David N. Pham

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\textbf{g-QUASI-FROBENIUS LIE ALGEBRAS}

\textbf{DAVID N. PHAM}

\textbf{Abstract.} A Lie version of Turaev’s \(\overline{G}\)-Frobenius algebras from 2-dimensional homotopy quantum field theory is proposed. The foundation for this Lie version is a structure we call a \(g\)-\textit{quasi-Frobenius Lie algebra} for \(g\) a finite dimensional Lie algebra. The latter consists of a quasi-Frobenius Lie algebra \((q, \beta)\) together with a left \(g\)-module structure which acts on \(q\) via derivations and for which \(\beta\) is \(g\)-invariant. Geometrically, \(g\)-\textit{quasi-Frobenius Lie algebras} are the Lie algebra structures associated to symplectic Lie groups with an action by a Lie group \(G\) which acts via symplectic Lie group automorphisms. In addition to geometry, \(g\)-\textit{quasi-Frobenius Lie algebras} can also be motivated from the point of view of category theory. Specifically, \(g\)-\textit{quasi Frobenius Lie algebras} correspond to \textit{quasi Frobenius Lie objects} in \(\text{Rep}(g)\). If \(g\) is now equipped with a Lie bialgebra structure, then the categorical formulation of \(\overline{G}\)-Frobenius algebras given in \cite{16} suggests that the Lie version of a \(\overline{G}\)-Frobenius algebra is a quasi-Frobenius Lie object in \(\text{Rep}(D(g))\), where \(D(g)\) is the associated (semiclassical) Drinfeld double. We show that if \(g\) is a quasitriangular Lie bialgebra, then every \(g\)-\textit{quasi-Frobenius Lie algebra} has an induced \(D(g)\)-action which gives it the structure of a \(D(g)\)-\textit{quasi-Frobenius Lie algebra}.

1. Introduction

Renewed interest in Frobenius algebras arose shortly after Witten’s introduction of \textit{Topological Quantum Field Theory} (TQFT) in \cite{28}. Shortly afterwards, Atiyah proposed a set of axioms for TQFT \cite{3}, thus making Witten’s work more accessible to the mathematical community. Working from Atiyah’s axioms, L. Abrams showed that 2-dimensional TQFTs are classified by commutative Frobenius algebras \cite{1}. Hence, in the 2-dimensional case, the algebraic structure of a TQFT is that of a Frobenius algebra.

The notion of a \((d+1)\)-dimensional TQFT was generalized to a \((d+1)\)-dimensional \textit{Homotopy Quantum Field Theory} (HQFT) by V. Turaev in \cite{25} by equipping closed \(d\)-manifolds and \((d+1)\)-dimensional cobordisms with homotopy classes of maps into a target space \(X\). In the special case when \(X\) is a \(K(\overline{G},1)\)-space for \(\overline{G}\) a finite group,
one finds that the 2-dimensional HQFTs are classified by Frobenius algebras with a $\mathcal{G}$-grading and a $\mathcal{G}$-action which satisfies a number of conditions \[15\ 25\]. These Frobenius algebras came to be called $\mathcal{G}$-Frobenius algebras (or crossed $\mathcal{G}$-algebras).

In \[16\], a categorical formulation of $\mathcal{G}$-Frobenius algebras was presented where $\mathcal{G}$-Frobenius algebras were shown to correspond to certain types of Frobenius objects in $\text{Rep}(D(k[\mathcal{G}]))$, the braided monoidal category of finite dimensional left $D(k[\mathcal{G}])$-modules, where $D(k[\mathcal{G}])$ is the Drinfeld double of the group ring $k[\mathcal{G}]$ with its usual Hopf structure. Now the semiclassical analogue of $D(k[\mathcal{G}])$ (or more generally $D(H)$ for $H$ a finite dimensional Hopf algebra) is $D(\mathfrak{g})$, the Drinfeld double of a finite dimensional Lie bialgebra $(\mathfrak{g}, \gamma)$ \[7\ 9\ 10\ 18\]. The relationship between $\mathcal{G}$-Frobenius algebras and $D(k[\mathcal{G}])$ in \[16\] motivates the following question:

*With $(\mathfrak{g}, \gamma)$ fixed, what structure plays the role of a $\mathcal{G}$-Frobenius algebra for $D(\mathfrak{g})$?*

Since $D(\mathfrak{g})$ is the Lie version of $D(k[\mathcal{G}])$, the structure in question should be the Lie version of a $\mathcal{G}$-Frobenius algebra. To answer the aforementioned question, we introduce the notion of \textit{$\mathfrak{g}$-quasi-Frobenius Lie algebras} for $\mathfrak{g}$ a finite dimensional Lie algebra. A \textit{$\mathfrak{g}$-quasi-Frobenius Lie algebra} consists of a quasi-Frobenius Lie algebra $(\mathfrak{q}, \beta)$ together with a left $\mathfrak{g}$-module structure which acts on $\mathfrak{q}$ via derivations and for which $\beta$ is $\mathfrak{g}$-invariant. Geometrically, $\mathfrak{g}$-quasi-Frobenius Lie algebras are the Lie algebra structures of symplectic Lie groups with an action by a Lie group $G$ which acts via symplectic Lie group automorphisms. We call the aforementioned structures $G$-\textit{symplectic Lie groups}.

Interestingly, $\mathfrak{g}$-quasi-Frobenius Lie algebras have a categorical formulation. To obtain this formulation, we introduce the notion of a \textit{quasi-Frobenius Lie object} for any additive symmetric monoidal category. The work of Goyvaerts and Vercruysse on the categorification of Lie algebras \[12\] provides the foundation for defining quasi-Frobenius Lie objects. The latter then yields an alternate (yet equivalent) definition of a $\mathfrak{g}$-quasi-Frobenius Lie algebra: a $\mathfrak{g}$-\textit{quasi Frobenius Lie algebra is simply a quasi Frobenius Lie object in $\text{Rep}(\mathfrak{g})$}, where $\text{Rep}(\mathfrak{g})$ is the category of finite dimensional representations of $\mathfrak{g}$. Using the categorical formulation of \[16\] as motivation, we obtain the Lie version of a $\mathcal{G}$-Frobenius algebra: for a fixed finite dimensional Lie bialgebra $(\mathfrak{g}, \gamma)$, the Lie version of a $\mathcal{G}$-Frobenius algebra is a $D(\mathfrak{g})$-\textit{quasi-Frobenius Lie algebra object in $\text{Rep}(D(\mathfrak{g}))$}. In other words, with respect to $(\mathfrak{g}, \gamma)$, a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra is the Lie version of a $\mathcal{G}$-Frobenius algebra. The definition of $D(\mathfrak{g})$ implies that a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra is equivalent to a quasi-Frobenius Lie algebra $(\mathfrak{q}, \beta)$ which is both a $\mathfrak{g}$ and $\mathfrak{g}^*$-quasi-Frobenius Lie algebra where the $\mathfrak{g}$ and $\mathfrak{g}^*$ actions satisfy a certain compatibility condition.

The rest of the paper is organized as follows. In Section 2, we give a brief review of quasi-Frobenius Lie algebras, symplectic Lie groups, Lie bialgebras, and the Drinfeld double. In Section 3, we formally define $\mathfrak{g}$-quasi-Frobenius Lie algebras and prove a general result for their construction. We conclude the section with the categorical formulation of these structures. In Section 4, $G$-symplectic Lie groups are introduced. We show that $\mathfrak{g}$-quasi-Frobenius Lie algebras are the Lie algebra
structures of $G$-symplectic Lie groups. In addition, we show that the category of finite dimensional $\mathfrak{g}$-quasi-Frobenius Lie algebras is equivalent to the category of simply connected $G$-symplectic Lie groups where $G$ is also simply connected. In Section 5, we focus our attention on $D(\mathfrak{g})$-quasi-Frobenius Lie algebras. We show that if $\mathfrak{g}$ is a quasitriangular Lie bialgebra, then every $\mathfrak{g}$-quasi-Frobenius Lie algebra has an induced $D(\mathfrak{g})$-action which extends the original $\mathfrak{g}$-action and gives the underlying quasi-Frobenius Lie algebra the structure of a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra. In particular, for any finite dimensional Lie algebra $\mathfrak{g}$ (viewed as a Lie bialgebra with co-bracket $\gamma \equiv 0$), every $\mathfrak{g}$-quasi-Frobenius Lie algebra is a $D(\mathfrak{g})$-quasi-Frobenius Lie algebra, where $D(\mathfrak{g})$ is the Drinfeld double of $(\mathfrak{g}, 0)$.

2. Preliminaries

In this section, we briefly review some of the relevant background for the current paper. Throughout this section, $k$ is a field of characteristic zero.

2.1. Quasi-Frobenius Lie Algebras. The definition of a Frobenius Lie algebra is modeled after the definition of a Frobenius algebra. Formally, a Frobenius Lie algebra is defined as follows:

**Definition 2.1.** A Frobenius Lie algebra over $k$ is a pair $(\mathfrak{g}, \alpha)$ where $\mathfrak{g}$ is a Lie algebra and $\alpha: \mathfrak{g} \to k$ is a linear map with the property that the skew-symmetric bilinear form $\beta$ on $\mathfrak{g}$ defined by

$$\beta(x, y) := \alpha([x, y]) \quad \forall \, x, y \in \mathfrak{g}$$

is nondegenerate.

As a consequence of the Jacobi identity, the skew-symmetric bilinear form $\beta$ in Definition 2.1 satisfies the following identity:

$$\beta([x, y], z) + \beta([y, z], x) + \beta([z, x], y) = 0, \quad \forall \, x, y, z \in \mathfrak{g}. \tag{2.1}$$

Equation (2.1) is equivalent to the statement that $\beta$ is a 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in $k$ (where $\mathfrak{g}$ acts trivially on $k$). This motivates the following generalization of Definition 2.1:

**Definition 2.2.** A quasi-Frobenius Lie algebra over $k$ is a pair $(\mathfrak{g}, \beta)$ where $\mathfrak{g}$ is a Lie algebra over $k$ and $\beta$ is a nondegenerate 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in $k$ (where $\mathfrak{g}$ acts trivially on $k$).

**Remark 2.3.** A quasi-Frobenius Lie algebra $(\mathfrak{g}, \beta)$ is a Frobenius Lie algebra iff $\beta$ is exact, i.e., $\beta(x, y) = (\delta \alpha)(x, y) := \alpha([x, y])$ for some linear map $\alpha: \mathfrak{g} \to k$.

**Proposition 2.4.** Every 2-dimensional non-abelian Lie algebra admits the structure of a Frobenius Lie algebra. In particular, every 2-dimensional non-abelian quasi-Frobenius Lie algebra is Frobenius.

**Proof.** Let $\mathfrak{g}$ be a 2-dimensional non-abelian Lie algebra. Then $\mathfrak{g}$ admits a basis $u_1, u_2$ such that $[u_1, u_2] = u_2$. Let $\alpha: \mathfrak{g} \to k$ be the linear map defined by $\alpha(u_1) = 0$ and $\alpha(u_2) = 1$. Then $(\mathfrak{g}, \alpha)$ is a Frobenius Lie algebra.
If \((g, \beta)\) is a quasi-Frobenius Lie algebra, set \(\alpha(u_1) = 0\) and \(\alpha(u_2) = \beta(u_1, u_2)\). Then it’s easy to see that \(\beta(x, y) = \alpha([x, y])\) for all \(x, y \in g\). Hence, \((g, \beta)\) is Frobenius.

\[\square\]

Remark 2.5. Since every finite dimensional quasi-Frobenius Lie algebra \((g, \beta)\) is also a symplectic vector space, it follows that the dimension of \(g\) is necessarily even.

Proposition 2.6. Let \(g\) be a Lie algebra of dimension \(n\) over \(k\) and let \(e_1, e_2, \ldots, e_n\) be a basis of \(g\). Then the following statements are equivalent:

1. There exists \(\alpha \in g^*\) such that \((g, \alpha)\) is a Frobenius Lie algebra.
2. There exists \(\alpha \in g^*\) such that \(\det(\alpha([e_i, e_j])) \neq 0\).
3. \(\det([e_i, e_j]) \neq 0\), where \([e_i, e_j] \in g\) are viewed as elements of the symmetric algebra \(S(g)\).

Proof. (1) \(\iff\) (2) Immediate.

(2) \(\Rightarrow\) (3) Recall that \(S(g)\) is naturally isomorphic to the polynomial ring in \(n\)-variables where the variables are taken to be the basis \(e_1, e_2, \ldots, e_n\). Extend the linear map \(\alpha : g \to k\) to a unit preserving algebra map \(\alpha : S(g) \to k\) via

\[\alpha(v_1v_2\cdots v_r) := \alpha(v_1)\alpha(v_2)\cdots\alpha(v_r)\]

for \(v_1, \ldots, v_r \in g\). Then

\[\alpha(\det([e_i, e_j])) = \det(\alpha([e_i, e_j])) \neq 0,\]

which implies that \(\det([e_i, e_j]) \neq 0\).

(2) \(\iff\) (3) Let \(p = \det([e_i, e_j]) \in S(g)\). Since \(p = p(e_1, \ldots, e_n) \neq 0\) and \(k\) is infinite, there exists \(\lambda_i \in k\) such that \(p(\lambda_1, \ldots, \lambda_n) \neq 0\) (see Theorem 3.76 of [26]). Let \(\alpha : g \to k\) be the linear map defined by \(\alpha(e_i) = \lambda_i\) for \(i = 1, \ldots, n\). As before, extend \(\alpha : g \to k\) to an algebra map \(\alpha : S(g) \to k\). Then

\[\det(\alpha([e_i, e_j])) = \alpha(\det([e_i, e_j]))\]

\[= \alpha(p(e_1, \ldots, e_n))\]

\[= p(\alpha(e_1), \ldots, \alpha(e_n))\]

\[= p(\lambda_1, \ldots, \lambda_n)\]

\[\neq 0.\]

We now recall two examples. The first is Frobenius and the second is quasi-Frobenius but not Frobenius [6, 22].

Example 2.7. Let \(g\) be the 4-dimensional Lie algebra with basis \(\{x_1, \ldots, x_4\}\) and non-zero commutator relations:

\[\begin{align*}
[x_1, x_2] &= \frac{1}{2}x_2 + x_3, & [x_1, x_3] &= \frac{1}{2}x_3, & [x_1, x_4] &= x_4, & [x_2, x_3] &= x_4.
\end{align*}\]

Then \(\det([x_i, x_j]) = (x_4)^4 \neq 0\), where \([x_i, x_j]\) are regarded as elements of the symmetric algebra \(S(g)\). By Proposition 2.6 there exists a linear map \(\alpha : g \to k\) for which \((g, \alpha)\) is a Frobenius Lie algebra.
Example 2.8. Let \( q \) be the 4-dimensional Lie algebra with basis \( \{ x_1, \ldots, x_4 \} \) and non-zero commutator relations:
\[
[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4.
\]
Since \( \det([x_1, x_j]) = 0 \), \( q \) cannot be Frobenius by Proposition 2.6. However, it does admit the structure of a quasi-Frobenius Lie algebra. As an example of this, let \( \beta \) be the nondegenerate, skew-symmetric bilinear form given by
\[
\beta = x_1^* \wedge x_4^* + x_2^* \wedge x_3^*
\]
where \( \{ x_1^*, \ldots, x_4^* \} \) is the dual basis. A direct calculation shows that \( \beta \) satisfies the 2-cocycle condition. Hence, \((q, \beta)\) is quasi-Frobenius.

Definition 2.9. Let \((g_1, \beta_1)\) and \((g_2, \beta_2)\) be quasi-Frobenius Lie algebras. A quasi-Frobenius Lie algebra homomorphism from \((g_1, \beta_1)\) to \((g_2, \beta_2)\) is a Lie algebra homomorphism \( \varphi : g_1 \to g_2 \) such that \( \varphi^* \beta_2 = \beta_1 \), that is,
\[
\beta_1(u, v) = \beta_2(\varphi(u), \varphi(v)), \quad \forall \, u, v \in g_1.
\]
If \( \varphi : g_1 \to g_2 \) satisfies \((2.2)\) and is also a Lie algebra isomorphism, then \( \varphi \) is an isomorphism of quasi-Frobenius Lie algebras.

Proposition 2.10. Let \( \varphi : (g_1, \beta_1) \to (g_2, \beta_2) \) be a quasi-Frobenius Lie algebra map. If \( \dim g_1 = \dim g_2 < \infty \), then \( \varphi \) is an isomorphism of quasi-Frobenius Lie algebras.

Proof. Since \( \dim g_1 = \dim g_2 < \infty \), it suffices to show that \( \varphi \) is injective. Let \( u \in g_1 \) be any nonzero element. Since \( \beta \) is nondegenerate, there exists \( v \in g_1 \) such that \( \beta(u, v) \neq 0 \). Hence,
\[
\beta_2(\varphi(u), \varphi(v)) = \beta_1(u, v) \neq 0,
\]
which implies that \( \varphi(u) \neq 0 \). This completes the proof. \( \square \)

2.2. Symplectic Lie Groups. In this section, we recall the correspondence between symplectic Lie groups \([4, 8]\) and quasi-Frobenius Lie algebras.

Definition 2.11. A symplectic Lie group is a pair \((G, \omega)\) where \( G \) is a Lie group and \( \omega \) is a left-invariant symplectic form on \( G \).

The next result shows that the Lie algebra of a symplectic Lie group is naturally a quasi-Frobenius Lie algebra.

Proposition 2.12. Let \((G, \omega)\) be a symplectic Lie group. Then \((g, \omega_e)\) is a quasi-Frobenius Lie algebra.

Proof. Let \( \mathcal{X}_l(G) \) denote the space of left-invariant vector fields on \( G \) and endow \( g := T_eG \) with the Lie algebra structure of \( \mathcal{X}_l(G) \). Also, let \( \tilde{x} \) denote the left-invariant vector field associated with \( x \in g \). We now show that \((g, \omega_e)\) is a quasi-Frobenius Lie algebra. Since \( \omega_g|_{T_gG} \) is nondegenerate for all \( g \in G \) (in particular for \( g = e \)), it only remains to show that \( \omega_e \) is a 2-cocycle of \( g \) with values in \( \mathbb{R} \) (where \( g \) acts trivially on \( \mathbb{R} \)).
First, note that for any \( x, y \in \mathfrak{g} \), \( \omega(\tilde{x}, \tilde{y}) \) is a constant function on \( G \). Indeed, for \( g \in G \)

\[
(\omega(\tilde{x}, \tilde{y}))(g) := \omega_g(\tilde{x}_g, \tilde{y}_g) = \omega_g((l_g)_* x, (l_g)_* y) = (l_g \omega)_e(x, y) = \omega_e(x, y)
\]

where the last equality follows from the fact that \( \omega \) is left-invariant. This fact along with the fact the \( \omega \) is closed implies that \( \omega_e \in Z^2(\mathfrak{g}; \mathbb{R}) \):

\[
0 = d\omega(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{x}(\omega(\tilde{y}, \tilde{z})) - \tilde{y}(\omega(\tilde{x}, \tilde{z})) + \tilde{z}(\omega(\tilde{x}, \tilde{y})) - \omega([\tilde{x}, \tilde{y}], \tilde{z}) - \omega([\tilde{x}, \tilde{z}], \tilde{y}) - \omega([\tilde{y}, \tilde{z}], \tilde{x}).
\]

Evaluating the last equality at \( e \in G \) and multiplying by \(-1\) gives the 2-cocycle condition on \( \omega_e \):

\[
\omega_e([x, y], z) + \omega_e([z, x], y) + \omega_e([y, z], x) = 0.
\]

Hence, \((\mathfrak{g}, \omega_e)\) is a quasi-Frobenius Lie algebra. \(\Box\)

**Proposition 2.13.** Let \( G \) be a Lie group whose Lie algebra \( \mathfrak{g} \) carries the structure of a quasi-Frobenius Lie algebra with 2-cocycle \( \beta \). Define \( \tilde{\beta} \in \Omega^2(G) \) by

\[
\tilde{\beta}_g := (l_g^{-1})^* \beta \in \wedge^2 T^*_g G, \quad \forall \ g \in G
\]

where \( l_g : G \to G \) is left translation by \( g \). Then \((G, \tilde{\beta})\) is a symplectic Lie group whose associated quasi-Frobenius Lie algebra is \((\mathfrak{g}, \tilde{\beta}_e) = (\mathfrak{g}, \beta)\).

**Proof.** It follows immediately from the definition that \( \tilde{\beta} \) is left-invariant, that is, \((l_g)^* \tilde{\beta} = \tilde{\beta}\) for all \( g \in G \). Moreover, since \( \beta \) is nondegenerate, \( \tilde{\beta} \) must be nondegenerate as well. To see that \( d\tilde{\beta} = 0 \), it suffices to show that \( d\tilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) = 0 \) for all left-invariant vector fields \( \tilde{x}, \tilde{y}, \) and \( \tilde{z} \). Since \( \tilde{\beta} \) is left-invariant, it follows that \( \tilde{\beta}(\tilde{x}, \tilde{y}) = \tilde{\beta}_e(x, y) = \beta(x, y) \) is a constant function on \( G \) for all left-invariant vector fields \( \tilde{x} \) and \( \tilde{y} \), where \( \tilde{x}_e = x \) and \( \tilde{y}_e = y \). In particular,

\[
\tilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) = \beta([x, y], z).
\]

The proof of Proposition 2.12 shows that if \( \tilde{\beta} \) is left-invariant, we have

\[
d\tilde{\beta}(\tilde{x}, \tilde{y}, \tilde{z}) = -\tilde{\beta}([\tilde{x}, \tilde{y}], \tilde{z}) - \tilde{\beta}([\tilde{z}, \tilde{x}], \tilde{y}) - \tilde{\beta}([\tilde{y}, \tilde{z}], \tilde{x}) = -\beta([x, y], z) - \beta([z, x], y) - \beta([y, z], x).
\]

Since \( \beta \in Z^2(\mathfrak{g}; \mathbb{R}) \), the last equality must be zero. Hence, \((G, \tilde{\beta})\) is a symplectic Lie group. \(\Box\)

**Definition 2.14.** Let \((G, \omega)\) and \((H, \sigma)\) be symplectic Lie groups. A homomorphism of symplectic Lie groups is a Lie group homomorphism \( \varphi : G \to H \) such that \( \varphi^* \sigma = \omega \).
Lemma 2.15. Let $(G, \omega)$ and $(H, \sigma)$ be symplectic Lie groups and let $\varphi: G \to H$ be a Lie group homomorphism. Then $\varphi^* \sigma = \omega$ iff $(\varphi^* \sigma)_e = \omega_e$.

Proof. ($\Rightarrow$) Suppose $(\varphi^* \sigma)_e = \omega_e$. By definition, $(\varphi^* \sigma)_g = \omega_g$ for all $g \in G$. In particular, the equality holds for $g = e$.

($\Leftarrow$) Now suppose $(\varphi^* \sigma)_e = \omega_e$. Let $g \in G$ and $x, y \in T_g G$. Then

$$(\varphi^* \sigma)_g(x, y) = \sigma_{\varphi(g)}(\varphi_{*g}(x), \varphi_{*g}(y))$$

$$= [(l_{\varphi(g^{-1})})^* \sigma_e](\varphi_{*g}(x), \varphi_{*g}(y))$$

$$= \sigma_e((l_{\varphi(g^{-1})} \circ \varphi)_{*g}(x), (l_{\varphi(g^{-1})} \circ \varphi)_{*g}(y))$$

$$= \sigma_e((\varphi \circ l_{g^{-1}})_{*g}(x), (\varphi \circ l_{g^{-1}})_{*g}(y))$$

$$= (\varphi^* \sigma)_e((l_{g^{-1}})_{*g}(x), (l_{g^{-1}})_{*g}(y))$$

$$= \omega_e((l_{g^{-1}})_{*g}(x), (l_{g^{-1}})_{*g}(y))$$

$$= [(l_{g^{-1}})^* \omega_e](x, y)$$

$$= \omega_g(x, y),$$

where the second and last equalities follow from the left-invariance of $\sigma$ and $\omega$ respectively and the fourth equality follows from the fact that $\varphi$ is a group homomorphism. This completes the proof. $\square$

Proposition 2.16. Let $\varphi: (G, \omega) \to (H, \sigma)$ be a homomorphism of symplectic Lie groups. Then $\varphi_{*, e}: (\mathfrak{g}, \omega_e) \to (\mathfrak{h}, \sigma_e)$ is a homomorphism of quasi-Frobenius Lie algebras.

Proof. This follows immediately from the properties of $\varphi$. $\square$

Proposition 2.17. Let $\psi: (\mathfrak{g}, \beta) \to (\mathfrak{h}, \sigma)$ be a homomorphism of quasi-Frobenius Lie algebras. Let $G$ be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and let $H$ be any Lie group whose Lie algebra is $\mathfrak{h}$. Let $(G, \beta)$ and $(H, \sigma)$ be the symplectic Lie groups associated to $(\mathfrak{g}, \beta)$ and $(\mathfrak{h}, \sigma)$ respectively (see Proposition 2.15). Then there exists a unique symplectic Lie group homomorphism $\hat{\psi}: (G, \beta) \to (H, \sigma)$ such that $\hat{\psi}_{*, e} = \psi$.

Proof. Since $G$ is simply connected, there exists a unique Lie group homomorphism $\hat{\psi}: G \to H$ such that $\hat{\psi}_{*, e} = \psi$. It only remains to show that $\hat{\psi}^* \sigma = \beta$. By Lemma 2.15, it suffices to show that $(\hat{\psi}^* \sigma)_e = \beta_e = \beta$. To do this, let $x, y \in \mathfrak{g}$. Then

$$(\hat{\psi}^* \sigma)_e(x, y) = \sigma_{\hat{\psi}(e)}(\hat{\psi}_{*, e}(x), \hat{\psi}_{*, e}(y))$$

$$= \sigma_e(\psi(x), \psi(y))$$

$$= (\psi^* \sigma)(x, y)$$

$$= \beta(x, y).$$
This completes the proof. □

**Theorem 2.18.** Let $\text{SCSLG}$ be the category of simply connected symplectic Lie groups and let $\text{qFLA}$ be the category of finite dimensional quasi-Frobenius Lie algebras. Let $F$ be the functor from $\text{SCSLG}$ to $\text{qFLA}$ which sends $(G, \omega)$ to $(\mathfrak{g}, \omega_e)$ and $\varphi: (G, \omega) \to (H, \sigma)$ to $\varphi^*: (\mathfrak{g}, \omega_e) \to (\mathfrak{h}, \sigma_e)$. Then $F$ is an equivalence of categories.

**Proof.** Theorem 2.18 follows from the well known correspondence between simply connected Lie groups and finite dimensional Lie algebras combined with Proposition 2.12, Proposition 2.13, Proposition 2.16, and Proposition 2.17. □

As an example, we now recall the symplectic Lie group structure on the affine Lie group $A(n, \mathbb{R})$ (c.f., [2, 21, 22]).

**Example 2.19.** Recall that $A(n, \mathbb{R})$ is the Lie group consisting of $(n+1) \times (n+1)$ matrices of the form

$$A(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{R}), \ v \in \mathbb{R}^n \right\}.$$ 

The associated Lie algebra is then

$$\mathfrak{a}(n, \mathbb{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(n, \mathbb{R}), \ v \in \mathbb{R}^n \right\}.$$ 

From the definition of $A(n, \mathbb{R})$, we see that $A(n, \mathbb{R})$ is even dimensional with dim $A(n, \mathbb{R}) = \dim \mathfrak{a}(n, \mathbb{R}) = n^2 + n = n(n+1)$. Let $E_{ij}$ denote the $(n+1) \times (n+1)$ matrix with 1 in the $(i, j)$-component and all other components zero. Then $\{E_{ij}\}_{1 \leq i \leq n, \ 1 \leq j \leq n+1}$ is a basis on $\mathfrak{a}(n, \mathbb{R})$. Let $\{E^*_{ij}\}_{1 \leq i \leq n, \ 1 \leq j \leq n+1}$ denote the corresponding dual basis. Define

$$\alpha = E^*_{12} + E^*_{23} + \cdots + E^*_{n,n+1}$$

and $\beta(X, Y) := -\delta \alpha(X, Y) = \alpha([X, Y])$ for all $X, Y \in \mathfrak{a}(n, \mathbb{R})$. Since

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj},$$

we see that

$$\beta(E_{ij}, E_{kl}) = \delta_{jk}\delta_{l,i+1} - \delta_{li}\delta_{j,k+1}.$$ 

Careful consideration of (2.3) shows that $\beta := -\delta \alpha \in Z^2(\mathfrak{a}(n, \mathbb{R}); \mathbb{R})$ is nondegenerate. Hence, $(\mathfrak{a}(n, \mathbb{R}), \alpha)$ is a Frobenius Lie algebra. (In particular, $(\mathfrak{a}(n, \mathbb{R}), \beta)$ is a quasi-Frobenius Lie algebra.) Let $\tilde{\beta} \in \Omega^2(A(n, \mathbb{R}))$ be the left-invariant 2-form on $A(n, \mathbb{R})$ associated to $\beta$. Then $(A(n, \mathbb{R}), \tilde{\beta})$ is a symplectic Lie group. Furthermore, since $\beta := -\delta \alpha$, it follows that $\tilde{\beta}$ is exact. Specifically,

$$\tilde{\beta} = -d\tilde{\alpha}$$

where $\tilde{\alpha} \in \Omega^1(A(n, \mathbb{R}))$ is the left-invariant 1-form on $A(n, \mathbb{R})$ associated to $\alpha$. 


2.3. Lie bialgebras & the Drinfeld Double.

**Definition 2.20.** A Lie bialgebra over a field \( k \) is a pair \((g, \gamma)\) where \( g \) is a Lie algebra over \( k \) and \( \gamma : g \rightarrow g \wedge g \subset g \otimes g \) is a skew-symmetric linear map such that

1. \( \gamma^* : g^* \otimes g^* \rightarrow g^* \) is a Lie bracket on \( g^* \), where the dual map \( \gamma^* \) is restricted to \( g^* \otimes g^* \subset (g \otimes g)^* \);
2. \( \gamma \) is a 1-cocycle on \( g \) with values in \( g \otimes g \), where \( g \) acts on \( g \otimes g \) via the adjoint action.

\( \gamma \) is called the cobracket or co-commutator.

Condition 2 in Definition 2.20 is equivalent to the condition

\[
\gamma([x,y]) = ad_x^2 \gamma(y) - ad_y^2 \gamma(x), \quad \forall \ x, y \in g
\]

where the linear map \( ad_x^2 : g \otimes g \rightarrow g \otimes g \) is the adjoint action of \( x \in g \) on \( g \otimes g \). Explicitly, \( ad_x^2 \) is defined via

\[
ad_x^2(y \otimes z) = ad_x(y) \otimes z + y \otimes ad_x(z) = [x,y] \otimes z + y \otimes [x,z]
\]

for \( y, z \in g \).

**Definition 2.21.** Let \((g, \gamma_g)\) and \((h, \gamma_h)\) be Lie bialgebras. A Lie bialgebra homomorphism from \((g, \gamma_g)\) to \((h, \gamma_h)\) is a Lie algebra map \( \varphi : g \rightarrow h \) such that

\[
(\varphi \otimes \varphi) \circ \gamma_g = \gamma_h \circ \varphi.
\]

**Example 2.22.** Any Lie algebra \( g \) can be turned into a Lie bialgebra by taking the cobracket \( \gamma \equiv 0 \). \((g, 0)\) is the trivial Lie bialgebra structure on \( g \).

The next result shows that the notion of a Lie bialgebra is self-dual for the finite dimensional case.

**Proposition 2.23.** Let \((g, \gamma_g)\) be a finite dimensional Lie bialgebra and let \( \gamma_g^* : g^* \rightarrow g^* \otimes g^* \) be the dual of the Lie bracket on \( g \). Then \((g^*, \gamma_g^*)\) is a Lie bialgebra, where the Lie bracket on \( g^* \) is given by the dual of \( \gamma_g \).

For a Lie algebra \( g \), the simplest way to obtain an element of \( Z^1_{ad}(g; g \otimes g) \) is to turn to the 0-cochains and take their coboundaries. This raises the following natural question: given \( r \in g \otimes g \), when does \( \delta r \in Z^1_{ad}(g; g \otimes g) \) define a Lie bialgebra structure on \( g \)? To answer this question, let

\[
r = \sum_i a_i \otimes b_i,
\]

and define

\[
[r, r] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}],
\]

(2.4)
where
\[(r_{12}, r_{13}) := \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j,\]
\[(r_{12}, r_{23}) := \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j,\]
\[(r_{13}, r_{23}) := \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].\]

**Definition 2.24.** A **coboundary Lie bialgebra** is a Lie bialgebra \((g, \gamma)\) such that \(\gamma = \delta r\) for some \(r \in g \otimes g\). The element \(r\) is called the \(r\)-matrix.

The next result provides a necessary and sufficient condition for an element \(r \in g \otimes g\) to define a Lie bialgebra structure on \(g\).

**Proposition 2.25.** Let \(g\) be a Lie algebra. Then \((g, \delta r)\) is a Lie bialgebra iff
(i) \(r + \sigma(r)\) is invariant under the adjoint action of \(g\) on \(g \otimes g\), where \(\sigma : g \otimes g \to g \otimes g\) is the unique linear map defined by \(x \otimes y \mapsto y \otimes x\) for \(x, y \in g\);
(ii) \([r, r]\) is invariant under the adjoint action of \(g\) on \(g \otimes g \otimes g\).

**Proof.** See pp. 51–54 of [7].

The simplest way to ensure that condition (ii) of Proposition 2.25 is satisfied is to demand that
\[\Box\]
Equation 2.8 is called the **classical Yang-Baxter equation** (CYBE). The CYBE motivates the following definition:

**Definition 2.26.** A coboundary Lie bialgebra \((g, \delta r)\) is **quasitriangular** if \(r\) is a solution of the CYBE. Furthermore, if \(r\) is skew-symmetric, that is, \(r \in g \wedge g \subset g \otimes g\), then \((g, \delta r)\) is said to be **triangular**.

**Example 2.27.** Let \(g\) be the two dimensional Lie algebra with basis \(x, y\) and commutator relation \([x, y] = x\). Define \(r = y \wedge x\). Then \((g, \delta r)\) is a triangular Lie bialgebra, where \(\gamma := \delta r\) is given explicitly by
\[\gamma(x) = 0, \quad \gamma(y) = x \wedge y.\]

Before turning to the Drinfeld double, we recall the following notion:

**Definition 2.28.** Let \(g\) be a Lie algebra and let \(\langle \cdot, \cdot \rangle\) be a bilinear form on \(g\). \(g\) is **ad-invariant** with respect to \(\langle \cdot, \cdot \rangle\) if
\[\langle [x, y], z \rangle = \langle x, [y, z] \rangle, \quad \forall \ x, y, z \in g.\]

Now let \((g, \gamma_g)\) be a finite dimensional Lie bialgebra and let \((g^*, \gamma_{g^*})\) be the associated dual Lie bialgebra. Consider the direct sum
\[g \oplus g^*\]
and equip it with the symmetric, nondegenerate bilinear form \(\langle \cdot, \cdot \rangle\) defined by
\[\langle x + \xi, y + \eta \rangle = \xi(y) + \eta(x),\]
where we write \( x + \xi \) and \( y + \eta \) for \((x, \xi), (y, \eta) \in g \oplus g^*\). The Drinfeld double of \((g, \gamma_g)\), denoted by \(D(g)\), is the unique quasitriangular Lie bialgebra which satisfies the following conditions:

1. As a vector space,
   \[ D(g) = g \oplus g^* \]

2. As a Lie algebra, \(D(g)\) is ad-invariant with respect to the inner product \(\langle \cdot, \cdot \rangle\) and contains \(g\) and \(g^*\) as Lie subalgebras.

3. The cobracket on \(D(g)\) is defined by \(\gamma_D := \gamma_g - \gamma_g^*\). Let \([\cdot, \cdot]_D\), \([\cdot, \cdot]_g\), and \([\cdot, \cdot]_{g^*}\) denote the Lie brackets on \(D(g)\), \(g\), and \(g^*\) respectively. Condition (2) implies that
   \[ [x, y]_D = [x, y]_g, \quad [\xi, \eta]_D = [\xi, \eta]_{g^*}, \quad [x, \xi]_D = ad^*_\xi x - ad^*_\eta x \]
for all \(x, y \in g\) and \(\xi, \eta \in g^*\), where \(ad^*_x\) denotes the coadjoint action of \(g\) on \(g^*\) and \(g^*\) on \(g\). Explicitly, \(ad^*_\xi : g^* \to g^*\) and \(ad^*_\eta : g \to g\) are defined by \(ad^*_\xi := -ad^*_\eta\) and \(ad^*_\eta := -ad^*_\xi\) where \(ad^*_\xi\) and \(ad^*_\eta\) are the ordinary duals of \(ad_x : g \to g\) and \(ad_\xi : g^* \to g^*\). In dealing with the Drinfeld double, we will drop the “\(D\)”, “\(g\)”, and “\(g^*\)” that appear as subscripts in the Lie brackets of \(D(g)\), \(g\), and \(g^*\) respectively. Condition (2) implies that the triple \((D(g), g, g^*)\) is a Manin triple with respect to the inner product \(\langle \cdot, \cdot \rangle\). In fact, there is a one to one correspondence between finite dimensional Lie bialgebras and Manin triples (see [7]).

Lastly, condition (3) implies that \(D(g)\) is quasitriangular with \(r\)-matrix
\[ r = \sum_i e_i \otimes e_i^* \]
where \(e_1, \ldots, e_n\) is any basis on \(g\) and \(e_1^*, \ldots, e_n^*\) is the corresponding dual basis.

**Example 2.29.** Let \((g, \gamma)\) be the 2-dimensional Lie bialgebra with basis \(x, y\) satisfying \([x, y] = x\) and cobracket \(\gamma(x) = 0\) and \(\gamma(y) = x \wedge y\). Let \(x^*, y^*\) denote the corresponding dual basis. The commutator relations on \(D(g)\) are
\[
[x, y] = x, \quad [x^*, y^*] = y^*, \quad [x, x^*] = -y^*, \quad [x, y^*] = 0
\]
\[
[y, x^*] = x^* + y, \quad [y, y^*] = -x.
\]
The \(r\)-matrix is \(r = x \otimes x^* + y \otimes y^*\).

### 3. \(g\)-quasi-Frobenius Lie Algebras

We begin with the formal definition:

**Definition 3.1.** A \(g\)-quasi-Frobenius Lie algebra is a triple \((q, \beta, \rho)\) such that \((q, \beta)\) is a quasi-Frobenius Lie algebra and \(\rho : g \to gl(q)\), \(x \mapsto \rho_x\) is a left \(g\)-module structure on \(q\) such that

1. \(\rho_x\) is a derivation on \(q\) for all \(x \in g\),
2. \(\beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0\) for all \(x \in g\), \(u, v \in q\) (\(g\)-invariance).

In this section, we prove a result for the general construction of \(g\)-quasi-Frobenius Lie algebras. Before doing so, we make the following observation:
**Proposition 3.2.** Let \((q, \beta)\) be a quasi-Frobenius Lie algebra and let \(\text{Aut}(q, \beta)\) be the automorphism group of \((q, \beta)\). Then \(\text{Aut}(q, \beta)\) is an embedded Lie subgroup of \(GL(q)\).

**Proof.** As a set, \(\text{Aut}(q, \beta) = \text{Aut}(q) \cap \text{Sp}(q, \beta)\) where \(\text{Aut}(q)\) is the group of automorphisms of the Lie algebra \(q\) and \(\text{Sp}(q, \beta)\) is the group of linear symplectomorphisms of \((q, \beta)\), where the latter is regarded as a symplectic vector space. Since \(\text{Aut}(q)\) and \(\text{Sp}(q, \beta)\) are both closed subgroups of \(GL(q)\), each being the zero set of a collection of polynomials, \(\text{Aut}(q, \beta)\) is also a closed subgroup of \(GL(q)\). By the closed subgroup theorem [27], \(\text{Aut}(q, \beta)\) is an embedded Lie subgroup of \(GL(q)\). □

**Proposition 3.3.** Let \((q, \beta)\) be a quasi-Frobenius Lie algebra and let \(\rho: G \to \text{Aut}(q, \beta) \subset GL(q), \ g \mapsto \rho_g\) be a Lie group homomorphism. Define

\[
\rho' := \rho_\ast e: g \to \mathfrak{gl}(q), \ x \mapsto \rho'_x.
\]

Then \((q, \beta, \rho')\) is a \(g\)-quasi-Frobenius Lie algebra. In particular, if \(G\) is any Lie subgroup of \(\text{Aut}(q, \beta)\), then \((q, \beta)\) admits the structure of a \(g\)-quasi-Frobenius Lie algebra.

**Proof.** Since \(\rho\) is a Lie group homomorphism, it immediately follows that \(\rho' : g \to \mathfrak{gl}(q)\) is a representation of \(g\) on \(q\). We now show that

\[
(3.1) \quad \rho_x([u, v]) = [\rho_x(u), v] + [u, \rho_x(v)]
\]

and

\[
(3.2) \quad \beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0
\]

for all \(x \in g\) and \(u, v \in q\). To do this, fix a basis \(e_1, e_2, \ldots, e_n\) on \(q\). Since \(\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v) \in q\), we have

\[
(3.3) \quad \rho_{\exp(tx)}(u) = \sum_i a_i(t)e_i, \quad \rho_{\exp(tx)}(v) = \sum_i b_i(t)e_i
\]

for some smooth functions \(a_i(t), b_i(t), i = 1, \ldots, n\). Hence,

\[
(3.4) \quad \rho'_x(u) = \sum_i \dot{a}_i(0)e_i, \quad \rho'_x(v) = \sum_i \dot{b}_i(0)e_i.
\]

Since \(\rho_g \in \text{Aut}(q, \beta)\) for all \(g \in G\), we have

\[
(3.5) \quad \rho_{\exp(tx)}([u, v]) = [\rho_{\exp(tx)}(u), \rho_{\exp(tx)}(v)].
\]
Substituting (3.3) into the right side of (3.5) and applying \( \frac{d}{dt} \big|_{t=0} \) to both sides of (3.5) gives
\[
\rho'_x([u,v]) = \frac{d}{dt} \bigg|_{t=0} \left[ \rho\exp(tx)(u), \rho\exp(tx)(v) \right]
\]
\[
= \frac{d}{dt} \bigg|_{t=0} \sum_{i,j} a_i(t) b_j(t)[e_i, e_j]
\]
\[
= \sum_{i,j} (\dot{a}_i(0) b_j(0)[e_i, e_j] + a_i(0) \dot{b}_j(0)[e_i, e_j])
\]
(3.6)
which proves (3.1).

For equation (3.2), note that
(3.7)
\[
\beta(\rho\exp(tx)(u), \rho\exp(tx)(v)) = \beta(u,v)
\]
since \( \rho_g \in \text{Aut}(q, \beta) \) for all \( g \in G \). Substituting (3.3) into the left side of (3.7) and applying \( \frac{d}{dt} \big|_{t=0} \) to both sides of (3.7) gives
\[
\beta(\rho'_x(u), v) + \beta(u, \rho'_x(v)) = 0.
\]
This completes the proof.

A trivial example of a \( g \)-quasi-Frobenius Lie algebra is obtained by equipping any quasi-Frobenius Lie algebra with the trivial \( g \)-action. We now consider a more interesting example which is an application of Proposition 3.3.

**Example 3.4.** Let \( q \) be the 4-dimensional Lie algebra \( \{e_1, e_2, e_3, e_4\} \) whose non-zero commutator relations are given by [9]:
\[
[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = 2e_4, \quad [e_2, e_3] = e_4.
\]

Let \( \alpha: q \rightarrow \mathbb{R} \) be the linear map defined by \( \alpha(e_i) = 0 \) for \( i = 1, 2, 3 \) and \( \alpha(e_4) = 1 \). Define \( \beta(u, v) := \alpha([u, v]) \) for all \( u, v \in q \). Then the matrix representation of \( \beta \) with respect to the basis \( \{e_1, e_2, e_3, e_4\} \) is
\[
(\beta_{ij}) = \begin{pmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-2 & 0 & 0 & 0
\end{pmatrix}.
\]

Hence, \( \beta \) is nondegenerate which shows that \( (q, \alpha) \) is a Frobenius Lie algebra. Let \( G \) be the set of linear isomorphisms on \( q \) whose matrix representations with respect to \( \{e_1, e_2, e_3, e_4\} \) is given by
(3.8)
\[
\left\{ \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b & c & 0 \\
0 & 0 & 1/b & 0 \\
a & 0 & 0 & 1
\end{pmatrix} \middle| \ a, c \in \mathbb{R}, \ b > 0 \right\}.
\]

A direct calculation shows that \( G \) is a 3-dimensional non-abelian, connected Lie subgroup of \( \text{Aut}(q, \beta) \). Let \( \rho: G \rightarrow \text{Aut}(q, \beta) \subset \text{GL}(q) \) be the inclusion map (which
is clearly a Lie group homomorphism). Proposition 3.3 implies that \((q, \beta, \rho')\) is a \(g\)-quasi-Frobenius Lie algebra, where \(\rho' := \rho_{*, e}\). As a Lie algebra, \(g\) has basis

\[
\begin{align*}
\mathbf{x}_1 := & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{x}_2 := & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathbf{x}_3 := & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

where we have identified \(G\) with its matrix representations in (3.8). The non-zero commutator relations are

\[[\mathbf{x}_2, \mathbf{x}_3] = 2\mathbf{x}_3.\]

Let \(a = a_1 \mathbf{x}_1 + a_2 \mathbf{x}_2 + a_3 \mathbf{x}_3 \in g\). Since \(\rho: G \to \text{Aut}(q, \beta) \subset GL(q)\) is just the inclusion map, it follows that the matrix representation of \(\rho'_a: q \to q\) with respect to the basis \(\{e_1, e_2, e_3, e_4\}\) is simply

\[
(3.10) \quad \rho'_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 \\ 0 & 0 & -a_2 & 0 \\ a_1 & 0 & 0 & 0 \end{pmatrix}.
\]

Since \((q, \beta, \rho')\) is a \(g\)-quasi-Frobenius Lie algebra by Proposition 3.3, \(\rho'_a\) acts on \(q\) via derivations and satisfies

\[\beta(\rho'_a(u), v) + \beta(u, \rho'_a(v)) = 0\]

for all \(u, v \in q\).

For later use, we conclude this section with the following natural definition:

**Definition 3.5.** Let \((q, \beta, \phi)\) and \((r, \sigma, \mu)\) be \(g\)-quasi-Frobenius Lie algebras. A homomorphism from \((q, \beta, \phi)\) to \((r, \sigma, \mu)\) is a homomorphism \(\psi: (q, \beta) \to (r, \sigma)\) of quasi-Frobenius Lie algebras which is also \(g\)-equivariant, that is,

\[\psi \circ \phi_x = \mu_x \circ \psi\]

for all \(x \in g\).

**3.1. Categorical Formulation.** In this section, we apply the idea of categorification to quasi-Frobenius Lie algebras. The upshot of this is the notion of a quasi-Frobenius Lie object, which can be viewed as the analogue of a Frobenius object in the current setting. The starting point for this particular step is the categorification of Lie algebra due to Goyvaerts and Vercruysse [12]:

**Definition 3.6.** A Lie object in an additive symmetric monoidal category \((C, \otimes, I, \Phi, l, r, c)\) is a pair \((L, b)\) where \(L\) is an object of \(C\) and \(b: L \otimes L \to L\) is a morphism such that

\[
\begin{align*}
(i) \quad & b + b \circ c = 0_{L \otimes L, L}, \\
(ii) \quad & b \circ (\text{id}_L \otimes b) \circ (\text{id}_L \otimes (L \otimes L)) + c_L \otimes L, L \circ \Phi^{-1}_{L, L, L} + \Phi_L, L, L \circ c_L, L \otimes L) = 0_{L \otimes (L \otimes L), L}.
\end{align*}
\]
Remark 3.7. With regard to the notation in Definition 3.6, \( \otimes \) is the monoidal product; \( I \) is the unit object; \( \Phi \) is the associator; \( l \) and \( r \) are the left and right unit maps respectively; and \( c \) is the braiding.

Example 3.8. Let \( \text{Vect}_k \) be the symmetric monoidal additive category of finite dimensional vector spaces over \( k \). It follows readily from Definition 3.6 that a Lie object \((L, b)\) in \( \text{Vect}_k \) is precisely a finite dimensional Lie algebra \( L \) over \( k \) with Lie bracket \( [x, y] := b(x, y) \).

Definition 3.9. A quasi-Frobenius Lie object in an additive symmetric monoidal category \((C, \otimes, I, \Phi, l, r, c)\) is a triple \((L, b, \beta)\) such that

1. \((L, b)\) is a Lie object.
2. \( L \) has a left dual object \( L^* \) (where \( \varepsilon: L^* \otimes L \to I \) and \( \eta: I \to L \otimes L^* \) denote the evaluation and coevaluation morphisms respectively).
3. \( \beta: L \sim \to L^* \) is an isomorphism such that the induced morphism
   \[
   \beta := \varepsilon \circ (\beta \otimes \text{id}_L): L \otimes L \to I,
   \]
   satisfies
   \[
   \beta + \beta \circ c_{L,L} = 0_{L \otimes L,I}
   \]
   and
   \[
   \beta \circ (b \otimes \text{id}_L) \circ [\text{id}_{(L \otimes L) \otimes L} + \Phi_{L,L,L}^{-1} \circ c_{L \otimes L,L} + c_{L,L,L} \circ \Phi_{L,L,L}] = 0_{(L \otimes L) \otimes L,I}.
   \]

If there exists a morphism \( \alpha: L \to I \) such that \( \beta = \alpha \circ b \), then \((L, b, \beta)\) is called a Frobenius Lie object.

Example 3.10. Let \((L, b, \beta)\) be a quasi-Frobenius Lie object in \( \text{Vect}_k \). Then its easy to see that \( L \) is a quasi-Frobenius Lie algebra over \( k \) with Lie bracket \( [x, y] := b(x, y) \) and \( \beta: L \otimes L \to k \) (as defined in (3) of Definition 3.9) is the nondegenerate 2-cocycle in the Lie algebra cohomology of \( L \). Likewise, a Frobenius Lie object in \( \text{Vect}_k \) is just a Frobenius Lie algebra.

Proposition 3.11. The category \( \text{Rep}(\mathfrak{g}) \) of finite dimensional left \( \mathfrak{g} \)-modules over \( k \) is an additive symmetric monoidal category where every object has a left dual and

1. the monoidal product is the usual tensor product of left \( \mathfrak{g} \)-modules and \( \mathfrak{g} \)-linear maps;
2. the unit object is \( k \) with the trivial \( \mathfrak{g} \)-action;
3. the associator \( \Phi \) is the trivial one;
4. for any object \((V, \rho)\) in \( \text{Rep}(\mathfrak{g}) \), the left and right morphisms \( l_V: k \otimes V \sim \to V \) and \( r_V: V \otimes k \sim \to V \) are the trivial ones;
5. for objects \((V, \rho), (W, \phi)\) in \( \text{Rep}(\mathfrak{g}) \), the braiding \( c_{V,W}: V \otimes W \sim \to W \otimes V \) is simply the linear map that sends \( v \otimes w \in V \otimes W \) to \( w \otimes v \in W \otimes V \);
6. the left dual of an object \((V, \rho)\) in \( \text{Rep}(\mathfrak{g}) \) is the dual representation \((V^*, \rho^*)\) (i.e., \( \rho^*_x := -\rho^{t}_x: V^* \to V^* \) for \( x \in \mathfrak{g} \), where \( \rho^{t}_x \) is the dual or transpose of \( \rho_x: V \to V \)).
(vii) the evaluation morphism is \( \varepsilon: V^* \otimes V \to k \), \( \varepsilon(\xi, v) := \xi(v) \) and the coevaluation morphism is \( \eta: k \to V \otimes V^* \), \( 1 \mapsto \sum_i e_i \otimes \delta^i \) where \( e_i \) is any basis of \( V \) and \( \delta^i \) is the corresponding dual basis.

**Proof.** It is an easy exercise to verify that \((\text{Rep}(\mathfrak{g}), \otimes, k, \Phi, l, r, c)\) satisfies all the axioms of an additive symmetric monoidal category. \( \square \)

The next result establishes the categorical formulation of \( \mathfrak{g} \)-quasi-Frobenius Lie algebras.

**Proposition 3.12.** A quasi-Frobenius Lie object in \( \text{Rep}(\mathfrak{g}) \) is a \( \mathfrak{g} \)-quasi-Frobenius Lie algebra.

**Proof.** By definition, a quasi-Frobenius Lie object in \( \text{Rep}(\mathfrak{g}) \) consists of a representation \( (q, \rho) \) of \( \mathfrak{g} \) together with \( \mathfrak{g} \)-linear maps \( b: q \otimes q \to q \), \( \beta: q \to q^* \), which satisfy conditions (1) and (3) of Definition 3.9.

We begin by verifying that \( (q, \beta) \) is a quasi-Frobenius Lie algebra. To start, note that condition (1) of Definition 3.9 implies that \( q \) is a Lie algebra with Lie bracket \([u, v] := b(u, v)\). From Definition 3.9, the morphism \( \beta: q \otimes q \to k \) is given explicitly as

\[
\beta(u, v) = \varepsilon(\beta(u), v) = \beta(u)(v) .
\]

Condition (3) of Definition 3.9 implies that \( \beta \) is a 2-cocycle of \( q \) with values in \( k \) (where \( q \) acts trivially on \( k \)). Furthermore, since \( \beta: q \to q^* \) is an isomorphism, it follows that \( \beta \) is nondegenerate. Hence, \( (q, \beta) \) is a quasi-Frobenius Lie algebra.

Since \( \beta \) is \( \mathfrak{g} \)-linear (being a morphism of \( \text{Rep}(\mathfrak{g}) \)), we have

\[
\beta(\rho_x(u))(v) = \rho_x^*(\beta(u))(v) = -\beta(u)(\rho_x(v)) , \quad \forall u, v \in q
\]

where we recall that \( \rho_x^* := -\rho_x' \). Expressing the left and right most sides of (3.11) in terms of \( \beta \) gives

\[
\beta(\rho_x(u), v) = -\beta(u, \rho_x(v)) ,
\]

which proves the \( \mathfrak{g} \)-invariance of \( \beta \), that is, \( \beta(\rho_x(u), v) + \beta(u, \rho_x(v)) = 0 \).

Since \( b \) is also \( \mathfrak{g} \)-linear, we also have

\[
\rho_x([u, v]) = \rho_x(b(u \otimes v)) \\
= b(\rho_x(u \otimes v)) \\
= b(\rho_x(u) \otimes v) + b(u \otimes \rho_x(v)) \\
= [\rho_x(u), v] + [u, \rho_x(v)] ,
\]

where \( \rho_x \) in the second equality denotes the induced left \( \mathfrak{g} \)-module structure on \( q \otimes q \). Hence, \( (q, \beta, \rho) \) is a \( \mathfrak{g} \)-quasi-Frobenius Lie algebra. \( \square \)
4. The Geometry of $\mathfrak{g}$-quasi-Frobenius Lie algebras

4.1. $G$-Symplectic Lie groups.

**Definition 4.1.** Let $G$ be a Lie group. A $G$-symplectic Lie group is a triple $(Q, \omega, \varphi)$ where $(Q, \omega)$ is a symplectic Lie group and

$$\varphi: G \times Q \to Q, \ (g, q) \mapsto \varphi_g(q) := \varphi(g, q)$$

is a smooth left action on $Q$ such that $\varphi_g: (Q, \omega) \to (Q, \omega)$ is an isomorphism of symplectic Lie groups.

**Notation 4.2.** When dealing with multiple Lie groups, we will denote the identity element of each group simply as $e$ as opposed to $e_G$ for $G$, $e_Q$ for $Q$, and so on when there is no risk of confusion.

**Proposition 4.3.** Let $(Q, \omega, \varphi)$ be a $G$-symplectic Lie group with action

$$\varphi: G \times Q \to Q, \ (g, q) \mapsto \varphi_g(q) := \varphi(g, q).$$

Define

$$\varphi': G \to GL(q), \ g \mapsto \varphi'_g := (\varphi_g)_{*e}: q \to q$$

$$\varphi'': g \mapsto gl(q), \ x \mapsto \varphi''_x := (\varphi')_{*e}(x): q \to q.$$

Then

(i) $\varphi'$ is a representation of $G$ on $q$ such that $\varphi'_g \in \text{Aut}(q, \omega_e)$ for all $g \in G$.

(ii) $(q, \omega_e, \varphi'')$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra.

**Proof.** Since $\varphi$ is a left action of $G$ on $Q$ and $\varphi_g(e) = e$ for all $g \in G$, we have

$$\varphi'_g \circ \varphi'_h = (\varphi_g)_{*e} \circ (\varphi_h)_{*e}$$

$$= (\varphi_g \circ \varphi_h)_{*e}$$

$$= (\varphi_{gh})_{*e} = \varphi'_{gh}.$$ 

Hence, $\varphi'$ is a representation of $G$ on $q$. Furthermore, since $\varphi_g: Q \to Q$ is both a Lie group isomorphism and a symplectomorphism, it follows that $\varphi'_g: q \to q$ is a Lie algebra isomorphism and

$$\omega_e(u, v) = \left((\varphi_g)^*\omega\right)_e(u, v) = \omega_e((\varphi_g)_{*e}(u), (\varphi_g)_{*e}(v)) = \omega_e(\varphi'_g(u), \varphi'_g(v)).$$

which shows that $\varphi'_g \in \text{Aut}(q, \omega_e)$ for all $g \in G$. This proves (i).

Statement (ii) follows from an application of Proposition 3.3 to the quasi-Frobenius Lie algebra $(q, \omega_e)$ with Lie group homomorphism $\varphi': G \to \text{Aut}(q, \omega_e) \subset GL(q)$. This completes the proof. \[ \square \]

**Remark 4.4.** We will refer to $(q, \omega_e, \varphi'')$ in Proposition 4.3 as the $\mathfrak{g}$-quasi-Frobenius Lie algebra associated to the $G$-symplectic Lie group $(Q, \omega, \varphi)$.

The next result provides a means of constructing $G$-symplectic Lie groups.
Proposition 4.5. Let \((Q, \omega)\) be a simply connected symplectic Lie group, let \(G\) be a Lie group, and let \(\rho: G \to \text{Aut}(q, \omega_e)\), \(g \mapsto \rho_g\) be a Lie group homomorphism. Then there exists a unique smooth left-\(G\) action

\[
\hat{\rho}: G \times Q \to Q, \quad (g, q) \mapsto \hat{\rho}_g(q),
\]

such that \((Q, \omega, \hat{\rho})\) is a \(G\)-symplectic Lie group and \((\hat{\rho}_g)^* e = \rho_g\). In particular, if \(G\) is any Lie subgroup of \(\text{Aut}(q, \omega_e)\) and \(G \neq \{e\}\), then \((Q, \omega)\) admits the structure of a \(G\)-symplectic Lie group with a nontrivial \(G\)-action.

Proof. Let \(\rho: G \to \text{Aut}(q, \omega_e)\), \(g \mapsto \rho_g\) be a Lie group homomorphism. Since \(Q\) is simply connected and \(\rho_g \in \text{Aut}(q, \omega_e)\) for all \(g \in G\), it follows from Proposition 2.17 that there exists a unique homomorphism of symplectic Lie groups

\[
\hat{\rho}_g: (Q, \omega) \to (Q, \omega)
\]

such that \((\hat{\rho}_g)^* e = \rho_g\) for all \(g \in G\). Furthermore, for \(g, h \in G\), we have

\[
(\hat{\rho}_g \circ \hat{\rho}_h)^* e = (\hat{\rho}_g)^* e \circ (\hat{\rho}_h)^* e = \rho_g \circ \rho_h = \rho_{gh} = (\hat{\rho}_{gh})^* e.
\]

(4.1)

Since \(\hat{\rho}_g \circ \hat{\rho}_h\) and \(\hat{\rho}_{gh}\) are Lie group homomorphisms and \(Q\) is connected, equation (4.1) implies that

\[
\hat{\rho}_g \circ \hat{\rho}_h = \hat{\rho}_{gh}.
\]

(4.2)

Hence,

\[
\hat{\rho}: G \times Q \to Q, \quad (g, q) \mapsto \hat{\rho}_g(q)
\]

is a left (not necessarily smooth) \(G\)-action. We now show that \(\hat{\rho}\) is smooth. To do this, set \(\tilde{\rho}(g, q) = \hat{\rho}_g(q)\) for \(g \in G\), \(q \in Q\) and let \(U\) be an open neighborhood of \(0 \in q\) such that

\[
\exp|_U: U \cong \exp(U)
\]

is a diffeomorphism. The naturality of the exponential map implies that

\[
\tilde{\rho}(g, q) = \exp \circ \rho_g \circ (\exp|_U)^{-1}(q), \quad \forall (g, q) \in G \times \exp(U).
\]

(4.3)

Since the right side of (4.3) is smooth on \(G \times \exp(U)\), it follows that \(\tilde{\rho}|_{G \times \exp(U)}\) is also smooth. Now fix an arbitrary element \(q_0\) of \(Q\) and define

\[
f: G \to Q, \quad g \mapsto \tilde{\rho}(g, q_0).
\]

We now show that \(f\) is smooth. Since \(Q\) is connected, \(\exp(U)\) generates \(Q\). Hence, there exists \(q_{0,1}, \ldots, q_{0,k} \in \exp(U)\) such that

\[
q_0 = q_{0,1} q_{0,2} \cdots q_{0,k}.
\]

Since \(\tilde{\rho}_g: Q \to Q\) is a Lie group homomorphism for all \(g \in G\), we have

\[
f(g) := \tilde{\rho}(g, q_0) = \tilde{\rho}(g, q_{0,1}) \tilde{\rho}(g, q_{0,2}) \cdots \tilde{\rho}(g, q_{0,k}) \in Q.
\]

(4.4)
Since \((g, q_{0,i}) \in G \times \exp(U)\) for \(i = 1, \ldots, k\), it follows that the right side of (4.4) depends smoothly on \(g\). Hence, \(f\) is smooth. Now, for all \((g, q) \in G \times (q_0 \exp(U))\), we have
\[
\tilde{\rho}(g, q) = \tilde{\rho}(g, q_0 q_0^{-1} q) \\
= \tilde{\rho}(g, q_0) \tilde{\rho}(g, q_0^{-1} q) \\
= f(g)[(\tilde{\rho}|_{G \times \exp(U)} \circ (\text{id}_G \times l_{q_0^{-1}})(g, q)],
\]
where \(l_{q_0^{-1}}: Q \to Q\) is left translation by \(q_0^{-1}\). Since \(f\) and \(\tilde{\rho}|_{G \times \exp(U)}\) are both smooth, it follows that the right side of (4.5) is smooth on \(G \times (q_0 \exp(U))\). Hence, \(\tilde{\rho}|_{G \times (q_0 \exp(U))}\) is smooth. Since \(q_0 \in Q\) is arbitrary, it follows that \(\tilde{\rho}\) is smooth on \(G \times Q\). This completes the proof. \(\Box\)

We now illustrate Proposition 4.5 with a simple example:

**Example 4.6.** Let \(Q\) be the 2-dimensional non-abelian Lie group
\[
Q = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \ a > 0, \ b \in \mathbb{R} \right\}.
\]
Note that \(Q\) is simply connected, being diffeomorphic to \(\mathbb{R}_+ \times \mathbb{R}\). The associated Lie algebra is
\[
q = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix} \middle| \ \bar{a}, \ \bar{b} \in \mathbb{R} \right\}.
\]
A convenient basis for \(q\) is then
\[
e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
where we note that
\[
[e_1, e_2] = e_2.
\]
Let \(\alpha: q \to \mathbb{R}\) be the linear map defined by \(\alpha(e_1) = 0\) and \(\alpha(e_2) = 1\). Then \((q, \alpha)\) is a Frobenius Lie algebra. Let \(\hat{\beta}\) be the left-invariant symplectic form on \(Q\) defined by \(\hat{\beta}_e = \beta\), where \(\beta(u, v) := \alpha([u, v])\) for \(u, v \in q\).

For \(\lambda \in \mathbb{R}\), let \(\rho_\lambda: q \to q\) be the linear isomorphism defined by
\[
\rho_\lambda(e_1) := e_1 + \lambda e_2, \quad \rho_\lambda(e_2) := e_2.
\]
Then it is a straightforward exercise to show that \(\rho_\lambda \in \text{Aut}(q, \omega_e)\) and
\[
\rho: \mathbb{R} \xrightarrow{\sim} \text{Aut}(q, \omega_e), \quad \lambda \mapsto \rho_\lambda
\]
is a Lie group isomorphism. Proposition 4.5 implies that \((Q, \omega)\) admits the structure of an \(\mathbb{R}\)-symplectic Lie group with unique action \(\hat{\rho}: \mathbb{R} \times Q \to Q\) satisfying \((\hat{\rho}_\lambda)_\ast, e = \rho_\lambda\).

We now compute the action \(\hat{\rho}\) explicitly. Let \(u \in q\). Then
\[
u = \bar{a} e_1 + \bar{b} e_2 = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 0 \end{pmatrix}
\]
for some $a, b \in \mathbb{R}$. Using the naturality of the exponential map, we have
\( (4.11) \quad \hat{\rho}_\lambda \circ \exp(u) = \exp \circ \rho_\lambda(u). \)
A direct calculation shows that
\( (4.12) \quad \exp(u) = \begin{pmatrix} e^{\pi} & \mu(\bar{a}) \bar{b} \\ 0 & 1 \end{pmatrix}, \)
where $\mu: \mathbb{R} \to \mathbb{R}_+$ is the nonzero smooth function given by $\mu(t) = \frac{1}{t}(e^t - 1)$ for $t \neq 0$ and $\mu(0) = 1$. Note that every element of $Q$ is in the image of the exponential map. Indeed, given $q = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for $a > 0$, $b \in \mathbb{R}$, one simply sets $\bar{a} = \ln a$ and $\bar{b} = b/\mu(\ln a)$ in (4.12) to obtain $\exp(u) = q$. The left side of (4.11) is
\( (4.13) \quad \exp \circ \rho_\lambda(u) = \exp \begin{pmatrix} \bar{a} & \lambda \bar{a} + \bar{b} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{\pi} & \mu(\bar{a})(\lambda \bar{a} + \bar{b}) \\ 0 & 1 \end{pmatrix}. \)
Hence,
\( (4.14) \quad \hat{\rho}_\lambda \left( \begin{pmatrix} e^{\pi} & \mu(\bar{a}) \bar{b} \\ 0 & 1 \end{pmatrix} \right) = \left( \begin{pmatrix} e^{\pi} & \mu(\bar{a})(\lambda \bar{a} + \bar{b}) \\ 0 & 1 \end{pmatrix} \right). \)
Setting $\bar{a} = \ln a$ and $\bar{b} = b/\mu(\ln a)$ for $a > 0$ and $b \in \mathbb{R}$, we obtain
\( (4.15) \quad \hat{\rho}_\lambda \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} a & \lambda(a-1) + b \\ 0 & 1 \end{pmatrix}. \)
Since $(Q, \omega, \varphi)$ is an $\mathbb{R}$-symplectic Lie group by Proposition 4.5, $\hat{\rho}_\lambda$ is both a Lie group isomorphism and a symplectomorphism of $(Q, \omega)$ which satisfies $(\hat{\rho}_\lambda)^*e = \rho_\lambda$.

In anticipation of the next section, we introduce the following definition:

**Definition 4.7.** Let $(Q, \omega, \varphi)$ and $(R, \tau, \chi)$ be $G$-symplectic Lie groups. A homomorphism of $G$-symplectic Lie groups from $(Q, \omega, \varphi)$ to $(R, \tau, \chi)$ is a homomorphism $\Psi: (Q, \omega) \to (R, \tau)$ of symplectic Lie groups which is also $G$-equivariant, that is, $\Psi(\varphi_g(q)) = \chi_g(\Psi(q))$ for all $g \in G$ and $q \in Q$.

**4.2. The Equivalence.** In this section, we show that the category of finite dimensional $\mathfrak{g}$-quasi-Frobenius Lie algebras is equivalent to the category of simply connected $G$-symplectic Lie groups, where $G$ is also simply connected. We begin with the following result.

**Proposition 4.8.** Let $\Psi: (Q, \omega, \varphi) \to (R, \tau, \chi)$ be a homomorphism of $G$-symplectic Lie groups. Then
\[ \Psi_{*,e}: (\mathfrak{q}, \omega_e, \varphi'') \to (\mathfrak{r}, \tau_e, \chi'') \]
is a homomorphism of $\mathfrak{g}$-quasi-Frobenius Lie algebras, where $\varphi''$ and $\chi''$ are defined as in Proposition 4.3.
Proof. By definition, $\Psi : (Q, \omega) \to (R, \tau)$ is a homomorphism of symplectic Lie groups. This implies that

$$\Psi_{s,e} : (q, \omega_e) \to (r, \tau_e)$$

is a homomorphism of quasi-Frobenius Lie algebras. It only remains to show that $\Psi_{s,e}$ is $g$-equivariant. Since $\Psi$ is $G$-equivariant, we have

$$\Psi \circ \varphi_g = \chi_g \circ \Psi, \quad \forall \ g \in G.$$ 

This in turn implies that

$$\Psi_{s,e} \circ \varphi'_g = \chi'_g \circ \Psi_{s,e}, \quad \forall \ g \in G,$$

where $\varphi'_g := (\varphi_g)_{s,e} : q \to q$ and $\chi'_g := (\chi_g)_{s,e} : r \to r$. Let $x \in g$ and set $g = \exp(tx)$ in (4.17). Applying $\frac{d}{dt} |_{t=0}$ to both sides then gives

$$\Psi_{s,e} \circ \varphi''_x = \chi''_x \circ \Psi_{s,e}.$$ 

This in turn completes the proof. \hfill \Box

Lemma 4.9. Let $(q, \beta, \phi)$ be a $g$-quasi-Frobenius Lie algebra and let $G$ be the simply connected Lie group whose Lie algebra is $q$. Then there exists a unique Lie group homomorphism $f : G \to GL(q)$, $g \mapsto f_g$ such that $f_{s,e} = \phi$ and $f_g \in \text{Aut}(q, \beta)$ for all $g \in G$.

Proof. Since $G$ is simply connected and $\phi : g \to gl(q)$ is a Lie algebra map, there exists a unique Lie group homomorphism $f : G \to GL(q)$ such that $f_{s,e} = \phi$. We now show that $f_g \in \text{Aut}(q, \beta)$ for all $g \in G$. Fix $x \in q$. To simplify notation, let

$$f_t := f_{\exp(tx)} : q \to q.$$ 

Define $A : \mathbb{R} \times q \times q \to \mathbb{R}$ by

$$A(t, u, v) := \beta(f_t(u), f_t(v)) - \beta(u, v).$$

Since $f_{s,e} = \phi$ and $(q, \beta, \phi)$ is a $g$-quasi-Frobenius Lie algebra, we have

$$\frac{d}{dt} |_{t=0} A(t, u, v) = \beta(\phi_x(u), v) + \beta(u, \phi_x(v)) = 0, \quad \forall \ u, v \in q.$$ 

Furthermore, since $f$ is a group homomorphism and

$$\exp((t + s)x) = \exp(tx) \exp(sx),$$

we have

$$A(t + s, u, v) = A(t, s, u, v) + A(s, u, v), \quad \forall \ u, v \in q.$$ 

Equations (4.20) and (4.21) imply

$$\frac{d}{dt} |_{t=s} A(t, u, v) = \frac{d}{dt} |_{t=0} A(t + s, u, v) = 0 + 0 = 0.$$ 

Hence, for fixed $u, v \in q$, $A(t, u, v)$ is a constant. Since $A(0, u, v) = 0$, it follows that $A(t, u, v) = 0$ for all $t \in \mathbb{R}$. Hence,

$$\beta(f_t(u), f_t(v)) = \beta(u, v), \quad \forall \ t \in \mathbb{R}.$$ 

In particular,

$$\beta(f_{\exp(x)}(u), f_{\exp(x)}(v)) = \beta(u, v).$$
Now define $B: \mathbb{R} \times \mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$ by
\begin{equation}
B(t, u, v, w) = \beta([f_t(u), f_t(v)] - f_t([u, v]), f_t(w)).
\end{equation}

Equation (4.23) implies that
\begin{equation}
B(t, u, v, w) = \beta([f_t(u), f_t(v)], f_t(w)) - \beta([u, v], w).
\end{equation}

From (4.26), we also have
\begin{equation}
B(t, u, v, w) = \beta([\phi_x(u), \phi_x(v)], w) + \beta([u, \phi_x(v)], w) + \beta([u, \phi_x(w)], w)
\end{equation}
(4.27)
\begin{equation}
= 0, \quad \forall \ u, v, w.
\end{equation}

From (4.26), we also have
\begin{equation}
B(t + s, u, v, w) = \beta([f_{t+s}(u), f_{t+s}(v)], f_{t+s}(w)) - \beta([u, v], w)
\end{equation}
(4.28)
\begin{equation}
= B(t, f_s(u), f_s(v), f_s(w)) + \beta([f_s(u), f_s(v)], f_s(w)) - \beta([u, v], w)
\end{equation}
Equations (4.27) and (4.28) now imply
\begin{equation}
\frac{d}{dt} |_{t=0} B(t, u, v, w) = \frac{d}{dt} |_{t=0} B(t + s, u, v, w) = 0 + 0 + 0 = 0.
\end{equation}
(4.29)

From (4.29), it follows that for fixed $u, v, w$, $B(t, u, v, w)$ is a constant for all $t \in \mathbb{R}$. Hence, $B(t, u, v, w) = B(0, u, v, w) = 0$ for all $t \in \mathbb{R}$ and $u, v, w \in \mathfrak{q}$. In particular,
\begin{equation}
B(1, u, v, w) = \beta([f_1(u), f_1(v)] - f_1([u, v]), f_1(w)) = 0, \quad \forall \ u, v, w \in \mathfrak{q}.
\end{equation}
Since $\beta$ is non-degenerate and $f_1 := f_{\exp(x)} \in GL(\mathfrak{q})$, it follows that
\begin{equation}
f_{\exp(x)}([u, v]) = [f_{\exp(x)}(u), f_{\exp(x)}(v)].
\end{equation}
(4.31)

Since $G$ is connected, $x \in \mathfrak{g}$ is arbitrary, and $f$ is a group homomorphism, equations (4.24) and (4.31) imply that
\begin{equation}
\beta(f_g(u), f_g(v)) = \beta(u, v), \quad f_g([u, v]) = [f_g(u), f_g(v)]
\end{equation}
(4.32)
for all $g \in G$. Hence, $f_g \in \text{Aut}(\mathfrak{q}, \beta)$ for all $g \in G$. This completes the proof. 

**Proposition 4.10.** Let $(\mathfrak{q}, \beta, \phi)$ be a $\mathfrak{g}$-quasi-Frobenius Lie algebra. Let $G$ and $Q$ be the simply connected Lie groups associated to $\mathfrak{g}$ and $\mathfrak{q}$ respectively and let $\tilde{\beta} \in \Omega^2(Q)$ be the left-invariant 2-form associated to $\beta$. Then there exists a unique left action $\tilde{\phi}: G \times Q \to Q$ such that $(Q, \tilde{\beta}, \tilde{\phi})$ is a $G$-symplectic Lie group whose associated $\mathfrak{g}$-quasi-Frobenius Lie algebra is
\begin{equation}
(\mathfrak{q}, \tilde{\beta}_e, \tilde{\phi}''') = (\mathfrak{q}, \beta, \phi),
\end{equation}
where $\tilde{\phi}'''$ is defined as in Proposition 4.3.

**Proof.** By Proposition 2.13 $(Q, \tilde{\beta})$ is a symplectic Lie group. Since $G$ is simply connected, Lemma 4.9 shows that there exists a unique Lie group homomorphism
\begin{equation}
f: G \to GL(\mathfrak{q}), \quad g \mapsto f_g
\end{equation}
for all $g \in G$. This completes the proof.
such that $f_{*, e} = \phi : g \to g(q)$ and $f_g \in \text{Aut}(q, \beta)$ for all $g \in G$. Since $Q$ is simply connected, Proposition 4.5 shows that there exists a unique smooth left $G$-action

$$\phi : G \times Q \to Q, \quad (g, q) \mapsto \phi_g(q)$$

such that $(Q, \tilde{\beta}, \phi)$ is a $G$-symplectic Lie group and $(\phi_g)_{*, e} = f_g$. Setting $\phi_g := (\phi_g)_{*, e}$ as in Proposition 4.3, we have

$$\phi'' := \phi_{*, e} = f_{*, e} = \phi.$$ 

This completes the proof. \hfill \Box

**Proposition 4.11.** Let $\psi : (q, \beta, \phi) \to (r, \sigma, \mu)$ be a homomorphism of $\mathfrak{g}$-quasi-Frobenius Lie algebras. Let $G$ be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and let $(Q, \tilde{\beta}, \phi)$ and $(R, \tilde{\sigma}, \tilde{\mu})$ be the simply connected $G$-symplectic Lie groups associated to $(q, \beta, \phi)$ and $(r, \sigma, \mu)$ respectively by Proposition 4.10. Then there exists a unique homomorphism of $G$-symplectic Lie groups

$$\hat{\psi} : (Q, \tilde{\beta}, \phi) \to (R, \tilde{\sigma}, \tilde{\mu})$$

such that $\hat{\psi}_{*, e} = \psi$.

**Proof.** By Proposition 2.17 there exists a unique homomorphism of symplectic Lie groups $\hat{\psi} : (Q, \tilde{\beta}) \to (R, \tilde{\sigma})$ such that $\hat{\psi}_{*, e} = \psi$. We now verify that $\hat{\psi}$ is $G$-equivariant.

Let $\phi' : G \to \text{Aut}(q, \beta)$, $g \mapsto \phi'_g$ and $\tilde{\mu}' : G \to \text{Aut}(r, \sigma)$, $g \mapsto \tilde{\mu}'_g$ be defined as in Proposition 4.3. Fix $x \in \mathfrak{g}$. To simplify notation, let

$$(\phi'_t) := \phi'_{\exp(tx)}, \quad (\tilde{\mu}'_t) := \tilde{\mu}'_{\exp(tx)}.$$ 

Define $B : \mathbb{R} \times \mathfrak{q} \times \mathfrak{r} \to \mathbb{R}$ by

$$B(t, u, v) := \sigma\left(\psi \circ \phi'_t(u) - \tilde{\mu}'_t \circ \psi(u), \tilde{\mu}'_t(v)\right)$$

$$= \sigma\left(\psi \circ \phi'_t(u), \tilde{\mu}'_t(v)\right) - \sigma\left(\tilde{\mu}'_t \circ \psi(u), \tilde{\mu}'_t(v)\right)$$

$$= \sigma\left(\psi \circ \phi'_t(u), \tilde{\mu}'_t(v)\right) - \sigma(\psi(u), v),$$

(4.33) where the third equality follows from the fact that $\tilde{\mu}'_t \in \text{Aut}(r, \sigma)$. Hence,

$$\frac{d}{dt} \bigg|_{t=0} B(t, u, v) = \sigma(\psi \circ \phi_x(u), v) + \sigma(\psi(u), \mu_x(v))$$

$$= \sigma(\mu_x \circ \psi(u), v) + \sigma(\psi(u), \mu_x(v))$$

$$= 0, \quad \forall \ u \in \mathfrak{q}, \ v \in \mathfrak{r}$$

(4.34) where the second equality follows from the fact that $\psi$ is $\mathfrak{g}$-equivariant (i.e., $\psi \circ \phi_x = \mu_x \circ \psi$) and the third equality follows from the fact that $(r, \sigma, \mu)$ is a $\mathfrak{g}$-quasi-Frobenius Lie algebra with 2-cocycle $\sigma$ and $\mathfrak{g}$-action $\mu$. Next note that

$$B(t + s, u, v) = B(t, \phi'_s(u), \tilde{\mu}'_s(v)) + \sigma(\psi(\phi'_s(u)), \tilde{\mu}'_s(v)) - \sigma(\psi(u), v).$$

(4.35) Hence,

$$\frac{d}{dt} \bigg|_{t=s} B(t, u, v) = \frac{d}{dt} \bigg|_{t=0} B(t + s, u, v) = 0 + 0 - 0 = 0,$$

(4.36)
where the first zero follows from \([4.34]\). Hence,

\[
B(t, u, v) = B(0, u, v) = 0, \quad \forall t \in \mathbb{R}, \ u \in \mathfrak{q}, \ v \in \mathfrak{r}.
\]

In particular, \(B(1, u, v) = 0\) for all \(u, v \in \mathfrak{r}\). Since \(\sigma\) is nondegenerate and \(\overline{\mu}' : \mathfrak{r} \to \mathfrak{r}\) is also a linear isomorphism for all \(t\), it follows that

\[
\text{(4.37)} \quad \overline{\mu}'(t, u, v) = \overline{\mu}'(0, u, v) = 0, \quad \forall t \in \mathbb{R}, \ u \in \mathfrak{q}, \ v \in \mathfrak{r}.
\]

In particular, we have

\[
\text{(4.38)} \quad \psi \circ \overline{\phi}' = \overline{\mu}' \circ \psi, \quad \forall t \in \mathbb{R}.
\]

Since \(x \in \mathfrak{g}\) was arbitrary, \(\text{(4.39)}\) must hold for all \(u, v \in \mathfrak{r}\). Since \(\sigma\) is nondegenerate and \(\overline{\mu}' : \mathfrak{r} \to \mathfrak{r}\) is also a linear isomorphism for all \(t\), it follows that

\[
\text{(4.40)} \quad \psi \circ \overline{\phi}_g = \overline{\mu}_g \circ \psi, \quad \forall g \in G.
\]

Equation \(\text{(4.40)}\) combined with the fact that (1) \(\mathcal{Q}\) is connected, (2) \(\overline{\psi} \circ \overline{\phi}_g\) and \(\overline{\mu}_g \circ \overline{\psi}\) are both Lie group homomorphisms \(\forall g \in G\), and (3)

\[
\text{(4.41)} \quad (\overline{\psi} \circ \overline{\phi}_g)_{*e} = \overline{\phi}_g \circ \overline{\psi} = (\overline{\mu}_g \circ \overline{\psi})_{*e}, \quad \forall g \in G
\]

imply that \(\overline{\psi} \circ \overline{\phi}_g = \overline{\mu}_g \circ \overline{\psi}\) for all \(g \in G\). In other words, \(\overline{\psi}\) is \(G\)-equivariant and this completes the proof. \(\square\)

We conclude the paper with the following generalization of Theorem 2.18.

**Theorem 4.12.** Let \(G\) be a simply connected Lie group and let \(G\)-SCSLG be the category of simply connected \(G\)-symplectic Lie groups and let \(g\)-qFLA be the category of finite dimensional \(g\)-quasi-Frobenius Lie algebras. Let \(\hat{F}\) be the functor from \(G\)-SCSLG to \(g\)-qFLA which sends the object \((\mathcal{Q}, \omega, \phi)\) to \((q, \omega_e, \phi'')\), where \(\phi''\) is defined as in Proposition 4.3 and the morphism \(\overline{\psi} : (\mathcal{Q}, \omega, \phi) \to (\mathcal{R}, \tau, \chi)\) to

\[
\overline{\psi}_{*e} : (q, \omega_e, \phi'') \mapsto (r, \tau_e, \chi'').
\]

Then \(\hat{F}\) is an equivalence of categories.

**Proof.** Theorem 4.12 follows from Theorem 2.18, Proposition 4.3, Proposition 4.8, Proposition 4.10, and Proposition 4.11. \(\square\)

5. \(D(\mathfrak{g})\)-QUASI-FROBENIUS LIE ALGEBRAS

Let \((\mathfrak{g}, \gamma)\) be a finite dimensional Lie bialgebra. We begin with the following observation:

**Proposition 5.1.** Let \(V\) be a vector space over \(k\) and let \(\rho : D(\mathfrak{g}) \to \mathfrak{gl}(V)\) be a linear map (not necessarily a representation). Define

\[
\phi := \rho \mid_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(V), \quad \psi := \rho \mid_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{gl}(V).
\]

The following statements are equivalent.

(i) \(\rho\) is a representation of \(D(\mathfrak{g})\) on \(V\).
(ii) \( \varphi \) and \( \psi \) are representations of \( g \) and \( g^* \) on \( V \) which satisfy

\[
\psi_{ad_x^*} \xi - \varphi_{ad_x^*} x = \varphi_x \circ \psi_x - \psi_x \circ \varphi_x, \quad \forall \, x \in g, \, \xi \in g^*.
\]  

(5.1)

**Proof.** (i)\( \Rightarrow \) (ii) Since \( \rho \) is a representation of \( D(g) \) on \( V \), it follows immediately that \( \varphi \) and \( \psi \) must be representations of \( g \) and \( g^* \) on \( V \) respectively. For (5.1), we note that

\[
[x, \xi] = ad_x^* \xi - ad_x^* x, \quad \forall \, x \in g, \, \xi \in g^*.
\]

Since \( \rho \) is a representation and \( \varphi := \rho |_g \) and \( \psi := \rho |_{g^*} \), we have

\[
\psi_{ad_x^*} \xi - \varphi_{ad_x^*} x = \rho[x, \xi] = \frac{1}{2} \rho x \rho x - \rho \xi \rho x = \varphi_x \psi_x - \psi_x \varphi_x,
\]

which proves (5.1).

(i)\( \Leftarrow \) (ii) Let \( \alpha = x + \xi \in D(g) \). Then

\[
\rho[x+\xi, y+\eta] = \rho[x, y] + \rho[x, \eta] + \rho[\xi, y] + \rho[\xi, \eta]
\]

\[
= \varphi_{xy} - \varphi_{ad_x^*} y + \varphi_{ad_x^*} y - \psi_{ad_y^*} \xi + \psi_{ad_y^*} \xi - \psi_{ad_x^*} \xi\]

\[
= \varphi_{xy} - \varphi_y \circ \psi_{ad_x^*} x + \varphi_{ad_x^*} y - \psi_y \circ \psi_x - \psi_x \circ \varphi_x
\]

\[
+ \psi_x \circ \varphi_y - \varphi_y \circ \psi_x\]

\[
= (\varphi_x + \psi_x) \circ (\varphi_y + \psi_y) - (\varphi_y + \psi_y) \circ (\varphi_x + \psi_x)
\]

\[
= \rho x \rho y - \rho y \rho x - \rho x \rho y + \rho y \rho x.
\]

This proves that \( \rho : D(g) \rightarrow gl(V) \) is a representation of \( D(g) \) on \( V \).

**Proposition 5.2.** Let \((q, \beta)\) be a quasi-Frobenius Lie algebra and let \( \rho : D(g) \rightarrow gl(q) \) be a linear map (not necessarily a representation). Define \( \varphi := \rho |_g \) and \( \psi := \rho |_{g^*} \). Then \((q, \beta, \rho)\) is a \( D(g)\)-quasi-Frobenius Lie algebra iff the following conditions are satisfied:

(a) \( \psi_{ad_x^*} \xi - \varphi_{ad_x^*} x = \varphi_x \circ \psi_x - \psi_x \circ \varphi_x, \quad \forall \, x \in g, \, \xi \in g^* \)

(b) \((q, \beta, \varphi)\) is a \( g\)-quasi-Frobenius Lie algebra.

(c) \((q, \beta, \psi)\) is a \( g^*\)-quasi-Frobenius Lie algebra.

**Proof.** By Proposition 5.1 \( \rho \) is left \( D(g)\)-module structure on \( q \) iff \( \varphi \) and \( \psi \) are left \( g \) and \( g^* \)-module structures on \( q \) respectively which satisfy condition (a). Since \( D(g) = g \oplus g^* \) as a vector space, it follows that \( \rho : D(g) \rightarrow gl(q) \) satisfies conditions (i) and (ii) of Definition 3.1 iff \( \phi : g \rightarrow gl(q) \) and \( \psi : g^* \rightarrow gl(q) \) both satisfy conditions (i) and (ii) of Definition 3.1. This completes the proof.

**Proposition 5.3.** Let \( g \) be a finite dimensional quasitriangular Lie bialgebra with r-matrix \( r = \sum_i a_i \otimes b_i \). Let \( \varphi : g \rightarrow gl(V) \), \( x \mapsto \varphi(x) \) be a representation of \( g \) on \( V \). Define \( \psi : g^* \rightarrow gl(V) \), \( x \mapsto \psi(x) \) by

\[
(5.2) \quad \psi(x) := \sum_i \xi(a_i) \varphi(b_i), \quad \forall \, \xi \in g^*.
\]

Then \( \psi \) is a representation of \( g^* \) on \( V \).

**Proof.** We need to show that

\[
(5.3) \quad \psi([\xi, \eta]) = \psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi).
\]
We now expand the left side of (5.3):
\[
\psi([\xi, \eta]) = \sum_j [\xi, \eta](a_j)\varphi(b_j) = \sum_j (\xi \otimes \eta)((\delta r)(a_j))\varphi(b_j)
\]
(5.4)
\[
= \sum_{i,j} \xi([a_j, a_i])\eta(b_i)\varphi(b_j) + \sum_{i,j} \xi(a_i)\eta([a_j, b_i])\varphi(b_j).
\]

The right side of (5.3) expands as
\[
\psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi) = \sum_{i,j} \xi(a_i)\eta(a_j)\varphi(b_i)\varphi(b_j) - \sum_{i,j} \eta(a_j)\xi(a_i)\varphi(b_i)\varphi(b_i)
\]
(5.5)
\[
= \sum_{i,j} \xi(a_i)\eta(a_j)(\varphi([b_i, b_j])).
\]

The CYBE can be rewritten as
(5.6)
\[
\sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j] = \sum_{i,j} [a_j, a_i] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [a_j, b_i] \otimes b_j.
\]

Applying \(\xi \otimes \eta \otimes \varphi\) to both sides of (5.6) gives
(5.7)
\[
\sum_{i,j} \xi(a_i)\eta(a_j)\varphi([b_i, b_j]) = \sum_{i,j} \xi([a_j, a_i])\eta(b_i)\varphi(b_j) + \sum_{i,j} \xi(a_i)\eta([a_j, b_i])\varphi(b_j).
\]

Equations (5.4), (5.5), and (5.7) imply
\[
\psi(\xi)\psi(\eta) - \psi(\eta)\psi(\xi) = \psi([\xi, \eta]).
\]

This completes the proof.

\[\square\]

**Corollary 5.4.** Let \(\mathfrak{g}\) be a finite dimensional quasitriangular Lie bialgebra with \(r\)-matrix \(r = \sum_i a_i \otimes b_i\) and let \((\mathfrak{q}, \beta, \varphi)\) be a \(\mathfrak{g}\)-quasi-Frobenius Lie algebra. Define \(\psi: \mathfrak{g}^* \to \mathfrak{gl}(\mathfrak{q}), \xi \mapsto \psi(\xi)\) by
\[
\psi(\xi) := \sum_i \xi(a_i)\varphi(b_i),
\]
where \(\varphi(b_i) := \varphi_{b_i}: \mathfrak{q} \to \mathfrak{q}\). Then \((\mathfrak{q}, \beta, \psi)\) is a \(\mathfrak{g}^*\)-quasi-Frobenius Lie algebra.

**Proof.** Immediate.

\[\square\]

**Proposition 5.5.** Let \(\mathfrak{g}\) be a finite dimensional quasitriangular Lie bialgebra with \(r\)-matrix \(r = \sum_i a_i \otimes b_i\). Let \(\varphi: \mathfrak{g} \to \mathfrak{gl}(V), x \mapsto \varphi(x)\) be a representation of \(\mathfrak{g}\) on \(V\). Define \(\psi: \mathfrak{g}^* \to \mathfrak{gl}(V), \xi \mapsto \psi(\xi)\) according to Proposition 5.3. Define \(\rho: \mathfrak{D}(\mathfrak{g}) \to \mathfrak{gl}(V), a \mapsto \rho(a)\) by
(5.8)
\[
\rho(x + \xi) := \varphi(x) + \psi(\xi), \quad \forall \, x \in \mathfrak{g}, \, \xi \in \mathfrak{g}^*.
\]

Then \(\rho\) is a representation of \(\mathfrak{D}(\mathfrak{g})\) on \(V\).
Proof. By Proposition 5.1, it suffices to show that
\begin{equation}
\psi(ad_x^a \xi) - \varphi(ad_x^\xi x) = \varphi(x)\psi(\xi) - \psi(\xi)\varphi(x) .
\end{equation}
We begin by expanding the left side of (5.9). First,
\begin{equation}
\psi(ad_x^a \xi) = \sum_i \langle ad_x^a \xi \rangle (a_i) \varphi(b_i)
\end{equation}
(5.10) \[ = \sum_i \xi([a_i, x])\varphi(b_i) .\]
By Proposition 2.25, \[ \sum_i a_i \otimes b_i + \sum_i b_i \otimes a_i \] is invariant under the adjoint action of \( g \). Hence,
\begin{equation}
\sum_i [a_i, x] \otimes b_i = \sum_i a_i \otimes [x, b_i] + \sum_i [x, b_i] \otimes a_i + \sum_i b_i \otimes [x, a_i] .
\end{equation}
(5.11) Equations (5.10) and (5.11) now imply
\begin{equation}
\psi(ad_x^a \xi) = \sum_i \xi(a_i)\varphi([x, b_i]) + \sum_i \xi([x, b_i])\varphi(a_i) + \sum_i \xi(b_i)\varphi([x, a_i]) .
\end{equation}
(5.12) Next, we note that
\begin{equation}
ad_x^a x = \sum_i \xi(b_i)[x, a_i] + \sum_i \xi([x, b_i])a_i .
\end{equation}
(5.13) From (5.12) and (5.13), we have
\begin{equation}
\psi(ad_x^a \xi) - \varphi(ad_x^b \xi x) = \sum_i \xi(a_i)\varphi([x, b_i]) .
\end{equation}
(5.14) For the right side of (5.9), we have
\begin{equation}
\varphi(x)\psi(\xi) - \psi(\xi)\varphi(x) = \sum_i \xi(a_i)\varphi(x)\varphi(b_i) - \sum_i \xi(a_i)\varphi(b_i)\varphi(x)
\end{equation}
\[ = \sum_i \xi(a_i)\varphi([x, b_i])
\end{equation}
(5.15) where the last equality follows from (5.14). This completes the proof. \( \square \)

Theorem 5.6. Let \( g \) be a finite dimensional quasitriangular Lie bialgebra. Let \((q, \beta, \varphi)\) be any \( g\)-quasi-Frobenius Lie algebra. Then there exists a representation \( \rho : D(g) \rightarrow \mathfrak{gl}(q) \) such that \( \rho \mid_g = \varphi \) and \((q, \beta, \rho)\) is a \( D(g)\)-quasi-Frobenius Lie algebra.

Proof. Let \( r \in g \otimes g \) be the r-matrix associated to \( g \) and let \( \psi : g^* \rightarrow \mathfrak{gl}(q) \) be the representation of \( g^* \) on \( q \) determined by \( \varphi \) and \( r \) according to Proposition 5.3. By Corollary 5.4, \((q, \beta, \psi)\) is a \( g^*\)-quasi-Frobenius Lie algebra. Define \( \rho : D(g) \rightarrow \mathfrak{gl}(q) \) by
\[ \rho(x + \xi) := \varphi(x) + \psi(\xi) , \quad \forall \ x \in g , \ \xi \in g^* . \]
By Proposition 5.5, \( \rho \) is a representation of \( D(g) \) on \( q \). Since \((q, \beta, \varphi)\) and \((q, \beta, \psi)\) are \( g \) and \( g^*\)-quasi-Frobenius Lie algebras and \( \rho \mid_g = \varphi \) and \( \rho \mid_{g^*} = \psi \) (by definition), it follows that \((q, \beta, \rho)\) is a \( D(g)\)-quasi-Frobenius Lie algebra. \( \square \)
Corollary 5.7. Let \( \mathfrak{g} \) be any finite dimensional Lie algebra and let \((\mathfrak{q}, \beta, \varphi)\) be any \(\mathfrak{g}\)-quasi-Frobenius Lie algebra. Let \(D(\mathfrak{g})\) be the Drinfeld double of the Lie bialgebra \((\mathfrak{g}, \gamma)\) where \(\gamma \equiv 0\). Define \(\rho: D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})\) by \(\rho(x + \xi) = \varphi(x)\) for all \(x \in \mathfrak{g}, \xi \in \mathfrak{g}^*\). Then \((\mathfrak{q}, \beta, \rho)\) is a \(D(\mathfrak{g})\)-quasi-Frobenius Lie algebra.

Proof. \((\mathfrak{g}, \gamma)\) is naturally a quasitriangular Lie bialgebra with \(r\)-matrix \(r \equiv 0 \in \mathfrak{g} \otimes \mathfrak{g}\). Corollary 5.7 now follows as a special case of the proof of Theorem 5.6. \(\square\)

We conclude the paper with an example.

Example 5.8. Let \((\mathfrak{q}, \beta)\) be the 4-dimensional quasi-Frobenius Lie algebra from Example 3.4. For convenience, we recall its structure: \(\mathfrak{q}\) has basis \(\{e_1, e_2, e_3, e_4\}\) with non-zero commutator relations given by

\[
[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = 2e_4, \quad [e_2, e_3] = e_4,
\]

and the matrix representation of \(\beta\) with respect to \(\{e_1, e_2, e_3, e_4\}\) is

\[
(\beta_{ij}) = \begin{pmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-2 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \((\mathfrak{g}, \delta r)\) be the 2-dimensional triangular Lie bialgebra from Examples 2.27 and 2.29. Once again, we recall the structure for convenience. \(\mathfrak{g}\) has basis \(\{x, y\}\) with commutator relation \([x, y] = x\) and \(r\)-matrix \(r = y \wedge x\). Let \(\{x^*, y^*\}\) denote the corresponding dual basis. The commutator relations on \(D(\mathfrak{g})\) are

\[
[x, y] = x, \quad [x^*, y^*] = y^*, \quad [x, x^*] = -y^*, \quad [x, y^*] = 0 \\
[y, x^*] = x^* + y, \quad [y, y^*] = -x.
\]

Let \(\varphi: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{q})\) be the linear map defined by

\[
\varphi_x(e_1) = 0, \quad \varphi_x(e_2) = 0, \quad \varphi_x(e_3) = e_2, \quad \varphi_x(e_4) = 0 \\
\varphi_y(e_1) = 0, \quad \varphi_y(e_2) = -\frac{1}{2}e_2, \quad \varphi_y(e_3) = \frac{1}{2}e_3, \quad \varphi_y(e_4) = 0.
\]

Consideration of Example 3.4 (or a direct calculation) shows that \((\mathfrak{q}, \beta, \varphi)\) is a \(\mathfrak{g}\)-quasi-Frobenius Lie algebra. By Theorem 5.6 there exists a representation \(\rho: D(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{q})\) such that \(\rho |_{\mathfrak{g}} = \varphi\) and \((\mathfrak{q}, \beta, \rho)\) is a \(D(\mathfrak{g})\)-quasi-Frobenius Lie algebra. We now compute \(\rho\) explicitly. From the proof of Theorem 5.6 this amounts to computing the representation \(\psi: \mathfrak{g}^* \to \mathfrak{gl}(\mathfrak{q})\) which is determined by \(\varphi\) and \(r = y \wedge x\) according to Proposition 5.3

\[
\psi_{x^*} = -\varphi_y, \quad \psi_{y^*} = \varphi_x.
\]

\(\rho\) is then uniquely defined by \(\rho |_{\mathfrak{g}} = \varphi\) and \(\rho |_{\mathfrak{g}^*} = \psi\).

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Department of Mathematics and Computer Science,  
QCC CUNY, Bayside, NY 11364  
E-mail: dpham90@gmail.com