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DEFECTS AND TRANSFORMATIONS OF QUASI-COPULAS

MICHAL DIBALA, SUSANNE SAMINGER-PLATZ, RADKO MESIAR, AND ERICH PETER KLEMENT

Six different functions measuring the defect of a quasi-copula, i.e., how far away it is from a copula, are discussed. This is done by means of extremal non-positive volumes of specific rectangles (in a way that a zero defect characterizes copulas). Based on these defect functions, six transformations of quasi-copulas are investigated which give rise to six different partitions of the set of all quasi-copulas. For each of these partitions, each equivalence class contains exactly one copula being a fixed point of the transformation under consideration. Finally, an application to the construction of so-called imprecise copulas is given.

Keywords: copula, quasi-copula, transformation of quasi-copulas, imprecise copula

Classification: 26B25, 62E10, 26B35, 60E05, 62H10

1. INTRODUCTION

Copulas were introduced in [34] (see also [1, 14, 27, 33, 35]) in order to represent and construct joint distribution functions of random vectors by means of the related one-dimensional marginal distribution functions. As a more general concept, quasi-copulas were introduced in [2] and later characterized by means of their 1-Lipschitz property (with respect to the $L_1$-norm) in [16].

Quasi-copulas have interesting applications in several areas, such as fuzzy logic [18, 31], fuzzy preference modeling [9, 10] or similarity measures [8]. Other deep results concerning quasi-copulas can be found in [6, 19, 28].

While copulas are characterized by the non-negativity of the volume of each sub-rectangle of $[0, 1]^2$ which is a Cartesian product of two subintervals of $[0, 1]$, this is no more true for quasi-copulas. This defect of quasi-copulas can be described in several ways, indicating how far away they are from copulas. We introduce several such descriptions and apply them to transform the original quasi-copulas. Note that the sequence of iterative transformations always converges to a copula. This allows us to introduce an equivalence relation on the set of quasi-copulas by grouping quasi-copulas converging to the same copula into an equivalence class. An interesting application of our approach to the so-called imprecise copulas [25, 26, 30, 36] will also be given.

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The paper is organized as follows. In Section 2 some preliminary notions and examples concerning copulas and quasi-copulas are given. Six types of functions induced by quasi-copulas and characterizing their defects are introduced and discussed in Section 3. In Section 4, the corresponding transformations of quasi-copulas are studied. The concept of imprecise copulas is recalled, and their relations to the transformations in Section 4 are shown in Section 5.

2. COPULAS AND QUASI-COPULAS

In this paper, we restrict ourselves to the case of real functions defined on the unit square \([0, 1]^2\). Therefore, there is no need to use adjectives like binary, 2-dimensional or bivariate, even if, e.g., for copulas, multivariate generalizations exist.

**Definition 2.1.** A function \(C : [0, 1] \to [0, 1]\) is called a **copula** if the following conditions are satisfied:

(i) \(C\) is grounded, i.e., we have \(C(0, x) = C(x, 0) = 0\) for all \(x \in [0, 1]\);

(ii) \(1\) is a neutral element of \(C\), i.e., we have \(C(x, 1) = C(1, x) = x\) for all \(x \in [0, 1]\);

(iii) \(C\) is 2-increasing, i.e., for each rectangle \([a, b] \times [c, d] \subseteq [0, 1]^2\) we have

\[
V_C([a, b] \times [c, d]) = C(a, c) + C(b, d) - C(a, d) - C(b, c) \geq 0.
\] (2.1)

The set of all copulas will be denoted by \(\mathcal{C}\). Note that for each copula \(C\) we have \(W \leq C \leq M\), where the Fréchet–Hoeffding lower and upper bound \(W\) and \(M\) are given by \(W(x, y) = \max(x + y - 1, 0)\) and \(M(x, y) = x \wedge y\), respectively, and where the order on \(\mathcal{C}\) is the pointwise partial order inherited from the linear order on \([0, 1]\). This means that the partially ordered set \(\mathcal{C}\) has \(M\) as top element and \(W\) as bottom element, but \(\mathcal{C}\) is not a lattice since the supremum of two copulas is not necessarily a copula (see, e.g., \(C_1 \cup C_2\) in Example 2.3 below). For more details about copulas and their applications see [14, 27].

The value \(V_C([a, b] \times [c, d])\) given by (2.1) is called the **C-volume** of the rectangle \([a, b] \times [c, d]\). Observe that it formally can be defined for each function \(F : \mathbb{R}^2 \to \mathbb{R}\) and each rectangle \([a, b] \times [c, d] \subseteq \mathbb{R}^2\).

**Definition 2.2.** A function \(Q : [0, 1]^2 \to [0, 1]\) is called a **quasi-copula** if it satisfies conditions (i) and (ii) in Definition 2.1 and inequality (2.1) for all rectangles \([a, b] \times [c, d] \subseteq [0, 1]^2\) such that \(\{a, b, c, d\} \cap \{0, 1\} \neq \emptyset\).

The set of all quasi-copulas will be denoted by \(\mathcal{Q}\) and, evidently, we have \(\mathcal{C} \subseteq \mathcal{Q}\). Quasi-copulas which are not copulas, i.e., elements of \(\mathcal{Q} \setminus \mathcal{C}\), are called **proper quasi-copulas** (see, for instance, Example 2.4). From Definitions 2.1 and 2.2 it follows that they have a negative volume for some rectangle \([a, b] \times [c, d] \subseteq [0, 1]^2\).

Observe that a function \(Q : [0, 1]^2 \to [0, 1]\) is a quasi-copula if and only if it is monotone non-decreasing in each coordinate, grounded, has \(1\) as neutral element and is \(1\)-Lipschitz, i.e., for all \((x_1, x_2), (y_1, y_2) \in [0, 1]^2\) we have

\[
|Q(x_1, x_2) - Q(y_1, y_2)| \leq |x_1 - y_1| + |x_2 - y_2|.
\] (2.2)
From $\mathcal{C} \subseteq \mathcal{Q}$ it follows that also each copula is non-decreasing in each coordinate and 1-Lipschitz.

From a lattice-theoretic point of view, $\mathcal{Q}$ is the smallest complete lattice containing $\mathcal{C}$, i.e., for each quasi-copula $Q$ we have $Q = \inf\{C_i \mid i \in I\} = \sup\{C_j \mid j \in J\}$ for some families of copulas $(C_i)_{i \in I}$ and $(C_j)_{j \in J}$ (in fact, $\mathcal{Q}$ was shown in [29] to be order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{C}$). In particular, for each family of copulas both its infimum and its supremum are quasi-copulas, and the Fréchet–Hoeffding bounds $M$ and $W$ are the top and the bottom element of $\mathcal{Q}$. For more details on quasi-copulas see [16] and [17].

**Example 2.3.** Consider the copulas $C_1$ and $C_2$ illustrated in Figure 1 (top). The unit mass of copula $C_1$ is uniformly distributed on the line segments connecting the points $(0, \frac{1}{3})$ and $(\frac{2}{3}, 1)$, and $(\frac{2}{3}, 0)$ and $(1, \frac{1}{3})$, respectively. The unit mass of copula $C_2$ is uniformly distributed on the line segments connecting the points $(0, \frac{2}{3})$ and $(\frac{1}{3}, 1)$, and $(\frac{1}{3}, 0)$ and $(1, \frac{2}{3})$, respectively. In Figure 1 also the functions $C_1 \lor C_2$ and $C_1 \land C_2$ are visualized. Observe that $C_1 \lor C_2$ is a proper quasi-copula since, e.g., $V_{C_1 \lor C_2}(\left[\frac{1}{5}, \frac{2}{5}\right]^2) = -\frac{4}{9}$, while $C_1 \land C_2$ is a copula.
Example 2.4. The function $Q$ given by

$$Q(x, y) = \text{med}(x \cdot y, \frac{1}{2}, x + y - 1) = \begin{cases} x \cdot y & \text{if } x \cdot y \leq \frac{1}{2}, \\ \max(x + y - 1, \frac{1}{2}) & \text{otherwise,} \end{cases} \quad (2.3)$$

where med is the shortcut for the median, is a proper quasi-copula, and it is visualized in Figure 2. Observe that the minimal value of the $Q$-volume of a rectangle in $[0, 1]^2$ is attained for the square $[\frac{2}{3}, \frac{3}{4}]^2$ where $V_Q([\frac{2}{3}, \frac{3}{4}]^2) = -\frac{1}{18}$.

3. DEFECTS OF QUASI-COPULAS

Consider an arbitrary point $(x_0, y_0) \in [0, 1]^2$. Then it is clear that each rectangle $[a, b] \times [c, d] \subseteq [0, 1]^2$ (i.e., its edges are parallel to the axes of the unit square) which has $(x_0, y_0)$ as one of its vertices belongs (with the exception of line segments, i.e., when $a = b = x_0$ or $c = d = y_0$, or of the trivial rectangle consisting of the point $(x_0, y_0)$ only, i.e., when $a = b = x_0$ and $c = d = y_0$) to exactly one of the following sets:

$$\mathcal{R}_\uparrow(x_0, y_0) = \{[x_0, x_0 + \alpha] \times [y_0, y_0 + \beta] \subseteq [0, 1]^2 \mid \alpha, \beta \geq 0\},$$
$$\mathcal{R}_\downarrow(x_0, y_0) = \{[x_0, x_0 + \alpha] \times [y_0 - \beta, y_0] \subseteq [0, 1]^2 \mid \alpha, \beta \geq 0\},$$
$$\mathcal{R}_\leftarrow(x_0, y_0) = \{[x_0 - \alpha, x_0] \times [y_0 - \beta, y_0] \subseteq [0, 1]^2 \mid \alpha, \beta \geq 0\},$$
$$\mathcal{R}_\rightarrow(x_0, y_0) = \{[x_0 - \alpha, x_0] \times [y_0, y_0 + \beta] \subseteq [0, 1]^2 \mid \alpha, \beta \geq 0\}.$$
Definition 3.1. Let \( Q : [0, 1]^2 \to [0, 1] \) be a quasi-copula. Then we consider the following defect functions \( D_{\uparrow\uparrow}^Q, D_{\downarrow\uparrow}^Q, D_{\downarrow\downarrow}^Q, D_{\uparrow\downarrow}^Q : [0, 1]^2 \to \mathbb{R} \) given by

\[
D_{\uparrow\uparrow}^Q(x, y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\uparrow\uparrow}(x, y)\}, \\
D_{\downarrow\uparrow}^Q(x, y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\downarrow\uparrow}(x, y)\}, \\
D_{\downarrow\downarrow}^Q(x, y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\downarrow\downarrow}(x, y)\}, \\
D_{\uparrow\downarrow}^Q(x, y) = \inf\{V_Q(R) \mid R \in \mathcal{R}_{\uparrow\downarrow}(x, y)\}.
\]

(3.1) (northeast-defect of \( Q \)) \hspace{1cm} (3.2) (southeast-defect of \( Q \)) \hspace{1cm} (3.3) (southwest-defect of \( Q \)) \hspace{1cm} (3.4) (northwest-defect of \( Q \))

It is obvious that each of these defect functions is non-positive. As a consequence of the continuity of \( Q \), each infimum in Definition 3.1 is actually attained (and, therefore, can be replaced by a minimum).

Moreover, we have the following result which immediately follows from the Definitions 2.1 and 2.2.

Proposition 3.2. Let \( Q : [0, 1]^2 \to [0, 1] \) be a quasi-copula. Then \( Q \) is a copula if and only if one (and, subsequently, each) of the defect functions \( D_{\uparrow\uparrow}^Q, D_{\downarrow\uparrow}^Q, D_{\downarrow\downarrow}^Q, \) and \( D_{\uparrow\downarrow}^Q \) given by (3.1)–(3.4) is identically zero.

Based on Definition 3.1, it is possible to introduce additional defect functions, two of which are given below:

Definition 3.3. Let \( Q : [0, 1]^2 \to [0, 1] \) be a quasi-copula. Then the following defect functions \( D_{\text{M}}^Q, D_{\text{O}}^Q : [0, 1]^2 \to \mathbb{R} \) are given by

\[
D_{\text{M}}^Q = D_{\uparrow\downarrow}^Q \land D_{\downarrow\downarrow}^Q, \\
D_{\text{O}}^Q = D_{\downarrow\uparrow}^Q \land D_{\uparrow\uparrow}^Q.
\]

(3.5) (main-defect of \( Q \)) \hspace{1cm} (3.6) (opposite-defect of \( Q \))

Observe that the main-defect \( D_{\text{M}}^Q(x, y) \) of \( Q \) is related to rectangles in \( \mathcal{R}_{\uparrow\downarrow}(x, y) \cup \mathcal{R}_{\downarrow\downarrow}(x, y) \), i.e., having \((x, y)\) as lower left or upper right vertex. Similarly, the opposite-
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The defect $D_Q^Q(x,y)$ of $Q$ is related to $\mathcal{R}_{\downarrow}(x,y) \cup \mathcal{R}_{\uparrow}(x,y)$, i.e., to rectangles having $(x,y)$ as lower right or upper left vertex.

Obviously, all the six defect functions introduced in Definitions 3.1 and 3.3 vanish at the boundary points of the unit square $[0,1]^2$. On the other hand, for each proper quasi-copula each of these six defect functions is strictly negative on a subset of $[0,1]^2$ with positive Lebesgue measure.

Example 3.4. Put $Q = C_1 \lor C_2$, i.e., the proper quasi-copula considered in Example 2.3 and Figure 1 (bottom left). The support of $Q$ is given in Figure 3 (left). Note that on the thick line segments the mass $\frac{2}{3}$ is uniformly distributed, while on the dashed line segment the mass $\frac{1}{3}$ is uniformly distributed. The northeast-defect function $D_Q^\uparrow$ given by (3.1) is visualized in Figure 3 (right). Similarly the supports of all six defect functions of $Q$ can be illustrated (see Figure 4 and notice that the extra thick line in the support of $D_Q^\uparrow$ indicates that the line segment connecting the points $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$ carries twice as much mass). Finally, in Figure 5, the main-defect function $D_M^Q$ and the opposite-defect function $D_O^Q$ of the quasi-copula $Q$ are given.

All the defect functions discussed so far have their values in the interval $[-\frac{1}{3}, 0]$, and the extremal value can be attained only in vertices of the square $[\frac{1}{3}, \frac{2}{3}]^2$ (compare Example 3.4). We even have the following stronger result:
Theorem 3.5. Define the functions $D_\nearrow, D_\searrow, D_\swarrow, D_\nwarrow, D_M, D_O : [0, 1]^2 \to \mathbb{R}$ by

$$
D_\nearrow = \inf\{ D^Q_\nearrow | Q \in \mathcal{Q} \}, \quad D_\searrow = \inf\{ D^Q_\searrow | Q \in \mathcal{Q} \}, \\
D_\swarrow = \inf\{ D^Q_\swarrow | Q \in \mathcal{Q} \}, \quad D_\nwarrow = \inf\{ D^Q_\nwarrow | Q \in \mathcal{Q} \}, \\
D_M = \inf\{ D^Q_M | Q \in \mathcal{Q} \}, \quad D_O = \inf\{ D^Q_O | Q \in \mathcal{Q} \}.
$$

Then we have $D_M = D_\nearrow \land D_\swarrow$ and $D_O = D_\searrow \land D_\nwarrow$ and, for all $(x, y) \in [0, 1]^2$,

$$
D_\nearrow(x, y) = \max(-x, -y, \frac{x-1}{2}, \frac{y-1}{2}), \\
D_\searrow(x, y) = \max(x - 1, y - 1, -\frac{x}{2}, -\frac{y}{2}), \\
D_\swarrow(x, y) = \max(x - 1, -y, -\frac{x}{2}, \frac{y-1}{2}), \\
D_\nwarrow(x, y) = \max(-x, y -1, \frac{x-1}{2}, -\frac{y}{2}).
$$

Proof. The result for $D_M$ and $D_O$ follows directly from Definition 3.3.

Now fix a point $(x, y) \in [0, 1]^2$. Then for each quasi-copula $Q$ and each rectangle $R = [x_1, x] \times [y_1, y] \in \mathcal{R}(x, y)$ we have

$$
V_Q(R) + x = V_Q([x, x_1] \times [y, y_1]) + V_Q([0, x] \times [0, 1]) \\
= V_Q([0, x] \times [0, y]) + V_Q([0, x_1] \times [y_1, y]) + V_Q([0, x] \times [y_1, 1]).
$$

Each of the three rectangles considered in the last line shares an edge with the boundary of $[0, 1]^2$ and, therefore, has a nonnegative $Q$-volume. Therefore, $V_Q(R) \geq -x$ for each rectangle $R \in \mathcal{R}(x, y)$ and each $Q \in \mathcal{Q}$, implying $D_\nearrow(x, y) \geq -x$. In an analogous way $D_\searrow(x, y) \geq -y$ is shown.

For each quasi-copula $Q$ and each rectangle $R = [x_1, x] \times [y_1, y] \in \mathcal{R}(x, y)$ we get,
using similar arguments, also

\[ 2V_Q(R) + 1 - x = 2V_Q(R) + V_Q([x, 1] \times [0, 1]) \]

\[ = V_Q([x, 1] \times [y, 1]) + V_Q([x, 1] \times [0, y_1]) + V_Q([x, 1] \times [y, 1]) \]

\[ + V_Q([x_1, 1] \times [y_1, 1]) + V_Q([1, 1] \times [0, y]) \]

\[ \geq 0, \]

implying \( D \nearrow (x, y) \geq \frac{x+1}{2} \). Of course, \( D \nearrow (x, y) \geq \frac{y+1}{2} \) is shown in complete analogy.

Summarizing, we know now that \( D \nearrow (x, y) \geq \max(-x, -y, \frac{x+1}{2}, \frac{y+1}{2}). \)

Now fix a point \((x, y)\) satisfying \( x \leq \frac{1}{3} \) and \( x \leq y \leq 1 - 2x \). Then \( y - x \geq 0, \ y + 2x \leq 1 \) and \( 3x \leq 1 \), implying \( \max(-x, -y, \frac{x+1}{2}, \frac{y+1}{2}) = -x \). Denote now by \( Q_{x,y} \) the quasi-copula whose support is visualized in Figure 6.

Observe that the restriction of \( Q_{x,y} \) to the square \([0, 3x] \times [y - x, y + 2x]\) is a linear transformation of the proper quasi-copula \( C_1 \lor C_2 \) considered in Examples 2.3 and 3.4.

Because of \( V_{Q_{x,y}}([1, 2x] \times [y, y + x]) = -x \) we have \( D \nearrow (x, y) = -x \).

In a similar way, for each point \((x, y)\) the existence of a proper quasi-copula \( Q \) and of a rectangle \( R \in R \nearrow (x, y) \) with \( V_Q(R) = \max(-x, -y, \frac{x+1}{2}, \frac{y+1}{2}) \) is shown, proving the validity of \( D \nearrow (x, y) = \max(-x, -y, \frac{x+1}{2}, \frac{y+1}{2}) \).

Note that for each quasi-copula \( Q \in \mathcal{Q} \) also the function \( \hat{Q} : [0, 1]^2 \to [0, 1] \) given by

\[ \hat{Q}(x, y) = x + y - 1 + Q(1-x, 1-y) \]

(3.7)

is a quasi-copula and, moreover, the mapping \( Q \mapsto \hat{Q} \) is an involution, i.e., we have

\( \hat{(Q)} = Q. \)

This implies \( D \nwarrow (x, y) = D \nearrow (1-x, 1-y) \) for all \((x, y) \in [0, 1]^2\) and, subsequently, \( D \nwarrow (x, y) = \max(x-1, y-1, -\frac{x+1}{2}, -\frac{y+1}{2}) \).

The remaining equalities for \( D \swarrow \) and \( D \searrow \) follow from results in [20]; for each quasi-copula \( Q \in \mathcal{Q} \) also the functions \( Q^{-}, Q_{-} : [0, 1]^2 \to [0, 1] \) given by

\[ Q^{-}(x, y) = x - Q(x, 1-y), \quad Q_{-}(x, y) = y - Q(1-x, y), \]

(3.8)
respectively, are quasi-copulas. Therefore $D_{\wedge}(x, y) = D_{\nearrow}(1 - x, y)$ and $D_{\wedge}(x, y) = D_{\searrow}(x, 1 - y)$ for each $(x, y) \in [0, 1]^2$, completing the proof. □

**Remark 3.6.** Note that there is no quasi-copula $Q \in Q$ such that $D_{\nearrow}^Q = D_{\nearrow}$ (an analogous statement holds for each of the other defect functions). Assuming the contrary, i.e., $D_{\nearrow}^Q = D_{\nearrow}$ for some $Q \in Q$, this means in particular

$$D_{\nearrow}^Q \left(\frac{1}{3}, \frac{1}{3}\right) = D_{\searrow} \left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3}.$$ 

Then, as a consequence of [14, Theorem 7.4.4], we necessarily get $Q \left(\frac{1}{3}, \frac{1}{3}\right) = 0$ and

$$Q \left(\frac{1}{3}, \frac{2}{3}\right) = Q \left(\frac{2}{3}, \frac{1}{3}\right) = Q \left(\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}.$$ 

From the latter equality it follows easily that $Q$ coincides with the Fréchet–Hoeffding lower bound $W$ on the square $\left[\frac{1}{3}, \frac{2}{3}\right]^2$, implying that $D_{\nearrow}^Q \left(\frac{2}{3}, \frac{2}{3}\right) = D_{\searrow}^W \left(\frac{2}{3}, \frac{2}{3}\right) = 0$, i.e.,

$$D_{\nearrow}^Q \left(\frac{2}{3}, \frac{2}{3}\right) = -\frac{1}{6} < 0 = D_{\nearrow}^Q \left(\frac{2}{3}, \frac{2}{3}\right).$$

Because of some symmetries of the defects introduced in (3.1)–(3.6) we may restrict our considerations to, say, northeast-defects of quasi-copulas only.

**Remark 3.7.** Using similar arguments as in the proof of Theorem 3.5, it is possible to show that for each quasi-copula $Q \in Q$ and for the quasi-copulas $\hat{Q}$, $Q^-$ and $Q^-$ considered in (3.7) and (3.8) the following equalities hold for all $(x, y) \in [0, 1]^2$:

$$D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(1 - x, 1 - y), \quad D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(1 - x, 1 - y),$$

$$D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(x, 1 - y), \quad D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(x, 1 - y),$$

$$D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(1 - x, y), \quad D_{\nearrow}^Q(x, y) = D_{\nearrow}^Q(1 - x, y).$$

As a consequence, the main- and the opposite-defect of a quasi-copula $Q$ can be expressed by northeast-defects of the quasi-copulas $Q$, $\hat{Q}$, $Q^-$, and $Q^-$, i.e., for all $(x, y) \in [0, 1]^2$ we have

$$D_{\wedge}^Q(x, y) = D_{\wedge}^Q(x, y) \wedge D_{\nearrow}^Q(1 - x, 1 - y),$$

$$D_{\searrow}^Q(x, y) = D_{\searrow}^Q(1 - x, y) \wedge D_{\nearrow}^Q(x, 1 - y).$$

An important tool for the construction of new quasi-copulas from given ones is the so-called ordinal sum. Based on earlier results in the context of partially ordered sets [4] and of abstract semigroups [5], the concept of an ordinal sum of triangular norms was introduced in [24, 32] (compare also [1, 21, 33]), and it can be carried over to the case of (quasi-)copulas in a straightforward way.

The following example shows that, for an ordinal sum of quasi-copulas, also the corresponding defect functions given in Definitions 3.1 and 3.3 have an ordinal sum structure:
Example 3.8. Let \( \{a_i, b_i\} \in I \) be a pairwise disjoint family of non-empty open subintervals of \([0,1]\), \((Q_i)_{i \in I}\) be a family of quasi-copulas, and assume that the quasi-copula \( Q = (\langle a_i, b_i, Q_i \rangle)_{i \in I} \) is the ordinal sum of the summands \((a_i, b_i, Q_i)_{i \in I}\) given by

\[
Q(x, y) = \begin{cases} 
    a_i + (b_i - a_i) \cdot Q_i \left( \frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \right) & \text{if } (x, y) \in [a_i, b_i]^2, \\
    M(x, y) & \text{otherwise}. 
\end{cases}
\]  

(3.9)

Then also the structure of the defect functions \( D_{Q^{\uparrow}}, D_{Q^{\downarrow}}, D_{Q^{\leftarrow}}, D_{Q^{\rightarrow}}, D_{M}, D_{O} \) given in (3.1)–(3.6) is that of an ordinal sum. We give here the exact formula for \( D_{Q^{\uparrow}} \) only (remember that \( D_{M^{\uparrow}}(x, y) = 0 \) for all \((x, y) \in [0,1]^2\)):

\[
D_{Q^{\uparrow}}(x, y) = \begin{cases} 
    (b_i - a_i) \cdot D_{Q^{\uparrow}} \left( \frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \right) & \text{if } (x, y) \in [a_i, b_i]^2, \\
    D_{M}(x, y) & \text{otherwise}. 
\end{cases}
\]

4. DEFECT-BASED TRANSFORMATIONS OF QUASI-COPULAS

Several constructions and transformations of copulas and quasi-copulas have been considered so far (see, e.g., \([3, 7, 11, 13, 15, 20, 22, 23]\)). Here we use the defect functions given in Definitions [3.1] and [3.3] to introduce new types of transformations of quasi-copulas.

For a quasi-copula \( Q \), consider the functions \( Q^{\uparrow}, Q^{\downarrow}, Q^{\leftarrow}, Q^{\rightarrow}, Q_M, Q_O : [0,1]^2 \to [0,1] \) defined by, respectively,

\[
Q^{\uparrow} = Q - D_{Q^{\uparrow}}, \quad Q^{\downarrow} = Q - D_{Q^{\downarrow}}, \\
Q^{\leftarrow} = Q + D_{Q^{\leftarrow}}, \quad Q^{\rightarrow} = Q + D_{Q^{\rightarrow}}, \\
Q_M = Q - D_{M}, \quad Q_O = Q + D_{O}.
\]

(4.1)
A natural question arises: given an arbitrary quasi-copula \( Q \), is each of the functions \( Q \uparrow, Q \downarrow, Q \searrow, Q \swarrow, Q_M, \) and \( Q_O \) given above a (quasi-)copula?

**Example 4.1.** Consider again the quasi-copula \( Q = C_1 \lor C_2 \) introduced in Example 2.3 and discussed in Example 3.4. Observe that the function \( Q \uparrow \) visualized in Figure 7 (left) is a copula (actually, it turns out to be a shuffle of the Fréchet–Hoeffding upper bound \( M \)). For more details about shuffles of \( M \) see [12, 14, 27]. In analogy, the functions \( Q \downarrow, Q \searrow, Q \swarrow, Q_M, \) and \( Q_O \) are shuffles of \( M \) and, therefore, copulas (for a visualization of their supports see Figure 8).

In general, however, we don’t obtain copulas using these transformations.

**Example 4.2.** Consider the proper quasi-copula \( Q \) given by (2.3) in Example 2.4 (see also Figure 2). Then the function \( Q \downarrow \) (see Figure 7 right) is a proper quasi-copula (observe that, e.g., \( V_{Q \downarrow}([3/4, 4/5]^2) = -\frac{1}{54} \)). Also the other functions \( Q \uparrow, Q \searrow, Q \swarrow, Q_M, \) and \( Q_O \) are proper quasi-copulas.

**Theorem 4.3.** Let \( Q \in \mathcal{Q} \) be a quasi-copula. Then each of the six functions \( Q \uparrow, Q \downarrow, Q \searrow, Q \swarrow, Q_M, \) and \( Q_O \) given in (4.1)–(4.3) is a quasi-copula.

**Proof.** Fix an arbitrary quasi-copula \( Q \in \mathcal{Q} \). Then each of the six functions \( Q \uparrow, Q \downarrow, \)
Q_\wedge, Q_\Join, Q_M, and Q_O satisfies the boundary conditions, i.e., it is grounded and has 1 as neutral element.

Now fix an arbitrary point \((x, y) \in [0, 1]^2\) and \(\varepsilon \in [0, 1-x[.\) Then we have \(Q(x, y) \leq Q_\Join(x, y)\) and \(Q(x + \varepsilon, y) \leq Q_\Join(x + \varepsilon, y)\).

If \(D_Q^\Join(x, y) = 0\) then \(Q_\Join(x, y) = Q(x, y) \leq Q(x + \varepsilon, y) \leq Q_\Join(x + \varepsilon, y)\). If \(D_Q^\Join(x, y) < 0\) then the continuity of \(Q\) implies the existence of a rectangle \(R = [x, x_1] \times [y, y_1] \in R_\Join(x, y)\) such that \(D_Q^\Join(x, y) = V_Q(R)\).

Suppose first that \(x_1 \leq x + \varepsilon\). Then we get

\[
Q_\Join(x + \varepsilon, y) - Q_\Join(x, y) \geq Q(x + \varepsilon, y) - Q(x_1, y) + Q(x_1, y_1) - Q(x, y_1) \geq 0,
\]

i.e., \(Q_\Join(x, y) \leq Q_\Join(x + \varepsilon, y)\).

On the other hand, if \(x_1 > x + \varepsilon\) then

\[
Q_\Join(x, y) = Q(x, y_1) + Q(x_1, y) - Q(x_1, y_1)
\]

\[
= Q(x + \varepsilon, y) - (Q(x_1, y_1) - Q(x + \varepsilon, y_1) - Q(x_1, y) + Q(x + \varepsilon, y))
\]

\[
= (Q(x + \varepsilon, y_1) - Q(x, y_1))
\]

\[
\leq Q(x + \varepsilon, y) - V_Q([x + \varepsilon, x_1] \times [y, y_1])
\]

\[
\leq Q_\Join(x + \varepsilon, y).
\]

Therefore, \(Q_\Join\) is monotone non-decreasing in its first coordinate. The monotonicity in the second coordinate is shown analogously. Using similar arguments, the monotonicity of the functions \(Q_\Join, Q_\wedge,\) and \(Q_\vee\) is verified.

Recall that for the quasi-copula \(Q^-\) given by \(Q^-(x, y) = x - Q(x, 1-y)\) we have \(D_Q^\Join(x, y) = D_Q^- (x, y)\) for all \((x, y) \in [0, 1]^2\). As a consequence, for each \(\varepsilon \in [0, 1-x[\) we obtain

\[
0 \leq Q_\Join(x + \varepsilon, y) - Q_\Join(x, y)
\]

\[
= Q(x + \varepsilon, y) - D_Q^\Join(x + \varepsilon, y) - Q(x, y) + D_Q^\Join(x, y)
\]

\[
= x + \varepsilon - Q^-(x + \varepsilon, 1-y) - D_Q^- (x + \varepsilon, 1-y) - x + Q^-(x, 1-y) + D_Q^- (x, 1-y)
\]

\[
= \varepsilon + Q_\wedge(x, 1-y) - Q_\wedge(x + \varepsilon, 1-y)
\]

\[
\leq \varepsilon,
\]

where the latter inequality follows from the monotonicity of \(Q_\wedge\), thus proving the 1-Lipschitz property of \(Q^-\) in the first coordinate. In a similar way, the 1-Lipschitz property of \(Q_\Join\) in the second coordinate can be shown, completing the proof that \(Q_\Join\) is a quasi-copula.

In the same way, one verifies that \(Q_\Join, Q_\wedge,\) and \(Q_\wedge\) are quasi-copulas. Since \(Q\) is a lattice, also \(Q_M = Q_\Join \lor Q_\wedge\) and \(Q_O = Q_\wedge \land Q_\wedge\) are quasi-copulas.

Observe that for each quasi-copula \(Q \in Q\) we have the inequalities

\[
Q_O \leq Q_\wedge \leq Q \leq Q_\Join \leq Q_M,
\]

\[
Q_O \leq Q_\wedge \leq Q \leq Q_\Join \leq Q_M.
\]
If $Q \in \mathcal{Q}$ is a proper quasi-copula then we obtain the strict inequalities

$$Q \prec Q, \quad Q \prec Q, \quad Q \prec Q, \quad Q \prec Q.$$ 

**Example 4.4.** Let $(]a_i, b_i[)_{i \in I}$ be a pairwise disjoint family of non-empty open subintervals of $[0,1]$, $(Q_i)_{i \in I}$ be a family of quasi-copulas, and assume that the quasi-copula $Q = (\langle a_i, b_i, Q_i \rangle)_{i \in I}$ is the ordinal sum of the summands $(]a_i, b_i[, Q_i)_{i \in I}$ given by (3.9).

Then each of the six quasi-copulas $Q \prec, Q \prec, Q \prec, Q \prec, Q \prec, Q \prec$ given in (4.1)–(4.3) is an ordinal sum of quasi-copulas, i.e., $Q = (\langle a_i, b_i, (Q_i) \prec \rangle)_{i \in I}$, and analogously for $Q \prec, Q \prec, Q \prec, Q \prec, Q \prec, Q \prec$.

Based on the results of Theorem 4.3 it is possible to define the following six transformations $\prec, \prec, \prec, \prec, M, O : \mathcal{Q} \to \mathcal{Q}$ by

$$\langle Q \rangle = Q \prec, \quad \langle Q \rangle = Q \prec, \quad \langle Q \rangle = Q \prec, \quad \langle Q \rangle = Q \prec, \quad M(Q) = Q \prec, \quad O(Q) = Q \prec.$$ 

Of course, any composition of these transformations is again a transformation on $\mathcal{Q}$.

In particular, if we write $\langle^1 = \rangle, \langle^2 = \rangle \circ \langle, \ldots, \langle^n+1 = \rangle \circ \langle^n$, we obtain, for each quasi-copula $Q \in \mathcal{Q}$, the sequence $(\langle^n (Q))_{n \in \mathbb{N}}$ which is monotone non-decreasing, implying that its supremum $\langle^*(Q)$ coincides with its pointwise limit. Obviously, we have $\langle \circ \langle = \langle^*$ and, subsequently, $D_{\langle} (Q) = 0$, implying that $\langle^*(Q)$ is a copula. In a similar way, we can construct the copulas $\langle^*(Q)$ and $M^*(Q)$.

For the transformations $\prec, \prec, \prec, \prec,$ and $O$, the sequences $(\prec^n (Q))_{n \in \mathbb{N}}, (\prec^n (Q))_{n \in \mathbb{N}},$ and $(O^n(Q))_{n \in \mathbb{N}}$ are monotone non-increasing, and their respective limits (i.e., infima) $\prec_*(Q), \prec_*(Q),$ and $O_*(Q)$ are copulas, too.

This allows us to construct six different partitions of the set of all quasi-copulas, considering the six equivalence relations $\sim, \sim, \sim, \sim, \sim, \sim$ on $\mathcal{Q}$ which are defined by

$$Q_1 \sim Q_2 \iff \langle^*(Q_1) \sim \langle^*(Q_2);$$
$$Q_1 \sim Q_2 \iff \langle^*(Q_1) \sim \langle^*(Q_2);$$
$$Q_1 \sim M Q_2 \iff M^*(Q_1) = M^*(Q_2);$$
$$Q_1 \sim Q_2 \iff \langle_*(Q_1) \sim \langle_*(Q_2);$$
$$Q_1 \sim Q_2 \iff \langle_*(Q_1) \sim \langle_*(Q_2);$$
$$Q_1 \sim O Q_2 \iff O^*(Q_1) = O^*(Q_2).$$

Obviously, each of the respective equivalence classes contains exactly one element which is a copula.

**Example 4.5.** Consider again the proper quasi-copula $Q$ given by (2.3) in Example 2.4 (see also Example 4.2 and Figures 2 and 7). After some computation we get for each $n \in \mathbb{N}$

$$\langle^n (Q)(x, y) = \begin{cases} x \cdot y & \text{if } x \cdot y \leq \frac{1}{2}, \\
 \max(x + y - 1, K_n(x, y)) & \text{otherwise}, \end{cases}$$
where
\[ K_n(x, y) = \frac{2^{n-2}}{\sqrt{2} \cdot 2^{n-1} - 1} \cdot \sqrt{2}xy + 1 + \frac{2^{n-1}}{2}. \]

Since, for \( x \cdot y \geq \frac{1}{2} \), we have \( \lim_{n \to \infty} K_n(x, y) = \frac{1 + \log(2xy)}{2} \), the copula \( \vee^*(Q) \) is given by
\[ \vee^*(Q)(x, y) = \begin{cases} x \cdot y & \text{if } x \cdot y \leq \frac{1}{2}, \\ \max(x + y - 1, \frac{1 + \log(2xy)}{2}) & \text{otherwise}. \end{cases} \]

5. AN APPLICATION TO IMPRECISE COPULAS

Imprecise copulas were studied in [30] and [26] (see also [25, 36]) in order to construct two-dimensional probability boxes (briefly \( p \)-boxes), which are represented by ordered pairs of comparable distribution functions, from two given one-dimensional \( p \)-boxes.

**Definition 5.1.** A pair \((A, B)\) of functions \(A, B : [0, 1]^2 \to [0, 1]\) is called an imprecise copula if \(A\) and \(B\) are grounded and have 1 as neutral element, and if for each rectangle \([a, b] \times [c, d] \subseteq [0, 1]^2\) we have
\[
A(b, d) + B(a, c) - A(a, d) - A(b, c) \geq 0; \quad (IC1)
\]
\[
B(b, d) + A(a, c) - A(a, d) - A(b, c) \geq 0; \quad (IC2)
\]
\[
B(b, d) + B(a, c) - B(a, d) - A(b, c) \geq 0; \quad (IC3)
\]
\[
B(b, d) + B(a, c) - A(a, d) - B(b, c) \geq 0. \quad (IC4)
\]

It is not difficult to check that, for each imprecise copula \((A, B)\), we have \(A, B \in Q\), i.e., both \(A\) and \(B\) are quasi-copulas, and \(A \leq B\).

The properties (IC1)–(IC4) in Definition 5.1 can be equivalently expressed in the following form:
\[
B \geq A_{\uparrow}; \quad (IC1^*)
\]
\[
B \geq A_{\uparrow}; \quad (IC2^*)
\]
\[
A \leq B_{\downarrow}; \quad (IC3^*)
\]
\[
A \leq B_{\downarrow}. \quad (IC4^*)
\]

Obviously, \((IC1^*)\) and \((IC2^*)\) are simultaneously satisfied if and only if we have \(B \geq A_{\uparrow} \lor A_{\uparrow} = A_M\). Similarly, \(A \leq B_{\downarrow} \land B_{\downarrow} = B_O\) is equivalent to the joint validity of \((IC3^*)\) and \((IC4^*)\).

Summarizing these observations, the following result is immediate.

**Theorem 5.2.** A pair \((A, B)\) of quasi-copulas \(A, B : [0, 1]^2 \to [0, 1]\) is an imprecise copula if and only if \(B \geq A_M\) and \(A \leq B_O\).

Evidently, each pair \((C_1, C_2)\) of copulas satisfying \(C_1 \leq C_2\) is an imprecise copula, in particular, each pair \((C, C)\) with \(C \in \mathcal{C}\). However, for a proper quasi-copula \(Q\), the pair \((Q, Q)\) is never an imprecise copula because of \(Q_O < Q < Q_M\).

The partially ordered set of imprecise copulas forms an upper semi-lattice with top element \((W, M)\):
Theorem 5.3. Let \( (A_i, B_i)_{i \in I} \) be a family of imprecise copulas. Then also

\[
\bigvee_{i \in I} (A_i, B_i) = \left( \bigwedge_{i \in I} A_i, \bigvee_{i \in I} B_i \right)
\]

is an imprecise copula.

Proof. Evidently, both \( A = \bigwedge_{i \in I} A_i \) and \( B = \bigvee_{i \in I} B_i \) are quasi-copulas. Note first that for each \( i \in I \) and for each rectangle \( [a, b] \times [c, d] \subseteq [0, 1]^2 \) we have \( A_i(b, d) + B_i(a, c) - A_i(a, d) - A_i(b, c) \geq 0 \) and, therefore, also

\[
A_i(b, d) + B_i(a, c) \geq A_i(a, d) + A_i(b, c) \geq A(a, d) + A(b, c).
\]

As a consequence, \( A(b, d) + B(a, c) \geq A(a, d) + A(b, c) \), i.e., (IC1) holds for the pair \((A, B)\). The validity of the other inequalities (IC2)–(IC4) for the pair \((A, B)\) is shown in a similar way. \( \square \)

As an immediate corollary of Theorem 5.3 we have the following result already mentioned in [26, 30]:

Corollary 5.4. For each family \((C_i)_{i \in I}\) of copulas the pair \((\underline{C}, \overline{C})\), where the two functions \(\underline{C}, \overline{C} : [0, 1]^2 \to [0, 1]\) are given by

\[
\underline{C} = \bigwedge_{i \in I} C_i \quad \text{and} \quad \overline{C} = \bigvee_{i \in I} C_i,
\]

is an imprecise copula.

The problem whether all imprecise copulas can be obtained in this way (already posed in [26, 30]) is still open, namely, whether for each each imprecise copula \((A, B)\) there is a family \((C_i)_{i \in I}\) of copulas such that \( A = \underline{C} \) and \( B = \overline{C} \).

Example 5.5. For each quasi-copula \( Q \in \mathcal{Q} \) and for each \( n \in \mathbb{N} \) the two pairs \((M^n(Q), M^*(Q))\) and \((O^*(Q), O^n(Q))\) are imprecise copulas.

CONCLUSION

We have introduced different functions measuring the defect of a quasi-copula expressed by means of extremal non-positive volumes of specific rectangles (such that defect zero characterizes copulas). These defect functions were applied to define new transformations of quasi-copulas, each of them leading to a new quasi-copula comparable with the original one, and each of them having copulas as the only fixed points. Therefore, starting with any quasi-copula \( Q \), the iterative application of each of these transformations has a limit which necessarily is a copula, and which can be seen as an attractor of \( Q \) under the respective transformation. Then, for each transformation, the domains of attraction of copulas form a partition of the set of all quasi-copulas. Finally, an application to the construction of so-called imprecise copulas was added.

Several problems are still open and will be the subject of future research.
First of all, starting with a quasi-copula $Q$ and considering Examples 4.1, 4.2 and Theorem 4.3, what are (sufficient) conditions for the six functions $Q\nearrow$, $Q\searrow$, $Q\swarrow$, $Q\searrow$, $Q_M$, and $Q_O$ to be copulas?

Second, are there (for some of the transformations under consideration) any singleton members of the corresponding partition of quasi-copulas, i.e., is there a copula which does not admit any proper quasi-copula in its domain of attraction (up to the trivial cases of the Fréchet–Hoeffding upper bound $M$ with respect to the transformations $\nearrow$, $\searrow$, and $O$, and of the Fréchet–Hoeffding lower bound $W$ with respect to the transformations $\searrow$, $\nearrow$, and $M$)?

A third problem is to think about similar defect functions and transformations of quasi-copulas in the case of dimensions higher than two. A positive solution of this problem could help to solve the still open problem of imprecise copulas of higher dimensions.

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