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Commentationes Mathematicae Universitatis Carolinae, Vol. 57 (2016), No. 4, 527–536

Persistent URL: http://dml.cz/dmlcz/145947

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Abstract. A trilinear alternating form on dimension n can be defined based on a Steiner triple system of order n. We prove some basic properties of these forms and using the radical polynomial we show that for dimensions up to 15 nonisomorphic Steiner triple systems provide nonequivalent forms over $GF(2)$. Finally, we prove that Steiner triple systems of order n with different number of subsystems of order $(n-1)/2$ yield nonequivalent forms over $GF(2)$.

Keywords: trilinear alternating form; Steiner triple system; radical polynomial

Classification: 15A69

1. Introduction

Let $f : V^3 \to F$ be a trilinear alternating form on a vector space V over a field F, $\dim V = n < \infty$. Two forms f_1 and f_2 on V are *equivalent* if there is an automorphism of V satisfying $f_1(u, v, w) = f_2(\phi(u), \phi(v), \phi(w))$ for all $u, v, w \in V$. Classification of classes of this equivalence seems to be a very difficult problem (unlike in the bilinear case) even for small dimensions of V and not much has been done in this respect. This classification is known for the case $n \leq 7$ for a large family of fields including all finite fields (see [1]) and Gurevich [2], D. Djokovic [3] and L. Noui [4] solved the case $n = 8$ for $F = \mathbf{C}$, $F = \mathbf{R}$ and F algebraically closed field of arbitrary characteristics, respectively. Classification of 8-dimensional forms over $GF(2)$ can be found in [7].

This paper concerns forms constructed from Steiner triple systems and the main results are over the two-element field. The reason is that the original motivation for this research comes from the theory of doubly even binary codes, of which trilinear alternating forms over the two-element field appear as important invariants.

Let S be a Steiner triple system on a set $X = \{1, \dots, n\}$ and let V be a vector space with a basis $\{b_i, i \in X\}$. A trilinear alternating form given by $f(b_i, b_j, b_k) = 1$ if $\{i, j, k\} \in S$, $i < j < k$, and $f(b_i, b_j, b_k) = 0$ if $\{i, j, k\} \notin S$ is called Steiner form. We show that these forms are nondegenerate and indecomposable. Moreover, using the classification of Steiner triple systems we show that there is an invariant distinguishing among all Steiner forms up to dimension 15 over $GF(2)$.

DOI 10.14712/1213-7243.2015.182

2. Preliminaries

Let V be an n-dimensional vector space over a finite field F and fix a basis $B = \{b_1, \dots, b_n\}$ of V. Denote by $B^* = \{b_1^*, \dots, b_n^*\}$ its dual basis (defined as usually by $b_i^*(b_j) = \delta_{ij}$). Given B and B^* as above, a trilinear alternating form f can be expressed as

$$f_B = \sum_{1 \le i_1 < i_2 < i_3 \le n} \alpha_{i_1 i_2 i_3} b_{i_1}^* \wedge b_{i_2}^* \wedge b_{i_3}^*,$$

where the index B indicates the dependence of the presentation upon the chosen basis. Denote by Δ the set

$$\Delta = \{(i_1, i_2, i_3) \mid 1 \le i_1 < i_2 < i_3 \le n, \alpha_{i_1 i_2 i_3} \neq 0\}.$$

Since the coefficients α of all forms presented in the paper are either 0 or 1, we shall write forms as

$$f_B = \sum_{\Delta} \underline{i_1 i_2 i_3}.$$

Similar notation is also used for bilinear alternating forms.

Proposition 2.1. *Let g be a bilinear alternating form on a vector space V of dimension n. Then there exists a basis $B = \{b_1, \dots, b_n\}$ and $k \le n$ such that*

$$g_B = \underline{12} + \underline{34} + \dots + \underline{(k-1)k}.$$

Let f be a trilinear alternating form on V. We shall use the symbol $f[v]$ for the bilinear form $f(v, -, -)$ and similarly $f[v_1, v_2]$ shall denote the linear form $f(v_1, v_2, -)$. The group of automorphisms of f is denoted $\mathrm{Aut}(f)$. The set $\{x \in V; f[x] = 0\}$ is called the *radical* of f and will be denoted $\mathrm{Rad}\, f$. If it contains only the zero vector then f is called *nondegenerate*. The *radical of* v is the subspace

$$\mathrm{Rad}(v) = \{u \in V; f[v, u] = 0\}.$$

The *rank* of $v \in V$ is the codimension of $\mathrm{Rad}(v)$ in V

$$r(v) = n - \dim \mathrm{Rad}(v).$$

To show nonequivalence of forms (over finite fields) we shall use an invariant introduced in [7], called the *radical polynomial*

$$P(f) = \sum_{v \in V} x^{r(v)} y^{n-r(v)}.$$

$P(f)$ is a homogenous polynomial of degree n and if written in the form

$$P(f) = \sum_{i=0}^{n-1} \alpha_i x^i y^{n-i} \qquad (1)$$

then every α_i is a nonnegative integer and we have $\sum_{i=0}^{n-1} \alpha_i = q^n$. Since for every $u \in V$ we have $u \in \mathrm{Rad}(u)$, the rank $r(u)$ of any vector u is less than n and the sum in (1) runs only to $n-1$. Moreover, by Proposition 2.1 we get α_i is equal to zero whenever i is odd. In the Appendix the y parts of the polynomials are omitted.

Fix a trilinear alternating form f on a vector space V. We say that vectors $u, v \in V$ are *orthogonal*, denoted by $u \perp v$, if $u \in \mathrm{Rad}(v)$. This relation is clearly reflexive and symmetric. Two subspaces V_1 and V_2 of V are said to be *orthogonal* if $v_1 \perp v_2$ for any $v_1 \in V_1$ and $v_2 \in V_2$. We say that a nondegenerate form f on V is *decomposable* if $V = W_1 \oplus \cdots \oplus W_m$, $m \geq 2$, and W_i is orthogonal to W_j whenever $i \neq j$. Unlike in the bilinear case, most of the trilinear alternating forms are indecomposable.

3. Steiner forms

A pair $S = (X, \mathcal{T})$ is called *Steiner triple system* if X is an n-element set of points (we shall assume $X = \{1, \dots, n\}$), $n \geq 3$, and $\mathcal{T}$ is a system of three-element subsets of X such that every pair of points is contained in exactly one triple in $\mathcal{T}$.

Theorem 3.1. *Steiner triple system on n points exists iff $n = 1, 3$ mod 6, $n \geq 3$.*

We shall also use the quasigroup notation for Steiner triple systems, where $x \cdot y = z$ if $\{x, y, z\} \in \mathcal{T}$ and $x \cdot x = x$ for every $x \in X$.

Let $S = (X, \mathcal{T})$ be an STS on n points. Define a trilinear alternating form f_S on an n-dimensional vector space V (over a field F) with a basis $B = \{b_1, \dots, b_n\}$:

$$f_S = \sum_{\{i,j,k\} \in \mathcal{T}} b_i^* \wedge b_j^* \wedge b_k^*.$$

We assume $i < j < k$ in the definition of f_S above. Call this form a *Steiner form* and B its Steiner basis. It is not clear how many Steiner bases there are, and even whether it is possible that a form has two Steiner bases such that the Steiner systems are nonisomorphic.

Proposition 3.2. *Any Steiner form is nondegenerate.*

Proof: Let f_S be a Steiner form and let $B = \{b_1, \dots, b_n\}$ be its Steiner basis. Consider a nonzero vector $v = \sum \alpha_i b_i$ and assume without loss of generality that $\alpha_1 \neq 0$. Choose any triple of S containing 1, say $\{1, j, k\}$ and assume $j < k$. Then we have $f_S(v, b_j, b_k) = \sum \alpha_i f_S(b_i, b_j, b_k) = \alpha_1$ and thus $v \notin \mathrm{Rad}\, f_S$. □

Lemma 3.3. *Let f be a decomposable form. Then the rank of any vector is at most $n-2$.*

Proof: Consider an orthogonal decomposition $V = \bigoplus_{i=1}^m W_i$, $m \geq 2$, and a nonzero vector w. After changing the order of W_i's, we can assume $w = \sum_{i=1}^p w_i$, where $w_i \in W_i$ and $p \leq m$. Since $f[w, w_j] = \sum_i f[w_i, w_j] = 0$, we get that the radical of w contains the subspace $\langle w_1, \dots, w_p\rangle$. Moreover, $\mathrm{Rad}(w)$ contains also subspaces $W_{p+1}, \dots, W_m$ and thus the dimension of the radical of w is at least m. Hence the rank of w is at most $n - m < n - 1$. □

Proposition 3.4. *Any Steiner form is indecomposable.*

PROOF: Let f_S be a Steiner form and let $B = \{b_1, \dots, b_n\}$ be its Steiner basis and put $k = (n-1)/2$. By definition of f_S the bilinear form $f_S[b_j]$ is equal to

$$f_S[b_j] = \underline{r_1 s_1} + \dots + \underline{r_k s_k},$$

where $\{j, r_i, s_i\} \in \mathcal{T}$, $i = 1, \dots k$, and thus the rank of any b_j is equal to $n - 1$. Hence f_S is indecomposable by Lemma 3.3. □

From now on we assume that the underlying field is $GF(2)$. A natural question that arises is: are two STS isomorphic if and only if their Steiner forms are equivalent? We shall give the answer up to dimension 15, but the result was obtained using a computer. Table 1 shows the number of nonisomorpic STS for small n. The trivial case $n = 3$ is omitted.

n	Number of STS on n points	
7	1	(Fano Plane)
9	1	(Affine Plane)
13	2	
15	80	
19	11084874829	

TABLE 1. Number of STS

Example. The only Steiner triple system S on seven points is the Fano plane. Its radical polynomial is equal to

$$y^7 + 7x^2y^5 + 56x^4y^3 + 64x^6y.$$

Notice the coefficient 7 at x^2y^5, which is by Proposition 3.10 the number of Steiner subsystems of order 3. Using the radical polynomial classification in [7] we see that the Fano form is equivalent to the form $f_6 = \underline{123} + \underline{145} + \underline{167} + \underline{357}$ (notation used both in [1] and [7]) and that in this case the Steiner basis is not the most efficient to express the form.

It is clear that for any Steiner triple system $\mathrm{Aut}(S) \le \mathrm{Aut}(f_S)$ holds. For the Fano plane we have $|\mathrm{Aut}(S)| = 168$. In [1], the group of automorphisms $\mathrm{Aut}(f_6)$ is computed ($|\mathrm{Aut}(f_6)| = 688128 = 4096 \cdot 168$) and we get $\mathrm{Aut}(\text{Fano Plane}) \neq \mathrm{Aut}(\mathrm{f}_{\text{Fano Plane}})$. Hence there are 4096 Steiner bases.

The radical polynomial of the only Steiner form on dimension 9 is

$$y^9 + 21x^4y^5 + 210x^6y^3 + 280x^8y$$

and is given here just for the sake of completeness.

For n equal to 13 there are two nonisomorphic Steiner triple systems and their radical polynomials are

$$y^{13} + 25x^6y^7 + 476x^8y^5 + 4634x^{10}y^3 + 3056x^{12}y,$$
$$y^{13} + 26x^6y^7 + 442x^8y^5 + 4615x^{10}y^3 + 3108x^{12}y.$$

Steiner triple systems used as an input for the program computing these polynomials (as well as the ones for $n = 15$) were taken from the Loops package of GAP (programmed by Gábor Nagy and Petr Vojtěchovský) and thus the order of the polynomials corresponds to the order of STS in [5], which they used as a source.

There are 80 nonisomorphic STS on 15 points and radical polynomials of these forms are given in Appendix A. They are pairwise different. Current version of the program computes radical polynomial of a 19-dimensional form approximately 30 seconds, using one core of a modern computer[1].

Theorem 3.5. *Let n be at most 15. Then two Steiner triple systems on n points are isomorphic iff their Steiner forms over $GF(2)$ are equivalent.*

Looking at the polynomials in Appendix A one can see that the coefficient at x^2 is precisely the number of Steiner subsystems of order 7 of the corresponding STS. We shall show this fact in general in Theorem 3.12, the proof uses several statements, some of them are trivial or well known.

Lemma 3.6. *Let $S = (X, \mathcal{T})$ be a Steiner triple system on n points and suppose that it contains a subsystem $S' = (X', \mathcal{T}')$ of order $(n-1)/2$. Then*

(1) *for every triple $T \in \mathcal{T}$ we have either $|T \cap X'| = 3$ or $|T \cap X'| = 1$;*
(2) *for every $x \in X'$ the mapping $L_x : y \mapsto x \cdot y$ is a permutation of both X' and $X \setminus X'$;*
(3) *for every $x \in X \setminus X'$ the mapping $L_x : y \mapsto x \cdot y$ is a bijection of X' and $X \setminus (X' \cup \{x\})$.*

Denote by V^+ the $(n-1)$-dimensional subspace generated by vectors $b_i + b_j$, $1 \le i, j \le n$.

Lemma 3.7. *Let f_S be a Steiner form over $GF(2)$ derived from a Steiner triple system $S = (X, \mathcal{T})$ and let B be its Steiner basis. Let $X' \subseteq X$ be a subset of order $(n-1)/2$ such that the restriction of S to S' is a Steiner triple system. Then the vector $v = \sum_{i \notin X'} b_i$ is of rank 2 and $Rad(v) \subseteq V^+$.*

Proof: Put $k = (n-1)/2$ and denote by l_i the linear form the kernel of which is generated by the set $\{b_j, j \neq i\}$. Without loss of generality we can assume that $X' = \{1, \dots, k\}$ and thus $v = \sum_{i=k+1}^{n} b_i$. We show that

$$\text{(2)} \quad \mathrm{Rad}(v) \supseteq \{b_1 + b_2, b_2 + b_3, \dots, b_{k-1} + b_k\} \cup \{b_{k+1} + b_{k+2}, \dots, b_{n-1} + b_n\}.$$

[1] Intel Core i7 920

By symmetry it suffices to prove it for the first and the last member of the set in (2). Using twice Lemma 3.6 we get

$$\begin{aligned} f_S[v, b_1 + b_2] &= \sum_{i=k+1}^{n} (f_S[b_i, b_1] + f_S[b_i, b_2]) \\ &= \sum_{i=k+1}^{n} l_{i\cdot 1} + \sum_{i=k+1}^{n} l_{i\cdot 2} = \sum_{j=k+1}^{n} l_j + \sum_{m=k+1}^{n} l_m \equiv 0. \end{aligned}$$

Similarly, for the last member we get

$$\begin{aligned} f_S[v, b_{n-1} + b_n] &= \sum_{i=k+1}^{n} (f_S[b_i, b_{n-1}] + f_S[b_i, b_n]) \\ &= \sum_{i=k+1}^{n} l_{i\cdot(n-1)} + \sum_{i=k+1}^{n} l_{i\cdot n} = \sum_{\substack{i=k+1 \\ i\neq n-1}}^{n} l_{i\cdot(n-1)} + \sum_{\substack{i=k+1 \\ i\neq n-1}}^{n} l_{i\cdot n} \\ &= \sum_{j=1}^{k} l_j + \sum_{m=1}^{k} l_m \equiv 0, \end{aligned}$$

and thus the dimension of $\mathrm{Rad}(v)$ is at least $n-2$. By Proposition 3.2 the form f_S is nondegenerate and there must be equality in (2). □

Lemma 3.8. *Let g be a bilinear alternating form on V over $GF(2)$ and let B be a basis of V. If g is of rank 2 then B is a disjoint union $B = B_0 \cup B_1 \cup B_2 \cup B_3$ such that for $b, b' \in B$ we have $g(b, b') = 1$ if and only if $b \in B_i$, $b' \in B_j$, $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

Proof: Since $\mathrm{Rad}(g)$ has dimension $n-2$ the intersections of B with the four cosets of $\mathrm{Rad}(g)$ in V satisfy the statement provided that $B_0 = B \cap \mathrm{Rad}(g)$. □

Lemma 3.9. *Let n be an odd integer, $n \geq 1$. Then 3 divides $2n+2$ if and only if $n = 6m+5$.*

Proposition 3.10. *Let $S = (X, \mathcal{T})$ be a Steiner triple system of order n and f_S be the corresponding Steiner form over $GF(2)$ with a Steiner basis B. The rank of a vector v is equal to 2 if and only if $v = \sum_{i\in K} b_i$, where $X \setminus K$ is a Steiner subsystem of S of order $(n-1)/2$.*

Proof: The if part holds by Lemma 3.7 and we shall prove the other direction. Let $v = \sum_{i\in K} b_i$ be a nonzero vector of rank 2 and let k denote the size of K. Consider the bilinear form $f_S[v]$ with respect to the Steiner basis B as a graph G on the set B, where b_i is connected to b_j iff $f_S(v, b_i, b_j) = 1$. By Lemma 3.8 the graph G is a complete tripartite graph with possible isolated vertices in the subset B_0. But since the rank of any basis vector b_i is equal to $n-1$, the size of K is greater than 1 and thus the degree of any vertex is at least 1 and B_0 is empty.

We want to show that G is actually bipartite, so we shall now assume that the sets B_i, $i = 1,2,3$, are nonempty. The degree of a vertex b_i is equal to $k-1$ if $i \in K$ and to k if $i \notin K$. Since vertices in the same part of a complete tripartite (or bipartite) graph have the same degree, there are without loss of generality two possibilities, either $K = B_1$ or $K = B_1 \cup B_2$. First, assume $K = B_1$. The degree of all vertices in $B_2 \cup B_3$ is equal to k and thus we must have $|B_2| = |B_3| = (n-k)/2$. Moreover, this degree (which is k) is equal to $|B_1| + |B_3| = k + (n-k)/2$ implying $k = 0$, a contradiction. Second, assume $K = B_1 \cup B_2$. Again, we have $|B_1| = |B_2| = k/2$. The degree $(k-1)$ of any vertex in B_1 is equal to $|B_2| = |B_3| = k/2 + (n-k)$. This equation yields $k = (2n+2)/3$. By Lemma 3.9 we get $n = 6m+5$, a contradiction with Theorem 3.1.

Thus, G is a bipartite graph. Part K has k elements each with degree $k-1$, part $B \setminus K$ has $n-k$ elements of degree k, from which we get $k-1 = n-k$, equivalently $k = (n+1)/2$. It remains to prove that $X \setminus K$ is a Steiner subsystem of S. Consider b_i and b_j in $X \setminus K$. If $i \cdot j$ is in K then $f_S(v, b_i, b_j) = 1$, a contradiction with b_i and b_j being in the same part of G. □

Corollary 3.11. *Let f be a Steiner form on a vector space V over $GF(2)$. Then the set of vectors of rank at most two is a subspace of V.*

PROOF: Let $B = \{b_1, \ldots, b_n\}$ be a Steiner basis of f. By Proposition 3.10 and Lemma 3.7 the radical of any vector v of rank 2 satisfies $\mathrm{Rad}(v) \subseteq V^+$. Let v_1 and v_2 be distinct vectors of rank 2. Since $f[v_1 + v_2] \not\equiv 0$, the subspaces $\mathrm{Rad}(v_1)$ and $\mathrm{Rad}(v_2)$ must be distinct (we use that the field is $GF(2)$). Thus we have $\dim(\mathrm{Rad}(v_1) + \mathrm{Rad}(v_2)) = \dim V^+ = n-1$, which implies $n - 3 = \dim(\mathrm{Rad}(v_1) \cap \mathrm{Rad}(v_2)) \subseteq \dim(\mathrm{Rad}(v_1 + v_2))$. But the codimension of $\mathrm{Rad}(v_1+v_2)$ is even and we get the result. □

Theorem 3.12. *Let S_1 and S_2 be Steiner triple systems of order n. If they have distinct number of subsystems of order $(n-1)/2$ then the corresponding Steiner forms over $GF(2)$ are nonequivalent.*

PROOF: By Proposition 3.10 there is one-to-one correspondence between the number of Steiner subsystems of order $(n-1)/2$ and vectors of rank 2. □

Whether the other coefficients of radical polynomial can be computed from the underlying Steiner triple system remains an open question.

Appendix A

1. $1x^0 + 15x^2 + 560x^4 + 448x^6 + 15360x^8 + 0x^{10} + 0x^{12} + 16384x^{14}$
2. $1x^0 + 7x^2 + 96x^4 + 568x^6 + 5472x^8 + 10240x^{10} + 0x^{12} + 16384x^{14}$
3. $1x^0 + 3x^2 + 40x^4 + 420x^6 + 3120x^8 + 8704x^{10} + 8192x^{12} + 12288x^{14}$
4. $1x^0 + 3x^2 + 20x^4 + 192x^6 + 2216x^8 + 8320x^{10} + 8704x^{12} + 13312x^{14}$
5. $1x^0 + 3x^2 + 28x^4 + 392x^6 + 2520x^8 + 8832x^{10} + 8704x^{12} + 12288x^{14}$
6. $1x^0 + 3x^2 + 4x^4 + 140x^6 + 1324x^8 + 7104x^{10} + 12928x^{12} + 11264x^{14}$
7. $1x^0 + 3x^2 + 4x^4 + 480x^6 + 2008x^8 + 6976x^{10} + 15104x^{12} + 8192x^{14}$

8. $1x^0 + 1x^2 + 14x^4 + 144x^6 + 1592x^8 + 6248x^{10} + 13248x^{12} + 11520x^{14}$
9. $1x^0 + 1x^2 + 8x^4 + 56x^6 + 950x^8 + 5320x^{10} + 14912x^{12} + 11520x^{14}$
10. $1x^0 + 1x^2 + 8x^4 + 90x^6 + 1012x^8 + 5736x^{10} + 14912x^{12} + 11008x^{14}$
11. $1x^0 + 1x^2 + 2x^4 + 30x^6 + 446x^8 + 4196x^{10} + 17900x^{12} + 10192x^{14}$
12. $1x^0 + 1x^2 + 6x^4 + 48x^6 + 719x^8 + 5313x^{10} + 15672x^{12} + 11008x^{14}$
13. $1x^0 + 1x^2 + 8x^4 + 126x^6 + 1344x^8 + 5304x^{10} + 14208x^{12} + 11776x^{14}$
14. $1x^0 + 1x^2 + 10x^4 + 120x^6 + 1452x^8 + 5456x^{10} + 13696x^{12} + 12032x^{14}$
15. $1x^0 + 1x^2 + 4x^4 + 68x^6 + 834x^8 + 4916x^{10} + 16576x^{12} + 10368x^{14}$
16. $1x^0 + 1x^2 + 28x^4 + 266x^6 + 2312x^8 + 7504x^{10} + 9856x^{12} + 12800x^{14}$
17. $1x^0 + 1x^2 + 4x^4 + 150x^6 + 1292x^8 + 5112x^{10} + 16480x^{12} + 9728x^{14}$
18. $1x^0 + 1x^2 + 4x^4 + 54x^6 + 820x^8 + 4624x^{10} + 16384x^{12} + 10880x^{14}$
19. $1x^0 + 1x^2 + 0x^4 + 44x^6 + 302x^8 + 4148x^{10} + 18416x^{12} + 9856x^{14}$
20. $1x^0 + 1x^2 + 0x^4 + 24x^6 + 310x^8 + 3628x^{10} + 17956x^{12} + 10848x^{14}$
21. $1x^0 + 1x^2 + 0x^4 + 0x^6 + 251x^8 + 2975x^{10} + 19292x^{12} + 10248x^{14}$
22. $1x^0 + 1x^2 + 0x^4 + 0x^6 + 205x^8 + 2883x^{10} + 19150x^{12} + 10528x^{14}$
23. $1x^0 + 0x^2 + 4x^4 + 21x^6 + 332x^8 + 3367x^{10} + 18103x^{12} + 10940x^{14}$
24. $1x^0 + 0x^2 + 4x^4 + 13x^6 + 292x^8 + 3110x^{10} + 17996x^{12} + 11352x^{14}$
25. $1x^0 + 0x^2 + 4x^4 + 25x^6 + 445x^8 + 3333x^{10} + 17856x^{12} + 11104x^{14}$
26. $1x^0 + 0x^2 + 5x^4 + 32x^6 + 513x^8 + 3760x^{10} + 17389x^{12} + 11068x^{14}$
27. $1x^0 + 0x^2 + 2x^4 + 15x^6 + 241x^8 + 2748x^{10} + 18841x^{12} + 10920x^{14}$
28. $1x^0 + 0x^2 + 2x^4 + 13x^6 + 222x^8 + 2668x^{10} + 18578x^{12} + 11284x^{14}$
29. $1x^0 + 0x^2 + 4x^4 + 18x^6 + 381x^8 + 3145x^{10} + 17823x^{12} + 11396x^{14}$
30. $1x^0 + 0x^2 + 2x^4 + 7x^6 + 200x^8 + 2370x^{10} + 18884x^{12} + 11304x^{14}$
31. $1x^0 + 0x^2 + 4x^4 + 24x^6 + 376x^8 + 3423x^{10} + 17988x^{12} + 10952x^{14}$
32. $1x^0 + 0x^2 + 2x^4 + 6x^6 + 173x^8 + 2338x^{10} + 18716x^{12} + 11532x^{14}$
33. $1x^0 + 0x^2 + 1x^4 + 8x^6 + 133x^8 + 2090x^{10} + 18699x^{12} + 11836x^{14}$
34. $1x^0 + 0x^2 + 1x^4 + 9x^6 + 150x^8 + 2182x^{10} + 18973x^{12} + 11452x^{14}$
35. $1x^0 + 0x^2 + 1x^4 + 6x^6 + 146x^8 + 2000x^{10} + 19158x^{12} + 11456x^{14}$
36. $1x^0 + 0x^2 + 0x^4 + 6x^6 + 144x^8 + 1845x^{10} + 18748x^{12} + 12024x^{14}$
37. $1x^0 + 0x^2 + 0x^4 + 0x^6 + 102x^8 + 1981x^{10} + 19012x^{12} + 11672x^{14}$
38. $1x^0 + 0x^2 + 0x^4 + 5x^6 + 105x^8 + 1561x^{10} + 19196x^{12} + 11900x^{14}$
39. $1x^0 + 0x^2 + 1x^4 + 2x^6 + 138x^8 + 1758x^{10} + 18916x^{12} + 11952x^{14}$
40. $1x^0 + 0x^2 + 1x^4 + 4x^6 + 163x^8 + 1989x^{10} + 18774x^{12} + 11836x^{14}$
41. $1x^0 + 0x^2 + 1x^4 + 5x^6 + 162x^8 + 1991x^{10} + 18924x^{12} + 11684x^{14}$
42. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 105x^8 + 1660x^{10} + 18901x^{12} + 12100x^{14}$
43. $1x^0 + 0x^2 + 0x^4 + 4x^6 + 147x^8 + 1721x^{10} + 18915x^{12} + 11980x^{14}$
44. $1x^0 + 0x^2 + 0x^4 + 4x^6 + 101x^8 + 1628x^{10} + 18778x^{12} + 12256x^{14}$
45. $1x^0 + 0x^2 + 0x^4 + 4x^6 + 102x^8 + 1630x^{10} + 19055x^{12} + 11976x^{14}$
46. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 87x^8 + 1534x^{10} + 19040x^{12} + 12104x^{14}$
47. $1x^0 + 0x^2 + 1x^4 + 0x^6 + 113x^8 + 1835x^{10} + 18806x^{12} + 12012x^{14}$
48. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 94x^8 + 1582x^{10} + 18765x^{12} + 12324x^{14}$
49. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 88x^8 + 1394x^{10} + 18740x^{12} + 12544x^{14}$
50. $1x^0 + 0x^2 + 0x^4 + 4x^6 + 95x^8 + 1707x^{10} + 18985x^{12} + 11976x^{14}$
51. $1x^0 + 0x^2 + 0x^4 + 3x^6 + 111x^8 + 1568x^{10} + 18925x^{12} + 12160x^{14}$

52. $1x^0 + 0x^2 + 0x^4 + 3x^6 + 107x^8 + 1559x^{10} + 18942x^{12} + 12156x^{14}$
53. $1x^0 + 0x^2 + 1x^4 + 2x^6 + 125x^8 + 1817x^{10} + 19090x^{12} + 11732x^{14}$
54. $1x^0 + 0x^2 + 1x^4 + 1x^6 + 131x^8 + 1828x^{10} + 18966x^{12} + 11840x^{14}$
55. $1x^0 + 0x^2 + 0x^4 + 3x^6 + 107x^8 + 1605x^{10} + 18852x^{12} + 12200x^{14}$
56. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 110x^8 + 1547x^{10} + 18968x^{12} + 12140x^{14}$
57. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 83x^8 + 1500x^{10} + 18847x^{12} + 12336x^{14}$
58. $1x^0 + 0x^2 + 1x^4 + 1x^6 + 109x^8 + 1755x^{10} + 18869x^{12} + 12032x^{14}$
59. $1x^0 + 0x^2 + 1x^4 + 3x^6 + 147x^8 + 2000x^{10} + 18888x^{12} + 11728x^{14}$
60. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 108x^8 + 1707x^{10} + 19134x^{12} + 11816x^{14}$
61. $1x^0 + 1x^2 + 0x^4 + 0x^6 + 142x^8 + 2604x^{10} + 19796x^{12} + 10224x^{14}$
62. $1x^0 + 0x^2 + 1x^4 + 0x^6 + 106x^8 + 1721x^{10} + 18935x^{12} + 12004x^{14}$
63. $1x^0 + 0x^2 + 1x^4 + 7x^6 + 124x^8 + 1938x^{10} + 19145x^{12} + 11552x^{14}$
64. $1x^0 + 0x^2 + 1x^4 + 3x^6 + 107x^8 + 1794x^{10} + 19078x^{12} + 11784x^{14}$
65. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 91x^8 + 1482x^{10} + 18993x^{12} + 12200x^{14}$
66. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 81x^8 + 1575x^{10} + 19101x^{12} + 12008x^{14}$
67. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 78x^8 + 1624x^{10} + 19059x^{12} + 12004x^{14}$
68. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 89x^8 + 1459x^{10} + 19010x^{12} + 12208x^{14}$
69. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 79x^8 + 1506x^{10} + 19012x^{12} + 12168x^{14}$
70. $1x^0 + 0x^2 + 0x^4 + 5x^6 + 103x^8 + 1624x^{10} + 18867x^{12} + 12168x^{14}$
71. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 78x^8 + 1478x^{10} + 19006x^{12} + 12204x^{14}$
72. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 83x^8 + 1538x^{10} + 18881x^{12} + 12264x^{14}$
73. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 82x^8 + 1575x^{10} + 19276x^{12} + 11832x^{14}$
74. $1x^0 + 0x^2 + 0x^4 + 4x^6 + 101x^8 + 1790x^{10} + 18808x^{12} + 12064x^{14}$
75. $1x^0 + 0x^2 + 0x^4 + 3x^6 + 86x^8 + 1571x^{10} + 18847x^{12} + 12260x^{14}$
76. $1x^0 + 0x^2 + 0x^4 + 10x^6 + 120x^8 + 1907x^{10} + 18630x^{12} + 12100x^{14}$
77. $1x^0 + 0x^2 + 0x^4 + 1x^6 + 67x^8 + 1513x^{10} + 18978x^{12} + 12208x^{14}$
78. $1x^0 + 0x^2 + 0x^4 + 2x^6 + 74x^8 + 1465x^{10} + 18650x^{12} + 12576x^{14}$
79. $1x^0 + 0x^2 + 0x^4 + 6x^6 + 80x^8 + 1783x^{10} + 18722x^{12} + 12176x^{14}$
80. $1x^0 + 0x^2 + 0x^4 + 0x^6 + 45x^8 + 870x^{10} + 19100x^{12} + 12752x^{14}$

References

[1] Cohen A.M., Helminck A.G., *Trilinear alternating forms on a vector space of dimension* 7, Comm. Algebra **16** (1988), no. 1, 1–25.

[2] Gurevich G.B., *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff Ltd., Groningen, 1964.

[3] Djokovic D., *Classification of trivectors of an eight dimensional real vector space*, Linear Multilinear Algebra **13** (1983), no. 3, 3–39.

[4] Noui L., *Transvecteur de rang* 8 *sur un corps algébriquement clos*, C.R. Acad. Sci. Paris Sér. I Math. **324** (1997), 611–614.

[5] Colbourn C.J., Rosa A., *Triple Systems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999.

[6] Hora J., *Orthogonal decompositions and canonical embeddings of multilinear alternating forms*, Linear Multilinear Algebra **52** (2004), no. 2, 121–132.

[7] Hora J., Pudlák P., *Classification of* 8*-dimensional trilinear alternating forms over* $GF(2)$, Comm. Algebra **43** (2015), no. 8, 3459–3471.

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(*Received* February 16, 2016*, revised* May 11, 2016)