)-dependent \(X\)-valued random variables which is Cesàro CUI. If

\[
\sum_{n=1}^{\infty} \frac{E\|X_n\|_{u_0}^2}{n^2} < \infty, \tag{2}
\]

then (1) holds.

**Proof.** For \(\varepsilon > 0\) arbitrarily small, by Cesàro CUI hypothesis there exists a compact subset \(K_\varepsilon\) of \(X\) such that

\[
\frac{1}{n} \sum_{i=1}^{n} E\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\} \leq \varepsilon \quad \text{for all } n.
\]

For \(n \geq 1\), we define the \(X\)-valued random variables as follows:

\[
Z_n(\omega) = \begin{cases} X_n(\omega) & \text{if } X_n(\omega) \in K_\varepsilon, \\ u_0 & \text{if } X_n(\omega) \notin K_\varepsilon. \end{cases}
\]

By the triangular inequality, we have

\[
d([n^{-1}, X_i^n]_{i=1}^n, [n^{-1}, EX_i^n]_{i=1}^n) \leq d([n^{-1}, X_i^n]_{i=1}^n, [n^{-1}, Z_i^n]_{i=1}^n) \\
+ d([n^{-1}, Z_i^n]_{i=1}^n, [n^{-1}, EZ_i^n]_{i=1}^n) \\
+ d([n^{-1}, EZ_i^n]_{i=1}^n, [n^{-1}, EX_i^n]_{i=1}^n) \\
:= (B_1) + (B_2) + (B_3).
\]

For \((B_1)\), we have

\[
d([n^{-1}, X_i^n]_{i=1}^n, [n^{-1}, Z_i^n]_{i=1}^n) \\
\leq \frac{1}{n} \sum_{i=1}^{n} d(X_i, Z_i) = \frac{1}{n} \sum_{i=1}^{n} \|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\} \\
= \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\})) \\
+ \frac{1}{n} \sum_{i=1}^{n} E(\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\}) \\
\leq \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin K_\varepsilon\})) + \varepsilon.
\]
It is clear that \( \{\|X_n\|_{u_0}I\{X_n \notin \mathcal{K}_\varepsilon\} - E(\|X_n\|_{u_0}I\{X_n \notin \mathcal{K}_\varepsilon\})\} : n \geq 1 \) is a sequence of blockwise \( m \)-dependent real-valued random variables. Moreover,

\[
\sum_{n=1}^{\infty} \frac{E(\|X_n\|_{u_0}I\{X_n \notin \mathcal{K}_\varepsilon\} - E(\|X_n\|_{u_0}I\{X_n \notin \mathcal{K}_\varepsilon\}))^2}{n^2} \leq \sum_{n=1}^{\infty} \frac{E\|X_n\|^2_{u_0}}{n^2} < \infty \quad \text{(by (2)).}
\]

Then by Theorem 1 of [5], we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0}I\{X_i \notin \mathcal{K}_\varepsilon\} - E(\|X_i\|_{u_0}I\{X_i \notin \mathcal{K}_\varepsilon\})) = 0 \quad \text{a.s.}
\]

Hence,

\[
\limsup_{n \to \infty} d([n^{-1}, X_i]_{i=1}^{n}, [n^{-1}, Z_i]_{i=1}^{n}) \leq \varepsilon \quad \text{a.s.}
\]

For \((B_2)\), it is clear that \( \{Z_n : n \geq 1\} \) is a sequence of blockwise and pairwise \( m \)-dependent \( X \)-valued random variables and \( Z_n \in \mathcal{K}_\varepsilon \cup \{u_0\} \) for all \( n \). By applying Proposition 3.1, we have \((B_2) \to 0 \) a.s. as \( n \to \infty \).

For \((B_3)\), we have

\[
d([n^{-1}, EZ_i]_{i=1}^{n}, [n^{-1}, EX_i]_{i=1}^{n}) \leq \frac{1}{n} \sum_{i=1}^{n} d(EZ_i, EX_i) \leq \frac{1}{n} \sum_{i=1}^{n} \text{Ed}(Z_i, X_i) = \frac{1}{n} \sum_{i=1}^{n} E(\|X_i\|_{u_0}I\{X_i \notin \mathcal{K}_\varepsilon\}) \leq \varepsilon.
\]

Combining the above parts, we obtain

\[
\limsup_{n \to \infty} d([n^{-1}, X_i]_{i=1}^{n}, [n^{-1}, EX_i]_{i=1}^{n}) \leq 2\varepsilon \quad \text{a.s.}
\]

The proof is completed.

For sequences of blockwise and pairwise \( m \)-dependent and Cesàro CUI \( X \)-valued random variables, we obtain the following result.
Theorem 3.3. Let \( \{X_n: n \geq 1\} \) be a sequence of blockwise and pairwise \( m \)-dependent \( X \)-valued random variables which are Cesàro CUI. Suppose that \( \{\|X_n\|_{u_0}: n \geq 1\} \) is stochastically dominated by a real-valued random variable \( X \). If

\[
E(|X|(|\log^+ |X|)^2) < \infty,
\]

then (1) holds.

Proof. We will use the same method and notation as in the proof of Theorem 3.2. We also have \((B_2) \to 0 \) a.s. as \( n \to \infty \) and \((B_3) \leq \epsilon \). Now we consider \((B_1)\). Note that \( \{\|X_n\|_{u_0}I\{X_n \notin K_\epsilon\}: n \geq 1\} \) is a sequence of blockwise and pairwise \( m \)-dependent random variables and for all \( n, t \geq 0 \),

\[
P(\|X_n\|_{u_0}I\{X_n \notin K_\epsilon\} \geq t) \leq P(\|X_n\|_{u_0} \geq t) \leq CP(|X| \geq t).
\]

Then by Theorem 1 of [8], we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0}I\{X_i \notin K_\epsilon\} - E(\|X_i\|_{u_0}I\{X_i \notin K_\epsilon\})) = 0 \quad \text{a.s.}
\]

Hence,

\[
\limsup_{n \to \infty} d([n^{-1}, X_i^n_{i=1}], [n^{-1}, Z_i^n_{i=1}]) \leq \epsilon \quad \text{a.s.}
\]

The proof is completed. \( \square \)

Remark 3.4. In general, the compactly uniform integrability is stronger than the uniform integrability. But, they are equivalent for the real case. Furthermore, if \( \{\|X_n\|_{u_0}: n \geq 1\} \) is stochastically dominated by a real-valued random variable \( X \) and \( E|X| < \infty \), then \( \{\|X_n\|_{u_0}: n \geq 1\} \) is uniformly integrable. Therefore, in the above theorem, if \( X = \mathbb{R} \) we obtain Theorem 1 of [8] with \( r = 1 \).

In the next part, we will establish the SLLN for sequences of blockwise \( m \)-dependent and identically distributed \( X \)-valued random variables. First, we will prove the following lemma.

Lemma 3.5. Let \( \{X, X_n: n \geq 1\} \) be a sequence of blockwise \( m \)-dependent and identically distributed real-valued random variables. If \( E|X| < \infty \), then

\[
\frac{1}{n} \sum_{i=1}^{n} X_i \to EX \quad \text{a.s. as } n \to \infty.
\]

Proof. Since \( \{X_i^+: n \geq 1\} \) and \( \{X_i^-: n \geq 1\} \) satisfy the hypotheses of the lemma and \( X_n = X_n^+ - X_n^- \), without loss of generality we can assume that \( X_n \geq 0 \).
Set \( Y_n = X_n I\{X_n \leq n\}, \ n \geq 1 \). Let \( F(x) \) be the distribution function of \( X \). We have

\[
\sum_{n=1}^{\infty} \frac{EY^2_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{0}^{n} x^2 \, dF(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \int_{k}^{k+1} x^2 \, dF(x)
\]

\[
= \sum_{k=0}^{\infty} \int_{k}^{k+1} x^2 \, dF(x) \sum_{n=k+1}^{\infty} \frac{1}{n^2}.
\]

On the other hand, we have

\[
\sum_{n=k+1}^{\infty} \frac{1}{n^2} \leq \frac{2}{k+1}, \quad k \in \mathbb{N}_0.
\]

Indeed, if \( k = 0 \), then

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 2.
\]

If \( k \geq 1 \), then

\[
\sum_{n=k+1}^{\infty} \frac{1}{n^2} \leq \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)} = \frac{1}{k} \leq \frac{2}{k+1}.
\]

Hence,

\[
\sum_{n=1}^{\infty} \frac{EY^2_n}{n^2} \leq 2 \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{k}^{k+1} x^2 \, dF(x) \leq 2 \sum_{k=0}^{\infty} \int_{k}^{k+1} x \, dF(x) = 2EX < \infty.
\]

This implies that

\[
\sum_{n=1}^{\infty} \frac{E(Y_n - EY_n)^2}{n^2} \leq \sum_{n=1}^{\infty} \frac{EY^2_n}{n^2} < \infty.
\]

Since \( \{Y_n: n \geq 1\} \) is a sequence of blockwise \( m \)-dependent random variables, \( \{Y_n - EY_n: n \geq 1\} \) is so as well. By Theorem 1 of [5], we have

\[
\frac{1}{n} \sum_{i=1}^{n} (Y_i - EY_i) \to 0 \quad \text{a.s. as } n \to \infty.
\]

On the other hand,

\[
\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(X_n > n) = \sum_{n=1}^{\infty} \int_{n}^{\infty} dF(x)
\]

\[
= \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \int_{i}^{i+1} dF(x) = \sum_{i=1}^{\infty} i \int_{i}^{i+1} dF(x)
\]

\[
\leq \sum_{i=1}^{\infty} \int_{i}^{i+1} x \, dF(x) \leq EX < \infty.
\]
It follows from $EY_n \uparrow EX$ as $n \to \infty$ and the Borel-Cantelli lemma that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to EX \quad \text{a.s. as } n \to \infty.$$ 

The lemma is proved. \hfill \square

Now we will establish the SLLN for sequences of blockwise $m$-dependent and identically distributed $X$-valued random variables.

**Theorem 3.6.** Let $\{X, X_n: n \geq 1\}$ be a sequence of blockwise $m$-dependent and identically distributed $X$-valued random variables. If

$$(5) \quad E\|X\|_{u_0} < \infty,$$

then

$$(6) \quad \left[n^{-1}, X_i\right]_{i=1}^{n} \to EX \quad \text{a.s. as } n \to \infty.$$ 

**Proof.** First, assume that $X$ is a simple function taking distinct values $x_1, x_2, \ldots, x_p$ on non-null sets $\Omega_1, \Omega_2, \ldots, \Omega_p$. For each $t = 1, \ldots, p$, we put

$$U_t^n(\omega) = \text{card}\{i: 1 \leq i \leq n, X_i(\omega) = x_t\} = \sum_{i=1}^{n} I\{X_i = x_t\}(\omega).$$

By Lemma 3.5, we have $n^{-1}U_t^n \to P(\Omega_t) > 0$ a.s. as $n \to \infty$. Hence, $U_t^n \to \infty$ a.s. as $n \to \infty$ and when $n$ is large enough, $U_t^n > 0$ a.s. For $n$ large enough, by property (2.1) we have

$$\left[n^{-1}, X_i\right]_{i=1}^{n} = \left[n^{-1}U_t^n, [(U_t^n)^{-1}, x_t]_{i=1}^{U_t^n}\right]_{l=1}^{p} \quad \text{a.s.}$$

On the other hand,

$$\left[n^{-1}U_t^n, [(U_t^n)^{-1}, x_t]_{i=1}^{U_t^n}\right]_{l=1}^{p} \to [P(\Omega_t), Kx_t]_{l=1}^{p} = EX \quad \text{a.s. as } n \to \infty.$$ 

So the conclusion (6) holds for simple random variables.

Let us consider the general case when $X$ is an integrable $X$-valued random variable. For each $\varepsilon > 0$, by Proposition 4.1 (d) of [7], there exists a sequence $\{\varphi_k: k \geq 1\}$ of (Borel measurable) maps such that $Ed(\varphi_k(X), X) \to 0$ as $k \to \infty$. Therefore, there exists $k_0$ such that $Ed(\varphi_{k_0}(X), X) \leq \varepsilon$. Furthermore, by the assumption and
Proposition 4.1 (a) of [7], \( \{ \varphi_{k_0}(X), \varphi_{k_0}(X_n) : n \geq 1 \} \) is a sequence of blockwise \( m \)-dependent identically distributed and simple random variables. We have

\[
E\|\varphi_{k_0}(X)\|_{u_0} \leq E\|X\|_{u_0} + \varepsilon < \infty.
\]

By the triangle inequality,

\[
d([n^{-1}, X_i]_{i=1}^n, EX) \leq d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n)
+ d([n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n, E\varphi_{k_0}(X))
+ d(E\varphi_{k_0}(X), EX) := (C_1) + (C_2) + (C_3).
\]

For \((C_1)\), we have

\[
d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n) \leq \frac{1}{n} \sum_{i=1}^n d(X_i, \varphi_{k_0}(X_i)).
\]

Note that \( \{d(X, \varphi_{k_0}(X)), d(X_n, \varphi_{k_0}(X_n)) : n \geq 1 \} \) is a sequence of blockwise \( m \)-dependent and identically distributed real-valued random variables. Thus by Lemma 3.5 again, we have

\[
\frac{1}{n} \sum_{i=1}^n d(X_i, \varphi_{k_0}(X_i)) \to Ed(X, \varphi_{k_0}(X)) \leq \varepsilon \quad \text{a.s. as } n \to \infty.
\]

For \((C_2)\), it follows from the first case and (7) that

\[
d([n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n, E\varphi_{k_0}(X)) \to 0 \quad \text{a.s. as } n \to \infty.
\]

For \((C_3)\), we have

\[
d(E\varphi_{k_0}(X), EX) \leq Ed(X, \varphi_{k_0}(X)) \leq \varepsilon.
\]

Combining the above parts, we obtain

\[
\limsup_{n \to \infty} d([n^{-1}, X_i]_{i=1}^n, EX) \leq 2\varepsilon \quad \text{a.s.}
\]

By the arbitrariness of \( \varepsilon > 0 \), we derive the conclusion (6).

By using Theorem 2 of [8] and a method similar to that used in the proof of Theorem 3.6, we derive SLLN for sequences of pairwise \( m \)-dependent and identically distributed \( \mathcal{X} \)-valued random variables. Therefore, the proof will be omitted.
**Theorem 3.7.** Let \( \{X, X_n: n \geq 1\} \) be a sequence of pairwise \( m \)-dependent and identically distributed \( \mathcal{X} \)-valued random variables. Then condition (5) implies (6).

**Remark 3.8.** Note that pairwise 0-dependence is equivalent to pairwise independence. Therefore, Theorem 5.1 of [7] is a special case of Theorem 3.7.

As mentioned in Section 2, \((k(\mathcal{X}), d_H)\) is a separable and complete convex combination space. We denote the expectation of an integrable random variable \( X \) in \((k(\mathcal{X}), d_H)\) by \( E_{k(\mathcal{X})}X \). Then, by applying the above theorems, we obtain the following corollaries immediately:

**Corollary 3.9.** Let \( \{X, X_n: n \geq 1\} \) be a sequence of blockwise \( m \)-dependent (or pairwise \( m \)-dependent) and identically distributed \( k(\mathcal{X}) \)-valued random variables. If 
\[
E\|X\|_{\{u_0\}} < \infty,
\]
then
\[
[n^{-1}, X_i]_{i=1}^n \rightarrow E_{k(\mathcal{X})}X \quad \text{a.s. as } n \rightarrow \infty.
\]

**Corollary 3.10.** Let \( \{X_n: n \geq 1\} \) be a sequence of blockwise \( m \)-dependent \( k(\mathcal{X}) \)-valued random variables which are Cesàro CUI. If
\[
\sum_{n=1}^{\infty} \frac{E\|X_n\|_{\{u_0\}}^2}{n^2} < \infty,
\]
then
\[
d_H([n^{-1}, X_i]_{i=1}^n, [n^{-1}, E_{k(\mathcal{X})}X_i]_{i=1}^n) \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.
\]

**Corollary 3.11.** Let \( \{X_n: n \geq 1\} \) be a sequence of blockwise and pairwise \( m \)-dependent \( k(\mathcal{X}) \)-valued random variables which are Cesàro CUI. Suppose that \( \{\|X_n\|_{\{u_0\}}: n \geq 1\} \) is stochastically dominated by a real-valued random variable \( X \). If
\[
E(|X|(|\log^+ |X||)^2) < \infty,
\]
then (10) holds.

**Acknowledgements.** The authors would like to thank the editors and referees for carefully reading the manuscript and for offering some very perceptive comments that helped us to improve this study.
References


Authors’ addresses: Nguyen Van Quang, Department of Mathematics, Vinh University, 182 Le Duan Street, Vinh City, Nghe An Province, Vietnam, e-mail: nvquang@hotmail.com; Pham Tri Nguyen, Fundamental Science Faculty, Electric Power University, 235 Hoang Quoc Viet, Hanoi City, Vietnam, e-mail: nguyench13@gmail.com.