CHANGING OF THE DOMINATION NUMBER OF A GRAPH:
EDGE MULTISUBDIVISION AND EDGE REMOVAL

VLADIMIR SAMODIVKIN, Sofia

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Abstract. For a graphical property $\mathcal{P}$ and a graph $G$, a subset $S$ of vertices of $G$ is a $\mathcal{P}$-set if the subgraph induced by $S$ has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_\mathcal{P}(G)$, is the minimum cardinality of a dominating $\mathcal{P}$-set. We define the domination multisubdivision number with respect to $\mathcal{P}$, denoted by $\text{msd}_\mathcal{P}(G)$, as a minimum positive integer $k$ such that there exists an edge which must be subdivided $k$ times to change $\gamma_\mathcal{P}(G)$. In this paper
(a) we present necessary and sufficient conditions for a change of $\gamma_\mathcal{P}(G)$ after subdividing an edge of $G$ once,
(b) we prove that if $e$ is an edge of a graph $G$ then $\gamma_\mathcal{P}(G_{e,1}) < \gamma_\mathcal{P}(G)$ if and only if $\gamma_\mathcal{P}(G - e) < \gamma_\mathcal{P}(G)$ ($G_{e,t}$ denotes the graph obtained from $G$ by subdivision of $e$ with $t$ vertices),
(c) we also prove that for every edge of a graph $G$ we have $\gamma_\mathcal{P}(G - e) \leq \gamma_\mathcal{P}(G_{e,3}) \leq \gamma_\mathcal{P}(G - e) + 1$, and
(d) we show that $\text{msd}_\mathcal{P}(G) \leq 3$, where $\mathcal{P}$ is hereditary and closed under union with $K_1$.

Keywords: dominating set; edge subdivision; domination multisubdivision number; hereditary graph property

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [14]. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G \rangle$. For a vertex $x$ of $G$, $N(x, G)$ denotes the set of all neighbors of $x$ in $G$, $N[x, G] = N(x, G) \cup \{x\}$ and the degree of $x$ is $\deg(x, G) = |N(x, G)|$. The maximum and minimum degrees of vertices in the graph $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For
a graph $G$, let $x \in X \subseteq V(G)$. A vertex $y$ is a private neighbor of $x$ with respect to $X$ if $N[y, G] \cap X = \{x\}$. The private neighbor set of $x$ with respect to $X$ is $\text{pm}_{G}[x, X] = \{y: N[y, G] \cap X = \{x\}\}$. For a graph $G$, the subdivision of the edge $e = uv \in E(G)$ with a vertex $x$ leads to a graph with the vertex set $V \cup \{x\}$ and the edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e,t}$ denote the graph obtained from $G$ by a subdivision of the edge $e$ with $t$ vertices (instead of the edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v))$. For $t = 1$ we write $G_e$.

Let $\mathcal{I}$ denote the set of all mutually non-isomorphic graphs. A graph property is any nonempty subset of $\mathcal{I}$. We say that a graph $G$ has the property $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. For example, we list some graph properties:

- $\mathcal{O} = \{H \in \mathcal{I}: H$ is totally disconnected$\}$;
- $\mathcal{C} = \{H \in \mathcal{I}: H$ is connected$\}$;
- $\mathcal{T} = \{H \in \mathcal{I}: \delta(H) \geq 1\}$;
- $\mathcal{M} = \{H \in \mathcal{I}: H$ has a perfect matching$\}$;
- $\mathcal{F} = \{H \in \mathcal{I}: H$ is a forest$\}$;
- $\mathcal{UK} = \{H \in \mathcal{I}:$ each component of $H$ is complete$\}$;
- $\mathcal{D}_k = \{H \in \mathcal{I}: \Delta(H) \leq k\}$.

A graph property $\mathcal{P}$ is called:

- (a) hereditary (induced-hereditary), if the fact that a graph $G$ has property $\mathcal{P}$ implies that all subgraphs (induced subgraphs) of $G$ also belong to $\mathcal{P}$, and
- (b) nondegenerate if $\mathcal{O} \subseteq \mathcal{P}$. Any set $S \subseteq V(G)$ such that the induced subgraph $\langle S, G \rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set.

Note that:

- (a) $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{D}_k$ are nondegenerate and hereditary properties;
- (b) $\mathcal{UK}$ is nondegenerate, induced-hereditary and is not hereditary;
- (c) all $\mathcal{C}$, $\mathcal{T}$ and $\mathcal{M}$ are neither induced-hereditary nor nondegenerate. For a survey on this subject we refer to Borowiecki et al. [2].

A set of vertices $D \subseteq V(G)$ is a dominating set of a graph $G$ if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of $G$. A dominating $\mathcal{P}$-set of $G$ with cardinality $\gamma_{\mathcal{P}}(G)$ is called a $\gamma_{\mathcal{P}}$-set of $G$. If a property $\mathcal{P}$ is nondegenerate, then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. Note that $\gamma_{\mathcal{I}}(G)$, $\gamma_{\mathcal{O}}(G)$, $\gamma_{\mathcal{C}}(G)$, $\gamma_{\mathcal{T}}(G)$, $\gamma_{\mathcal{M}}(G)$, $\gamma_{\mathcal{F}}(G)$ and $\gamma_{\mathcal{D}_k}(G)$ are well known as the domination number $\gamma(G)$, the independent domination number $i(G)$ ([5]), the connected domination number $\gamma_{\mathcal{C}}(G)$ ([24]), the total domination number $\gamma_t(G)$ ([3]), the paired-domination number $\gamma_{pr}(G)$ ([16]), the acyclic domination number $\gamma_a(G)$ ([17]) and the $k$-dependent domination number $\gamma^k(G)$ ([9]). The concept of domination with respect to any graph property $\mathcal{P}$ was introduced by
It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In [20], the present author began the study of the effects on $\gamma_P(G)$ when a graph $G$ is modified by deleting a vertex or by adding an edge ($P$ is nondegenerate). In this paper we concentrate on effects on $\gamma_P(G)$ when a graph is modified by deleting/subdividing an edge. An edge $e$ of a graph $G$ is called a $\gamma_P$-$ER^-$-critical edge of $G$ if $\gamma_P(G) > \gamma_P(G - e)$. Note that

(a) $\gamma$-$ER^-$-critical edges do not exist (see [13]),
(b) Grobler [11] was the first who began the investigation of $\gamma_P$-$ER^-$-critical edges when $P = O$,
(c) necessary and sufficient conditions for an edge of a graph $G$ to be $\gamma_P$-$ER^-$-critical, where $P$ is hereditary, may be found in [20].

One measure of the stability of the domination number of $G$ under edge subdivision is the domination subdivision number with respect to the property $P$, denoted $sd^+_\gamma_P(G)$, which is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase $\gamma_P(G)$. The following special cases for $sd^+_\gamma_P(G)$ have been investigated up to now:

(a) $sd^+_\gamma_I(G)$—the domination subdivision number defined by Velammal [25],
(b) $sd^+_\gamma_T(G)$—the total domination subdivision number introduced by Haynes et al. in [15],
(c) $sd^+_\gamma_M(G)$—the paired domination subdivision number introduced by Favaron et al. in [8],
(d) $sd^+_\gamma_c(G)$—the connected domination subdivision number introduced by Favaron et al. in [7], and
(e) $sd^+_\gamma_P(G)$—the domination subdivision number with respect to the nondegenerate property $P$ introduced by the present author in [23].

Here we focus on the existence of critical edges with respect to the subdivision/multisubdivision. Results in this direction, in the case when $P = \mathcal{I}$, were recently obtained by Lemańska, Tey and Zuazua [18] and Dettlaff, Raczek and Topp [6]. For any nondegenerate property $P \subseteq \mathcal{I}$ we define an edge $e$ of a graph $G$ to be

(i) a $\gamma_P$-$S^+$-critical edge of $G$ if $\gamma_P(G) < \gamma_P(G - e)$, and
(ii) a $\gamma_P$-$S^-$-critical edge of $G$ if $\gamma_P(G) > \gamma_P(G - e)$.

In Section 2:

(a) we present necessary and sufficient conditions for a change of $\gamma_P(G)$ after subdividing an edge of $G$ once, and
(b) we prove that an edge $e$ of a graph $G$ is $\gamma_H$-$S^-$-critical if and only if $e$ is $\gamma_H$-$ER^-$-critical, for any graph property $\mathcal{H} \subseteq \mathcal{I}$ which is induced-hereditary and closed under union with $K_1$. 

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In Section 3 we deal with changes of $\gamma_P(G)$ when an edge of $G$ is multiple subdivided. To present our results we need the following definitions.

For every edge $e$ of a graph $G$ let
\begin{itemize}
  \item $\text{msd}_P(e) = \min\{t: \gamma_P(G_{e,t}) \neq \gamma_P(G)\}$;
  \item $\text{msd}_P^+(e) = \min\{t: \gamma_P(G_{e,t}) > \gamma_P(G)\}$;
  \item $\text{msd}_P^-(e) = \min\{t: \gamma_P(G_{e,t}) < \gamma_P(G)\}$.
\end{itemize}

If $\gamma_P(G_{e,t}) \geq \gamma_P(G)$ for every $t \geq 1$, then we write $\text{msd}_P^-(e) = \infty$. If $\gamma_P(G_{e,t}) \leq \gamma_P(G)$ for every $t \geq 1$, then we write $\text{msd}_P^+(e) = \infty$.

**Definition 1.1.** For every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, we define the domination multisubdivision (plus domination multisubdivision, minus domination multisubdivision) number with respect to the property $\mathcal{P}$, denoted $\text{msd}_\mathcal{P}(G)$ ($\text{msd}_\mathcal{P}^+$, $\text{msd}_\mathcal{P}^-$, respectively) to be
\begin{itemize}
  \item $\text{msd}_\mathcal{P}(G) = \min\{\text{msd}_\mathcal{P}(e): e \in E(G)\}$,
  \item $\text{msd}_\mathcal{P}^+(G) = \min\{\text{msd}_\mathcal{P}^+(e): e \in E(G)\}$,
  \item $\text{msd}_\mathcal{P}^-(G) = \min\{\text{msd}_\mathcal{P}^-(e): e \in E(G)\}$,
\end{itemize}
respectively. If $\gamma_\mathcal{P}(G_{e,t}) \geq \gamma_\mathcal{P}(G)$ for every $t$ and every edge $e \in E(G)$, then we write $\text{msd}_\mathcal{P}^-(G) = \infty$.

The parameters $\text{msd}_\mathcal{P}^+(G)$ and $\text{msd}_\mathcal{P}^-(G)$ (in our designation) were introduced by Dettlaff, Raczek and Topp in [6] and by Avella-Alaminos, Dettlaff, Lemańska and Zuazua in [1], respectively. Note that in the case when $\mathcal{P} = \mathcal{I}$, clearly, $\text{msd}(G) = \text{msd}_\mathcal{I}^+(G)$, and $\text{msd}_\mathcal{I}^-(G) = \infty$. In Section 3 we prove that for every edge of a graph $G$ we have $\gamma_\mathcal{P}(G-e) \leq \gamma_\mathcal{P}(G_{e,3}) \leq \gamma_\mathcal{P}(G-e)+1$ and we present necessary and sufficient conditions for the validity of $\gamma_\mathcal{P}(G-e) = \gamma_\mathcal{P}(G_{e,3})$. Our main result in that section is that $\text{msd}_\mathcal{P}(G) \leq 3$ for any graph $G$ and any graph property $\mathcal{P}$ which is hereditary and closed under union with $K_1$.

2. **Single subdivision: critical edges**

We begin this section with a characterization of $\gamma_\mathcal{P}-S^+$-critical edges of a graph. Note that if a property $\mathcal{P}$ is induced-hereditary and closed under union with $K_1$ then $\mathcal{P}$ is nondegenerate.

**Theorem 2.1.** Let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with $K_1$. Let $G$ be a graph and $e = uv \in E(G)$. Then $\gamma_\mathcal{H}(G_e) \leq \gamma_\mathcal{H}(G) + 1$. If $e$ is a $\gamma_\mathcal{H}-S^+$-critical edge of $G$ then $\gamma_\mathcal{H}(G_e) = \gamma_\mathcal{H}(G) + 1$ and for each $\gamma_\mathcal{H}$-set $M$ of $G$ one of the following conditions holds:

\begin{enumerate}
  \item[(i)] $u, v \in V(G) - M$;
\end{enumerate}
(ii) \( u \in M, v \in pn_G[u, M] \) and \( pn_G[u, M] \) is not a subset of \( \{u, v\} \);
(iii) \( v \in M, u \in pn_G[v, M] \) and \( pn_G[u, M] \) is not a subset of \( \{u, v\} \).

If \( e \) is not \( \gamma_P\)-\( S^+ \)-critical and for each \( \gamma_H \)-set \( M \) of \( G \) one of (i), (ii) and (iii) holds then there is a dominating \( H \)-set \( R \) of \( G - uv \) with \( u, v \in R \) and \( |R| \leq \gamma_H(G) \).

**Proof.** Let \( x \in V(G_e) \) be the subdivision vertex and let \( M \) be a \( \gamma_H \)-set of \( G \). If \( u, v \notin M \) then \( M \cup \{x\} \) is a dominating \( H \)-set of \( G_e \) (\( H \) is closed under union with \( K_1 \)) and we have \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \). If both \( u \) and \( v \) are in \( M \) then \( M \) is a dominating \( H \)-set of \( G \) (\( H \) is hereditary), which implies \( \gamma_H(G_e) \leq \gamma_H(G) \). If \( u \in M, v \notin M \) and \( v \notin pn_G[u, M] \) then again \( M \) is a dominating \( H \)-set of \( G_e \) and hence \( \gamma_H(G_e) \leq \gamma_H(G) \). So, let \( u \in M, v \notin M \) and \( v \in pn_G[u, M] \). If either \( \{v\} \) or \( \{u, v\} \) coincides with \( pn_G[u, M] \) then \( (M - \{u\}) \cup \{x\} \) is a dominating \( H \)-set of \( G_e \); hence \( \gamma_H(G_e) \leq \gamma_H(G) \). If neither \( pn_G[u, M] = \{v\} \) nor \( pn_G[u, M] = \{u, v\} \) then \( M \cup \{v\} \) is a dominating \( H \)-set of \( G_e \) and we have \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \). Thus \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \) and if the equality is fulfilled then one of (i), (ii) and (iii) holds.

Now, let for each \( \gamma_H \)-set \( M \) of \( G \) one of (i), (ii) and (iii) holds. Assume \( \gamma_H(G_e) \leq \gamma_H(G) \) and let \( R \) be a \( \gamma_H \)-set of \( G_e \).

**Case 1:** \( u, v \notin R \). Hence \( x \in R \). If \( u, v \notin pn_{G_e}[x, R] \) then \( R - \{x\} \) is a dominating \( H \)-set of \( G \), a contradiction with \( \gamma_H(G_e) \leq \gamma_H(G) \). If \( u \in pn_{G_e}[x, R] \) and \( v \notin pn_{G_e}[x, R] \) then \( R_1 = (R - \{x\}) \cup \{u\} \) is a dominating \( H \)-set of \( G \) of cardinality \( |R_1| = |R| = \gamma_H(G_e) \). Since \( \gamma_H(G_e) \leq \gamma_H(G) \), we have that \( R_1 \) is a \( \gamma_H \)-set of \( G \). But then \( u \in R_1, v \notin R_1 \) and \( v \notin pn_G[u, R_1] \), contradicting (ii). If \( u, v \in pn_G[x, R] \) then as above \( R_1 \) is a \( \gamma_H \)-set of \( G \) and since \( u \in R_1 \) and \( \{u, v\} = pn_G[u, R_1] \), again we arrive at a contradiction with (ii).

**Case 2:** \( u \in R \) and \( v \notin R \). Hence \( x \notin R \), otherwise \( R - \{x\} \) is a dominating \( H \)-set of \( G \), contradicting \( \gamma_H(G_e) \leq \gamma_H(G) \). This implies that \( R \) is a \( \gamma_H \)-set of \( G \), \( u \in R \) and \( v \notin pn_G[u, R] \), a contradiction with (ii).

**Case 3:** \( u, v \in R \). Hence \( R \) is a dominating \( H \)-set of \( G - uv \) and \( |R| = \gamma_H(G_e) \leq \gamma_H(G) \). □

When we restrict our attention to the case where \( H = I \), we can describe more precisely when an edge of a graph \( G \) is \( \gamma \)-\( S^+ \)-critical.

**Corollary 2.2.** Let \( G \) be a graph and \( e = uv \in E(G) \). Then \( e \) is a \( \gamma \)-\( S^+ \)-critical edge of \( G \) if and only if for each \( \gamma \)-set \( M \) of \( G \) one of (i), (ii) and (iii) stated in Theorem 2.1 holds.

**Proof.** **Necessity:** The result immediately follows by Theorem 2.1.

**Sufficiency:** Assume \( \gamma(G_e) \leq \gamma(G) \). Then by Theorem 2.1, there is a dominating set \( R \) of \( G - uv \) with \( u, v \in R \) and \( |R| \leq \gamma(G) \). But it is a well known fact that if \( f \)
is an edge of a graph $G$ then always $\gamma(G-f) \geq \gamma(G)$. Hence $R$ is a $\gamma$-set of both $G$ and $G-e$ and $u,v \in R$, contradicting all (i), (ii) and (iii). \hfill \square \\

**Theorem 2.3.** Let $\mathcal{H} \subseteq \mathcal{I}$ be induced-hereditary and closed under union with $K_1$. An edge $e$ of a graph $G$ is $\gamma_{\mathcal{H}}$-S-critical if and only if $e$ is $\gamma_{\mathcal{H}}$-ER-critical.

**Proof.** As we have already shown, $\mathcal{H}$ is nondegenerate and then all $\gamma_{\mathcal{H}}(G-e)$, $\gamma_{\mathcal{H}}(G_e)$ and $\gamma_{\mathcal{H}}(G)$ exist. Let $v$ be the subdivision vertex of $G_e$.

**Sufficiency:** Let $e = xy$ be a $\gamma_{\mathcal{H}}$-ER-critical edge of $G$ and $M$ a $\gamma_{\mathcal{H}}$-set of $G-e$. Hence $\gamma_{\mathcal{H}}(G-e) < \gamma_{\mathcal{H}}(G)$ and $x,y \in M$. But then $M$ is a dominating $\mathcal{H}$-set of $G_e$, which leads to $\gamma_{\mathcal{H}}(G_e) < \gamma_{\mathcal{H}}(G-e) < \gamma_{\mathcal{H}}(G)$.

**Necessity:** Let $e = xy$ be a $\gamma_{\mathcal{H}}$-S-critical edge of $G$ and $M$ a $\gamma_{\mathcal{H}}$-set of $G_e$. Hence $\gamma_{\mathcal{H}}(G_e) < \gamma_{\mathcal{H}}(G)$. Assume $v \notin M$. Hence at least one of $x$ and $y$ is in $M$. If both $x,y \in M$ then $M$ is a dominating $\mathcal{H}$-set of $G-e$ and the result follows. If $x \notin M$ and $y \in M$ then $M$ is a dominating $\mathcal{H}$-set of $G$, a contradiction. Thus we may assume $v$ is in all $\gamma_{\mathcal{H}}$-sets of $G_e$. Since $\mathcal{H}$ is induced-hereditary, at least one of $x$ and $y$ is not in $M$. First let $x \in M$ and $y \notin M$. Then $y \in pn_{G_e}[v,M]$, which implies $M - \{v\}$ is a dominating $\mathcal{H}$-set of $G$, a contradiction. Hence neither $x$ nor $y$ are in $M$. If $x \notin M$ and $y \in M$ then $M - \{v\}$ is a dominating $\mathcal{H}$-set of $G$, a contradiction. Hence at least one of $x$ and $y$, say $y$, is in $pn_{G_e}[v,M]$. But then $(M - \{v\}) \cup \{y\}$ is a dominating $\mathcal{H}$-set of $G$, a contradiction. \hfill \square \\

Note that
(a) there do not exist $\gamma$-ER-critical edges (see [13]), and
(b) necessary and sufficient conditions for an edge of a graph $G$ to be $\gamma_{\mathcal{H}}$-ER-critical may be found in [20].

Now we define the following classes of graphs:
\begin{itemize}
  \item $(CS_p) \gamma_p(G) > \gamma_p(G_e)$ for every edge $e$ of $G$, and
  \item $(CER_p) \gamma_p(G) > \gamma_p(G-e)$ for every edge $e$ of $G$.
\end{itemize}

As an immediate consequence of Theorem 2.3 we obtain the next result.

**Corollary 2.4.** If $\mathcal{H} \subseteq \mathcal{I}$ is induced-hereditary and closed under union with $K_1$ then the classes of graphs $CS_p$ and $CER_p$ coincide.

Note that the class $CER_p$ in the case when $\mathcal{P} = \mathcal{O}$ was introduced by Grobler [11] and also considered in [12], [13], [4].
3. Multiple subdivision

We first state our theorems, then we pose a problem they generate, and after that we give the proofs.

Recall that $G_{e,t}$ denotes the graph obtained from a graph $G$ by the subdivision of the edge $e \in E(G)$ with $t$ vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For any graph $G$ and any nondegenerate property $\mathcal{P}$ let us denote by $V^-_{\mathcal{P}}(G)$ the set $\{v \in V(G): \gamma_{\mathcal{P}}(G - v) < \gamma_{\mathcal{P}}(G)\}$. Our first result shows that the value of the difference $\gamma_{\mathcal{P}}(G_{e,3}) - \gamma_{\mathcal{P}}(G - e)$ is either 0 or 1.

**Theorem 3.1.** Let $\mathcal{H} \subseteq \mathcal{I}$ be induced-hereditary and closed under union with $K_1$. If $e = uv$ is an edge of a graph $G$ then $\gamma_{\mathcal{H}}(G - e) \leq \gamma_{\mathcal{H}}(G_{e,3}) \leq \gamma_{\mathcal{H}}(G - e) + 1$.

Moreover, the following conditions are equivalent:

(A$_1$) $\gamma_{\mathcal{H}}(G - e) = \gamma_{\mathcal{H}}(G_{e,3})$;

(A$_2$) at least one of the following holds:

(i) $u \in V^-_{\mathcal{H}}(G - e)$ and $v$ belongs to some $\gamma_{\mathcal{H}}$-set of $G - u$;

(ii) $v \in V^-_{\mathcal{H}}(G - e)$ and $u$ belongs to some $\gamma_{\mathcal{H}}$-set of $G - v$.

If in addition $\mathcal{H}$ is hereditary then (A$_1$) and (A$_2$) are equivalent to

(A$_3$) $\gamma_{\mathcal{H}}(G - e) = 1 + \gamma_{\mathcal{H}}(G)$.

The main result in this section is the following.

**Theorem 3.2.** Let $e$ be an edge of a graph $G$ and let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with $K_1$.

(i) Then $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3})$ if and only if $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) + 1$.

(ii) If $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) + 1$ then $\text{msd}_{\mathcal{H}}(e) \leq 6$ and $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,1}) + 1 = \gamma_{\mathcal{H}}(G_{e,2}) + 1 = \gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G_{e,4}) = \gamma_{\mathcal{H}}(G_{e,5}) = \gamma_{\mathcal{H}}(G_{e,6}) - 1$.

(iii) Then $\text{msd}_{\mathcal{H}}(e) \leq 3$. In particular (Detlaff, Raczek and Topp [6] when $\mathcal{H} = \mathcal{I}$), $\text{msd}_{\mathcal{H}}(G) \leq 3$.

**Example 3.3.** It is easy to see that if $G = K_{3n_2 \ldots n_m}$, where $m \geq 2$ and $n_i \geq 3$ for $2 \leq i \leq m$, then $\gamma_{\mathcal{O}}(G) = \gamma_{\mathcal{O}}(G_{e,3}) = \gamma_{\mathcal{O}}(G - e) + 1 = 3$ for every edge $e$ of $G$. Hence by Theorem 3.2, $\text{msd}_{\mathcal{O}}(G) = \text{msd}_{\mathcal{O}}(G) = 1$ and $\text{msd}_{\mathcal{O}}^+(G) = 6$.

In view of Theorem 3.2 (iii), we can split the family of all graphs $G$ into three classes with respect to the value of $\text{msd}_{\mathcal{P}}(G)$, where $\mathcal{P} \subseteq \mathcal{I}$ is hereditary and closed under union with $K_1$. We define that a graph $G$ belongs to the class $S^i_{\mathcal{P}}$ whenever $\text{msd}_{\mathcal{P}}(G) = i$, $i \in \{1, 2, 3\}$. It is straightforward to verify that if $k \geq 1$ and $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I}$ then
\[ P_{3k}, C_{3k} \in S^1_P; P_{3k+2}, C_{3k+2} \in S^2_P; \text{ and } P_{3k+1}, C_{3k+1} \in S^3_P. \]

Thus, none of \( S^1_P, S^2_P \) and \( S^3_P \) is empty.

We conclude this part with an open problem.

**Problem 3.4.** Characterize the graphs belonging to \( S^i_P \), or find further properties of such graphs.

Remark that Dettlaff, Raczek and Topp recently characterized all trees belonging to \( S^1 \) and \( S^3 \) (see [6]).

### 3.1. Proofs.

For the proofs of Theorems 3.1 and 3.2, we need the following results.

**Theorem A** ([20]). Let \( \mathcal{H} \subseteq \mathcal{I} \) be nondegenerate and closed under union with \( K_1 \). Let \( G \) be a graph and \( v \in V(G) \).

1. If \( v \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G \) then \( \gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) \).
2. If \( \gamma_{\mathcal{H}}(G - v) < \gamma_{\mathcal{H}}(G) \) then \( \gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1 \). Moreover, if \( M \) is a \( \gamma_{\mathcal{H}} \)-set of \( G - v \) then \( M \cup \{v\} \) is a \( \gamma_{\mathcal{H}} \)-set of \( G \) and \( \{v\} = pm_G[v, M \cup \{v\}] \).

**Theorem B** ([20]). Let \( \mathcal{H} \subseteq \mathcal{I} \) be hereditary and closed under union with \( K_1 \). Let \( e = uv \) be an edge of a graph \( G \). If \( \gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G - e) \) then \( \gamma_{\mathcal{H}}(G - e) = \gamma_{\mathcal{H}}(G - e) - 1 \). Moreover, \( \gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1 \) if and only if at least one of the conditions (i) and (ii) stated in Theorem 3.1 holds.

**Theorem C** ([20]). Let \( e = xy \) be an edge of a graph \( G \) and let \( \mathcal{H} \subseteq \mathcal{I} \) be hereditary and closed under union with \( K_1 \). If \( \gamma_{\mathcal{H}}(G) > \gamma_{\mathcal{H}}(G - e) \) then:

1. no \( \gamma_{\mathcal{H}} \)-set of \( G - e \) is an \( \mathcal{H} \)-set of \( G \);
2. both \( x \) and \( y \) are in all \( \gamma_{\mathcal{H}} \)-sets of \( G - e \);
3. \( \gamma_{\mathcal{H}}(G - x) \geq \gamma_{\mathcal{H}}(G - e) \) and \( \gamma_{\mathcal{H}}(G - y) \geq \gamma_{\mathcal{H}}(G - e) \);
4. if \( \gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G - e) \) then \( y \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G - x \);
5. if \( \gamma_{\mathcal{H}}(G - y) = \gamma_{\mathcal{H}}(G - e) \) then \( x \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G - y \).

**Proof of Theorem 3.1.** Let \( D \) be a \( \gamma_{\mathcal{H}} \)-set of \( G - e \). Then since \( \mathcal{H} \) is closed under union with \( K_1 \), \( D \cup \{x_2\} \) is a dominating \( \mathcal{H} \)-set of \( G_{e,3} \). Hence \( \gamma_{\mathcal{H}}(G_{e,3}) \leq |D \cup \{y\}| \leq \gamma_{\mathcal{H}}(G - e) + 1 \).

For the left-hand side inequality, let \( \tilde{D} \) be a \( \gamma_{\mathcal{H}} \)-set of \( G_{e,3} \) and \( S = \tilde{D} \cap \{x_1, x_2, x_3\} \). If \( S = \{x_2\} \) then \( \tilde{D} - \{x_2\} \) is a dominating \( \mathcal{H} \)-set of \( G - e \) and \( \gamma_{\mathcal{H}}(G - e) \leq |	ilde{D} - \{x_2\}| = \gamma_{\mathcal{H}}(G_{e,3}) - 1 \). If \( S = \{x_1, x_2\} \) then \( pm_{G_{e,3}}[x_1, \tilde{D}] = \{u\} \) and hence \( \tilde{D}_1 = (\tilde{D} - \{x_1, x_2\}) \cup \{u\} \) is a dominating \( \mathcal{H} \)-set of \( G - e \), which implies \( \gamma_{\mathcal{H}}(G - e) \leq |\tilde{D}_1| < |\tilde{D}| = \gamma_{\mathcal{H}}(G_{e,3}). \)
Let $S = \{x_1\}$. If $u \in pm[x_1, \tilde{D}]$ then $\tilde{D}_2 = (\tilde{D} - \{x_1\}) \cup \{u\}$ is a dominating $H$-set of $G - e$ and hence $\gamma_H(G - e) \leq |\tilde{D}_2| = |\tilde{D}| = \gamma_H(G_{e,3})$. If $u \notin pm[x_1, \tilde{D}]$ then $\tilde{D} - \{x_1\}$ is a dominating $H$-set of $G - e$ and $\gamma_H(G - e) \leq |\tilde{D} - \{x_1\}| = \gamma_H(G_{e,3}) - 1$.

If $S = \{x_1, x_3\}$ then at least one of $pm_{G_{e,3}}[x_1, \tilde{D}] = \{x_1, u\}$ and $pm_{G_{e,3}}[x_3, \tilde{D}] = \{x_3, v\}$ holds, otherwise $(\tilde{D} - \{x_1, x_3\}) \cup \{x_2\}$ would be a dominating $H$-set of $G_{e,3}$, contradicting the choice of $\tilde{D}$. Say, without loss of generality, $pm_{G_{e,3}}[x_3, \tilde{D}] = \{x_3, v\}$. Then $\tilde{D}_3 = (\tilde{D} - \{x_3\}) \cup \{v\}$ is a $\gamma_H$-set of $G_{e,3}$ and $\tilde{D}_3 \cap \{x_1, x_2, x_3\} = \{x_1\}$. As above we obtain $\gamma_H(G - e) < \gamma_H(G_{e,3})$. By reason of symmetry, the left-hand side inequality is proved.

$(A_2) \Rightarrow (A_1)$ Let us assume without loss of generality that (i) holds. Let $D$ be a $\gamma_H(G - u)$-set and $v \in D$. By Theorem A, $D \cup \{u\}$ is a $\gamma_H$-set of $G - e$ and $\gamma_H(G_{e,3}) \leq |D \cup \{x_1\}| = \gamma_H(G - e)$. But we have already shown that $\gamma_H(G_{e,3}) \geq \gamma_H(G - e)$. Therefore $\gamma_H(G_{e,3}) = \gamma_H(G - e)$.

$(A_1) \Rightarrow (A_2)$ Suppose $\gamma_H(G - e) = \gamma_H(G_{e,3})$. Let $\tilde{D}$ be a $\gamma_H(G_{e,3})$-set and $S = D \cap \{x_1, x_2, x_3\}$. If $S = \{x_2\}$ then $\tilde{D} - \{x_2\}$ is a dominating $H$-set of $G - e$, a contradiction. If $S = \{x_1, x_2\}$ then clearly $pm_{G_{e,3}}[x_1, \tilde{D}] = \{u\}$, which implies that $(\tilde{D} - \{x_1, x_2\}) \cup \{u\}$ is a dominating $H$-set of $G - e$, a contradiction.

Let $S = \{x_1\}$. Hence $v \in \tilde{D}$. If $u \notin pm_{G_{e,3}}[x_1, \tilde{D}]$ then $\tilde{D} - \{x_1\}$ is a dominating $H$-set of $G - e$, a contradiction. If $u \in pm_{G_{e,3}}[x_1, \tilde{D}]$ then $D_1 = (\tilde{D} - \{x_1\}) \cup \{u\}$ is a $\gamma_H$-set of $G - e$, $u, v \in D_1, D_1 - \{u\}$ is a $\gamma_H$-set of $G - u$ (by Theorem A) and $v \in D_1 - \{u\}$. In addition it follows that $u \in V_{\tilde{H}}(G - e)$. Thus, (i) holds.

By symmetry we still have the case when $S = \{x_1, x_3\}$. If $u \notin pm_{G_{e,3}}[x_1, \tilde{D}]$ and $v \notin pm_{G_{e,3}}[x_3, \tilde{D}]$ then $\tilde{D} - \{x_1, x_3\}$ is a dominating $H$-set of $G - e$, a contradiction. If $u \in pm_{G_{e,3}}[x_1, \tilde{D}]$ and $v \notin pm_{G_{e,3}}[x_3, \tilde{D}]$ then $(\tilde{D} - \{x_1, x_3\}) \cup \{u\}$ is a dominating $H$-set of $G - e$, a contradiction. So, $u \in pm_{G_{e,3}}[x_1, \tilde{D}]$ and $v \in pm_{G_{e,3}}[x_3, \tilde{D}]$. Then $D_2 = (\tilde{D} - \{x_1, x_3\}) \cup \{u, v\}$ is a $\gamma_H$-set of $G - e$ and both $\{u\} = pm_{G - e}[x_1, D_2]$ and $\{v\} = pm_{G - e}[x_3, D_2]$ hold. Thus both (i) and (ii) are fulfilled.

$(A_2) \Leftrightarrow (A_3)$ By Theorem B.

Proof of Theorem 3.2. (i) Necessity: Let $\gamma_H(G) = \gamma_H(G_{e,3})$. By Theorem 3.1 we know that $\gamma_H(G - e) \leq \gamma_H(G_{e,3}) \leq \gamma_H(G - e) + 1$ and if $\gamma_H(G - e) = \gamma_H(G_{e,3})$ then $\gamma_H(G_{e,3}) = \gamma_H(G) + 1$. Thus $\gamma_H(G) = \gamma_H(G_{e,3}) = \gamma_H(G - e) + 1$.

Sufficiency: Let $\gamma_H(G - e) + 1 = \gamma_H(G)$. Assume $\gamma_H(G) \neq \gamma_H(G_{e,3})$. Now by Theorem 3.1, $\gamma_H(G_{e,3}) = \gamma_H(G - e)$. Applying again Theorem 3.1 we obtain $\gamma_H(G) = \gamma_H(G - e) - 1$, a contradiction. Thus, $\gamma_H(G) = \gamma_H(G_{e,3})$.

(ii) By (i), $\gamma_H(G) = \gamma_H(G_{e,3})$. Let $M$ be a $\gamma_H$-set of $G - e$ and $e = uv$. By Theorem C (ii), both $u$ and $v$ are in $M$. Then
(a) $M$ is a dominating $\mathcal{H}$-set of $G_{e,1}$ and $G_{e,2}$,
(b) $M \cup \{x_3\}$ is a dominating $\mathcal{H}$-set of $G_{e,4}$ and $G_{e,5}$, and
(c) $M \cup \{x_3, x_5\}$ is a dominating $\mathcal{H}$-set of $G_{e,6}$. Hence

\[(A) \quad \gamma_\mathcal{H}(G_{e,i}) \leq \gamma_\mathcal{H}(G-e) = \gamma_\mathcal{H}(G) - 1 \quad \text{for} \quad i = 1, 2; \quad \gamma_\mathcal{H}(G_{e,j}) \leq \gamma_\mathcal{H}(G-e) + 1 = \gamma_\mathcal{H}(G) \quad \text{for} \quad i = 4, 5; \quad \gamma_\mathcal{H}(G_{e,6}) \leq \gamma_\mathcal{H}(G-e) + 2 = \gamma_\mathcal{H}(G) + 1.\]

By Theorem C, $\min\{\gamma_\mathcal{H}(G - u), \gamma_\mathcal{H}(G - v)\} \geq \gamma_\mathcal{H}(G-e)$ and by Theorem A we have $\gamma_\mathcal{H}(G - \{u, v\}) = \gamma_\mathcal{H}((G - u) - v) \geq \gamma_\mathcal{H}(G - u) - 1 \geq \gamma_\mathcal{H}(G-e) - 1$. Suppose that $\gamma_\mathcal{H}(G - \{u, v\}) = \gamma_\mathcal{H}(G-e) - 1$. Then both $\gamma_\mathcal{H}(G - u) = \gamma_\mathcal{H}(G-e)$ and $\gamma_\mathcal{H}((G - u) - v) = \gamma_\mathcal{H}(G - u) - 1$ hold. By the second equality and Theorem A we deduce that $v$ belongs to some $\gamma_\mathcal{H}$-set of $G - u$. On the other hand, since $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G-e) + 1 > \gamma_\mathcal{H}(G - u), v$ belongs to no $\gamma_\mathcal{H}$-set of $G - u$, a contradiction. Thus,

\[(B) \quad \min\{\gamma_\mathcal{H}(G - u), \gamma_\mathcal{H}(G - v), \gamma_\mathcal{H}(G - \{u, v\})\} \geq \gamma_\mathcal{H}(G-e).\]

Let $D_t$ be a $\gamma_\mathcal{H}$-set of $G_{e,t}$ and $U_t = D_t \cap \{x_1, \ldots, x_t\}$, where $t = 1, \ldots, 6$.

**Case 1:** $t \in \{1, 2\}$. Assume $U_t \neq \emptyset$. Then $D_t - U_t$ is a dominating $\mathcal{H}$-set for at least one of the graphs $G-e$, $G-u$, $G-v$ and $G-\{u, v\}$. Using (B) we have

\[
\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G-e) + 1 \leq |D_t - U_t| + 1 = \gamma_\mathcal{H}(G_{e,t}) - |U_t| + 1 \leq \gamma_\mathcal{H}(G_{e,t}),
\]

contradicting (A). Thus $U_t$ is empty. But then $D_t$ is a dominating $\mathcal{H}$-set of $G - e$, which leads to $\gamma_\mathcal{H}(G_{e,t}) \geq \gamma_\mathcal{H}(G-e)$. Now by (A) the equality $\gamma_\mathcal{H}(G_{e,t}) = \gamma_\mathcal{H}(G-e)$ follows.

**Case 2:** $t \in \{4, 5\}$. Obviously $U_t \neq \emptyset$. As in Case 1 we obtain $\gamma_\mathcal{H}(G) \leq \gamma_\mathcal{H}(G_{e,t})$. Since by (A) $\gamma_\mathcal{H}(G_{e,t}) \leq \gamma_\mathcal{H}(G)$, we have $\gamma_\mathcal{H}(G_{e,t}) = \gamma_\mathcal{H}(G)$.

**Case 3:** $t = 6$. Clearly $|U_6| \geq 2$. As in Case 1 we obtain $\gamma_\mathcal{H}(G) \leq \gamma_\mathcal{H}(G_{e,6}) - |U_6| + 1$. Since $|U_6| \geq 2$, we have $\gamma_\mathcal{H}(G) \leq \gamma_\mathcal{H}(G_{e,6}) - 1$. Now by (A) we deduce that $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G_{e,6}) - 1$.

(iii) Immediately by (i) and (ii).

\[\square\]

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References


Author’s address: Vladimir Samodivkin, Departement of Mathematics, Faculty of Transportation Engineering, Civil Engineering and Geodesy, University of Architecture, 1 Hristo Smirnenski Blvd., 1046 Sofia, Bulgaria, e-mail: vl.samodivkin@gmail.com.