Kybernetika

Juan Fernández-Sánchez; Manuel Úbeda-Flores

Proving the characterization of Archimedean copulas via Dini derivatives

Kybernetika, Vol. 52 (2016), No. 5, 785–790

Persistent URL: http://dml.cz/dmlcz/145968

KYBERNETIKA — VOLUME *52* (2016), NUMBER 5, PAGES 785–790

PROVING THE CHARACTERIZATION OF ARCHIMEDEAN COPULAS VIA DINI DERIVATIVES

Juan Fernández-Sánchez and Manuel Úbeda-Flores

In this note we prove the characterization of the class of Archimedean copulas by using Dini derivatives.

Keywords: Archimedean copula, derived number, Dini derivative

Classification: 60E05, 62E10

1. INTRODUCTION

Copulas are n-dimensional distribution functions that concentrate the probability mass on $[0,1]^n$ and whose univariate margins are uniformly distributed on $[0,1]$. A (bivariate) *copula* is a function $C\colon [0,1]^2 \longrightarrow [0,1]$ which satisfies:

(C1) the boundary conditions $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in [0,1]$;

(C2) the *2-increasing property*, i. e., $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for all u_1,u_2,v_1,v_2 in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

In particular, copulas are Lipschitz continuous functions in each variable with constant 1.

The importance of copulas comes from *Sklar's Theorem* [17], which shows that the joint distribution H of a pair of random variables and the corresponding marginal distributions F and G are linked by a copula C in the following manner: $H(x,y) = C(F(x),G(y))$ for all x,y in $[-\infty,\infty]$. If F and G are continuous, then the copula is unique; otherwise, the copula is uniquely determined on $\operatorname{Range} F \times \operatorname{Range} G$ [2]. For a complete review on copulas and some of their applications, we refer to [6, 9, 15].

Let $\varphi\colon [0,1] \longrightarrow [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the *pseudo-inverse* of φ, i. e., $\varphi^{[-1]}(x) = \varphi^{-1}(\min(\varphi(0),x))$ for $x \in [0,\infty]$, and consider the function given by

$$C_\varphi(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v)), \quad (u,v) \in [0,1]^2. \tag{1}$$

The following result provides a characterization of the function given by (1) to be a copula [16].

DOI: 10.14736/kyb-2016-5-0785

Theorem 1.1. The function C_φ given by (1) is a copula if, and only if, φ is convex.

Copulas given by (1) are called *Archimedean* – the name is due to a property of associative operations [10, 15] – and φ is the *generator* of C_φ – for another different characterization of Archimedean copulas, see [10]. Archimedean copulas became popular since they model the dependence structure between risk factors, and are used in many applications, such as finance, insurance, or reliability (see, for example, [4, 13]) due to their simple forms and nice properties.

In [18], the author provides three characterizations of n-dimensional Archimedean copulas: algebraic, differential and diagonal. Our purpose in this note is to provide a new proof of Theorem 1.1 (Section 3) by using Dini derivatives, a known result of Lebesgue from Real Analysis (Section 2) – which allow to reconstruct a function from the Dini derivative D^+f when this is finite – and a characterization of copulas given by Jaworski and Durante [5].

2. PRELIMINARY RESULTS FROM REAL ANALYSIS

Derived numbers play an important role in several results on the differentiability of monotone functions. We recall their definition [14].

Definition 2.1. The number λ (finite or infinite) is said to be a *derived number* of the function f at the point x_0 if there exists a sequence $h_1, h_2, h_3, \ldots$ ($h_n \neq 0$ for all n) such that $h_n \to 0$ and

$$\lim_{n\to\infty} \frac{f(x_0+h_n)-f(x_0)}{h_n} = \lambda.$$

Symbolically, we say $\lambda = Df(x_0)$. If the (finite or infinite) derivative $f'(x_0)$ exists at the point x_0, then it will be a derived number $Df(x_0)$, and in this case, the function f will have no other derived numbers at the point x_0.

We note that in Definition 2.1 it is possible to use the term *derived number to the right* by imposing $h_n > 0$.

There are some particularly important derived numbers, the Dini derivatives, whose definition we recall now [11].

Definition 2.2. Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function, with $a < b$, and let x be a point in $[a,b[$. The limit

$$D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

is called the (*rightside upper*) *Dini derivative* of f at x. When it is substituted lim sup by lim inf, we obtain the (*rightside lower*) Dini derivative D_+f.

The following result provides conditions for which a function can be recovered as a definite integral of one of its Dini derivatives [8].

Lemma 2.3. If f is a continuous function that has a finite Dini derivative $D^+f(x)$ at every point x of $\mathbb{R}$, then

$$f(b) - f(a) = \int_a^b D^+f(x)\,\mathrm{d}x$$

for each interval $[a, b]$.

Observe that, as an immediate consequence of Lemma 2.3, we have that if $D^+f \equiv 0$ then f is constant.

We understand a *strictly increasing (decreasing) singular function* as a continuous and strictly increasing (decreasing) function with derivative zero almost everywhere. Since the Dini derivatives of a decreasing function cannot be positive, Lemma 2.3 implies that a strictly decreasing singular function on an interval has a dense set of points in which D^+f is equal to $-\infty$. Both Lemma 2.3 and these last observations remain true if we replace D^+f by D_+f. Furthermore, we have the following lemma [3].

Lemma 2.4. If f is a strictly singular function, then the inverse f^{-1} is also strictly singular.

3. A NEW PROOF OF THE CHARACTERIZATION OF ARCHIMEDEAN COPULAS

We begin this section with some additional notation. For every function $K\colon [x_1, x_2] \times [y_1, y_2] \longrightarrow \mathbb{R}$ and every $y \in [y_1, y_2]$, let K_y denote the function from $[x_1, x_2]$ onto $\mathbb{R}$ given by $K_y(x) = K(x, y)$.

The following result – whose proof can be found in [5] – provides a characterization of copulas in terms of Dini derivatives.

Lemma 3.1. A function $C\colon [0, 1]^2 \longrightarrow [0, 1]$ is a copula if, and only if, C satisfies (C1) and the following conditions:

1. C is continuous;
2. there exists a countable set $S \subset [0, 1]$ such that, for every $u \in [0, 1]\backslash S$, the following conditions hold:
   (a) $D^+C_v(u)$ is finite for every $v \in [0, 1]$;
   (b) $D^+C_{v_1}(u) \leq D^+C_{v_2}(u)$ whenever $0 \leq v_1 \leq v_2 \leq 1$.

We are now in position to provide a new proof of Theorem 1.1 by using Dini derivatives and derived numbers – compare, for example, with the ones given in [1, 7, 12].

P r o o f of Theorem 1.1. Suppose C_φ is a copula given by (1). Since φ is monotone, we have that φ is derivable almost everywhere. Let $u \in]0, 1[$ be a point such that $\varphi'(u)$ exists and $\varphi'(u) \neq 0$, and suppose $(C_\varphi)_v(u) \neq 0$. By writing

$$\frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{h} = \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi(u+h) - \varphi(u)} \cdot \frac{\varphi(u+h) - \varphi(u)}{h},$$

and by taking supremum limits when $h \to 0^+$ in both sides, the existence of the derivative of φ in u assures

$$D^+ \left(C_\varphi\right)_v (u) = \lambda \cdot \varphi'(u),$$

where λ is the inverse of a derived number of φ at $\left(C_\varphi\right)_v (u)$. To be precise, with $\tilde{h} := \left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)$, we get $\tilde{h} \to 0^+$ for $h \to 0^+$ – as C_φ is non-decreasing – and

$$\begin{aligned}
\lambda &= \limsup_{h\to 0^+} \frac{\left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)}{\varphi(u+h)-\varphi(u)} = \limsup_{h\to 0^+} \frac{\left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)}{\varphi(u+h)+\varphi(v)-[\varphi(u)+\varphi(v)]} \\
&= \limsup_{h\to 0^+} \frac{\left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)}{\varphi\left(\left(C_\varphi\right)_v (u+h)\right) - \varphi\left(\left(C_\varphi\right)_v (u)\right)} \\
&= \limsup_{h\to 0^+} \frac{\left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)}{\varphi\left(\left(C_\varphi\right)_v (u) + \left(C_\varphi\right)_v (u+h) - \left(C_\varphi\right)_v (u)\right) - \varphi\left(\left(C_\varphi\right)_v (u)\right)} \\
&= \limsup_{\tilde{h}\to 0^+} \frac{\tilde{h}}{\varphi\left(\left(C_\varphi\right)_v (u) + \tilde{h}\right) - \varphi\left(\left(C_\varphi\right)_v (u)\right)} \\
&= \frac{1}{\liminf\limits_{\tilde{h}\to 0^+} \dfrac{\varphi\left(\left(C_\varphi\right)_v (u) + \tilde{h}\right) - \varphi\left(\left(C_\varphi\right)_v (u)\right)}{\tilde{h}}} = \frac{1}{D_+\varphi\left(\left(C_\varphi\right)_v (u)\right)}.
\end{aligned}$$

From Lemma 3.1, we have $D^+ \left(C_\varphi\right)_{v_1} (u) \le D^+ \left(C_\varphi\right)_{v_2} (u)$ as long as $0 \le v_1 \le v_2 \le 1$, which implies

$$\frac{\varphi'(u)}{D_+\varphi\left(\left(C_\varphi\right)_{v_1} (u)\right)} \le \frac{\varphi'(u)}{D_+\varphi\left(\left(C_\varphi\right)_{v_2} (u)\right)},$$

and therefore $D_+\varphi\left(\left(C_\varphi\right)_{v_1} (u)\right) \le D_+\varphi\left(\left(C_\varphi\right)_{v_2} (u)\right)$ – since φ is strictly decreasing – i. e. $D_+\varphi$ is increasing in $]0,u[$.

We now prove that there exists a sequence $\{u_n\} \to 1$ as $n \to +\infty$ such that $\varphi'(u_n) \ne 0$ for every n. Suppose, on the contrary, this is not true, that is, we have $\varphi'(u) = 0$ almost everywhere in an interval $[a_0, 1] \subset [0,1]$ and φ is not derivable in the rest of the points, i. e. φ is a strictly decreasing singular function. From Lemma 2.4, we have that φ^{-1} is a strictly decreasing singular function in $[0, \varphi(a_0)]$. In this case, there exists a set of real points $\{x_n : n \in \mathbb{N}\}$ such that $\{x_n\} \to 0$ as $n \to +\infty$ with $D_+\left(\varphi^{-1}\right)(x_n) = -\infty$.

Now, let x be a real point such that $D_+\left(\varphi^{-1}\right)(x) = -\infty$, and let u and v be two real points such that $\varphi(u)+\varphi(v) = x$, with u such that any derived number to the right of φ at u is different from 0 – we note that the existence of u and v is due to the continuity of φ and as a consequence of the fact that the derived numbers to the right cannot be greater than $D_+\varphi$.

Since $\left(C_\varphi\right)_v$ verifies the Lipschitz condition with constant 1, we have

$$\beta \cdot D_+\varphi^{-1}\left(\varphi(u)+\varphi(v)\right) \le 1,$$

where β is a derived number to the right of φ at u. Since $D_+\varphi^{-1}\left(\varphi(u)+\varphi(v)\right) = -\infty$ and $\beta < 0$, that upper bound is not possible, so we obtain a contradiction; therefore, there exists a sequence $\{u_n\} \to 1$ as $n \to +\infty$ such that $\varphi'(u_n) \ne 0$ for every n.

All this reasoning leads to the fact that $D_+\varphi$ is non-decreasing in $]0,1[$. Therefore, if s and s' are two numbers in $[0,1]$ such that $s > s'$, from Lemma 2.3 we have

$$\begin{aligned}\frac{\varphi(s)+\varphi(s')}{2} - \varphi\left(\frac{s+s'}{2}\right) &= \frac{1}{2}\left[\varphi(s) - \varphi\left(\frac{s+s'}{2}\right)\right] - \frac{1}{2}\left[\varphi\left(\frac{s+s'}{2}\right) - \varphi(s')\right] \\ &= \frac{1}{2}\left(\int_{\frac{s+s'}{2}}^{s} D_+\varphi(t)\,\mathrm{d}t - \int_{s'}^{\frac{s+s'}{2}} D_+\varphi(t)\,\mathrm{d}t\right) \geq 0,\end{aligned}$$

and we conclude that φ is convex.

Conversely, we only need to follow the same steps backwards, which completes the proof. □

ACKNOWLEDGEMENT

The authors acknowledge the support by the Ministerio de Economía y Competitividad (Spain) under research project MTM2014-60594-P, and two anonymous referees for helpful comments.

(Received July 3, 2016)

REFERENCES

[1] C. Alsina, M. J. Frank, and B. Schweizer: Associative Functions: Triangular Norms and Copulas. World Scientific, Singapore 2006. DOI:10.1142/9789812774200

[2] E. de Amo, M. Díaz Carrillo, and J. Fernández Sánchez: Characterization of all copulas associated with non-continuous random variables. Fuzzy Sets Syst. *191* (2012), 103–112. DOI:10.1016/j.fss.2011.10.005

[3] L. Berg and M. Krüppel: De Rahm's singular function and related functions. Z. Anal. Anw. *19* (2000), 227–237. DOI:10.4171/zaa/947

[4] U. Cherubini, E. Luciano, and W. Vecchiato: Copula Methods in Finance. Wiley Finance Series, John Wiley and Sons Ltd., Chichester 2004. DOI:10.1002/9781118673331

[5] F. Durante and P. Jaworski: A new characterization of bivariate copulas. Comm. Statist. Theory Methods *39* (2010), 2901–2912. DOI:10.1080/03610920903151459

[6] F. Durante and C. Sempi: Principles of Copula Theory. Chapman and Hall/CRC, London 2015. DOI:10.1201/b18674

[7] C. Genest and J. MacKay: Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. Canad. J. Statist. *14* (1986), 145–159. DOI:10.2307/3314660

[8] J. W. Hagood and B. S. Thomson: Recovering a function from a Dini derivative. Amer. Math. Monthly *113* (2006), 34–46. DOI:10.2307/27641835

[9] P. Jaworski, F. Durante, W. Härdle, and T. Rychlik (editors): Copula Theory and its Applications. Lecture Notes in Statistics–Proceedings, Springer, Berlin–Heidelberg 2010. DOI:10.1007/978-3-642-12465-5

[10] C. H. Ling: Representation of associative functions. Publ. Math. Debrecen *12* (1965), 189–212.

[11] S. Łojasiewicz: An Introduction to the Theory of Real Functions. Third Edition. A Wiley-Interscience Publication, John Wiley and Sons Ltd., Chichester 1988.

[12] A. J. McNeil and J. Nešlehová: Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. Ann. Stat. *37* (2009), 3059–3097. DOI:10.1214/07-aos556

[13] A. J. McNeil, R. Frey, and P. Embrechts: Quantitative Risk Management: Concepts, Techniques, and Tools. Princeton University Press, Princeton 2005.

[14] L. P. Natanson: Theory of Functions of a Real Variable. Vol. I, revised edition. Frederick Ungar Publishing, New York 1961.

[15] R. B. Nelsen: An Introduction to Copulas. Second Edition. Springer, New York 2006. DOI:10.1007/0-387-28678-0

[16] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. North-Holland, New York 1983. Reprinted, Dover, Mineola NY, 2005.

[17] A. Sklar: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris *8* (1959), 229–231.

[18] W. Wysocki: Characterizations of Archimedean n-copulas. Kybernetika *51* (2015), 212–230. DOI:10.14736/kyb-2015-2-0212

Juan Fernández-Sánchez, Research Group of Mathematical Analysis, University of Almería, Carretera de Sacramento s/n, 04120 Almería. Spain.
e-mail: juanfernandez@ual.es

Manuel Úbeda-Flores, Department of Mathematics, University of Almería, Carretera de Sacramento s/n, 04120 Almería. Spain.
e-mail: mubeda@ual.es