Mathematica Bohemica

Vladimir D. Samodivkin

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Mathematica Bohemica, Vol. 142 (2017), No. 1, 9–20

Persistent URL: `http://dml.cz/dmlcz/146005`

CHANGING OF THE DOMINATION NUMBER OF A GRAPH: EDGE MULTISUBDIVISION AND EDGE REMOVAL

Vladimir Samodivkin, Sofia

Received February 25, 2015. First published October 18, 2016.
Communicated by Václav Koubek

Abstract. For a graphical property $\mathcal{P}$ and a graph G, a subset S of vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the minimum cardinality of a dominating $\mathcal{P}$-set. We define the domination multisubdivision number with respect to $\mathcal{P}$, denoted by $\mathrm{msd}_{\mathcal{P}}(G)$, as a minimum positive integer k such that there exists an edge which must be subdivided k times to change $\gamma_{\mathcal{P}}(G)$. In this paper

(a) we present necessary and sufficient conditions for a change of $\gamma_{\mathcal{P}}(G)$ after subdividing an edge of G once,

(b) we prove that if e is an edge of a graph G then $\gamma_{\mathcal{P}}(G_{e,1}) < \gamma_{\mathcal{P}}(G)$ if and only if $\gamma_{\mathcal{P}}(G-e) < \gamma_{\mathcal{P}}(G)$ ($G_{e,t}$ denotes the graph obtained from G by subdivision of e with t vertices),

(c) we also prove that for every edge of a graph G we have $\gamma_{\mathcal{P}}(G-e) \leqslant \gamma_{\mathcal{P}}(G_{e,3}) \leqslant \gamma_{\mathcal{P}}(G-e)+1$, and

(d) we show that $\mathrm{msd}_{\mathcal{P}}(G) \leqslant 3$, where $\mathcal{P}$ is hereditary and closed under union with K_1.

Keywords: dominating set; edge subdivision; domination multisubdivision number; hereditary graph property

MSC 2010: 05C69

1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [14]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x,G)$ denotes the set of all neighbors of x in G, $N[x,G] = N(x,G) \cup \{x\}$ and the degree of x is $\deg(x,G) = |N(x,G)|$. The maximum and minimum degrees of vertices in the graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For

DOI: 10.21136/MB.2017.0009-15

a graph G, let $x \in X \subseteq V(G)$. A vertex y is a *private neighbor of x with respect to X* if $N[y,G] \cap X = \{x\}$. The *private neighbor set of x with respect to X* is $pn_G[x,X] = \{y\colon N[y,G] \cap X = \{x\}\}$. For a graph G, the subdivision of the edge $e = uv \in E(G)$ with a vertex x leads to a graph with the vertex set $V \cup \{x\}$ and the edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e,t}$ denote the graph obtained from G by a subdivision of the edge e with t vertices (instead of the edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For $t = 1$ we write G_e.

Let $\mathcal{I}$ denote the set of all mutually non-isomorphic graphs. A *graph property* is any nonempty subset of $\mathcal{I}$. We say that a *graph G has the property $\mathcal{P}$* whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example, we list some graph properties:

▷ $\mathcal{O} = \{H \in \mathcal{I}\colon H$ is totally disconnected$\}$;
▷ $\mathcal{C} = \{H \in \mathcal{I}\colon H$ is connected$\}$;
▷ $\mathcal{T} = \{H \in \mathcal{I}\colon \delta(H) \geqslant 1\}$;
▷ $\mathcal{M} = \{H \in \mathcal{I}\colon H$ has a perfect matching$\}$;
▷ $\mathcal{F} = \{H \in \mathcal{I}\colon H$ is a forest$\}$;
▷ $\mathcal{UK} = \{H \in \mathcal{I}\colon$ each component of H is complete$\}$;
▷ $\mathcal{D}_k = \{H \in \mathcal{I}\colon \Delta(H) \leqslant k\}$.

A graph property $\mathcal{P}$ is called:

(a) hereditary (induced-hereditary), if the fact that a graph G has property $\mathcal{P}$ implies that all subgraphs (induced subgraphs) of G also belong to $\mathcal{P}$, and
(b) nondegenerate if $\mathcal{O} \subseteq \mathcal{P}$. Any set $S \subseteq V(G)$ such that the induced subgraph $\langle S, G\rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set.

Note that:

(a) $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{D}_k$ are nondegenerate and hereditary properties;
(b) $\mathcal{UK}$ is nondegenerate, induced-hereditary and is not hereditary;
(c) all $\mathcal{C}$, $\mathcal{T}$ and $\mathcal{M}$ are neither induced-hereditary nor nondegenerate. For a survey on this subject we refer to Borowiecki et al. [2].

A set of vertices $D \subseteq V(G)$ is a dominating set of a graph G if every vertex not in D is adjacent to a vertex in D. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of G. A dominating $\mathcal{P}$-set of G with cardinality $\gamma_{\mathcal{P}}(G)$ is called a *$\gamma_{\mathcal{P}}$-set of G*. If a property $\mathcal{P}$ is nondegenerate, then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. Note that $\gamma_{\mathcal{I}}(G)$, $\gamma_{\mathcal{O}}(G)$, $\gamma_{\mathcal{C}}(G)$, $\gamma_{\mathcal{T}}(G)$, $\gamma_{\mathcal{M}}(G)$, $\gamma_{\mathcal{F}}(G)$ and $\gamma_{\mathcal{D}_k}(G)$ are well known as the domination number $\gamma(G)$, the independent domination number $i(G)$ ([5]), the connected domination number $\gamma_c(G)$ ([24]), the total domination number $\gamma_t(G)$ ([3]), the paired-domination number $\gamma_{pr}(G)$ ([16]), the acyclic domination number $\gamma_a(G)$ ([17]) and the k-dependent domination number $\gamma^k(G)$ ([9]). The concept of domination with respect to any graph property $\mathcal{P}$ was introduced by

Goddard et al. [10] and has been studied, for example, in [19], [20], [21], [22], [23] and elsewhere.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In [20], the present author began the study of the effects on $\gamma_{\mathcal{P}}(G)$ when a graph G is modified by deleting a vertex or by adding an edge ($\mathcal{P}$ is nondegenerate). In this paper we concentrate on effects on $\gamma_{\mathcal{P}}(G)$ when a graph is modified by deleting/subdividing an edge. An edge e of a graph G is called a $\gamma_{\mathcal{P}}$-ER^--critical edge of G if $\gamma_{\mathcal{P}}(G) > \gamma_{\mathcal{P}}(G - e)$. Note that

(a) γ-ER^--critical edges do not exist (see [13]),

(b) Grobler [11] was the first who began the investigation of $\gamma_{\mathcal{P}}$-ER^--critical edges when $\mathcal{P} = \mathcal{O}$, and

(c) necessary and sufficient conditions for an edge of a graph G to be $\gamma_{\mathcal{P}}$-ER^--critical, where $\mathcal{P}$ is hereditary, may be found in [20].

One measure of the stability of the domination number of G under edge subdivision is the domination subdivision number with respect to the property $\mathcal{P}$, denoted $\mathrm{sd}^+_{\gamma_{\mathcal{P}}}(G)$, which is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase $\gamma_{\mathcal{P}}(G)$. The following special cases for $\mathrm{sd}^+_{\gamma_{\mathcal{P}}}(G)$ have been investigated up to now:

(a) $\mathrm{sd}^+_{\gamma_{\mathcal{I}}}(G)$—the domination subdivision number defined by Velammal [25],

(b) $\mathrm{sd}^+_{\gamma_{\mathcal{T}}}(G)$—the total domination subdivision number introduced by Haynes et al. in [15],

(c) $\mathrm{sd}^+_{\gamma_{\mathcal{M}}}(G)$—the paired domination subdivision number introduced by Favaron et al. in [8],

(d) $\mathrm{sd}^+_{\gamma_{\mathcal{C}}}(G)$—the connected domination subdivision number introduced by Favaron et al. in [7], and

(e) $\mathrm{sd}^+_{\gamma_{\mathcal{P}}}(G)$—the domination subdivision number with respect to the nondegenerate property $\mathcal{P}$ introduced by the present author in [23].

Here we focus on the existence of critical edges with respect to the subdivision/multisubdivision. Results in this direction, in the case when $\mathcal{P} = \mathcal{I}$, were recently obtained by Lemańska, Tey and Zuazua [18] and Dettlaff, Raczek and Topp [6]. For any nondegenerate property $\mathcal{P} \subseteq \mathcal{I}$ we define an edge e of a graph G to be

(i) a $\gamma_{\mathcal{P}}$-S^+-critical edge of G if $\gamma_{\mathcal{P}}(G) < \gamma_{\mathcal{P}}(G_e)$, and

(ii) a $\gamma_{\mathcal{P}}$-S^--critical edge of G if $\gamma_{\mathcal{P}}(G) > \gamma_{\mathcal{P}}(G_e)$.

In Section 2:

(a) we present necessary and sufficient conditions for a change of $\gamma_{\mathcal{P}}(G)$ after subdividing an edge of G once, and

(b) we prove that an edge e of a graph G is $\gamma_{\mathcal{H}}$-S$^-$-critical if and only if e is $\gamma_{\mathcal{H}}$-ER^--critical, for any graph property $\mathcal{H} \subseteq \mathcal{I}$ which is induced-hereditary and closed under union with K_1.

In Section 3 we deal with changes of $\gamma_{\mathcal{P}}(G)$ when an edge of G is multiple subdivided. To present our results we need the following definitions.

For every edge e of a graph G let

▷ $\text{msd}_{\mathcal{P}}(e) = \min\{t\colon\ \gamma_{\mathcal{P}}(G_{e,t}) \neq \gamma_{\mathcal{P}}(G)\}$;

▷ $\text{msd}^{+}_{\mathcal{P}}(e) = \min\{t\colon\ \gamma_{\mathcal{P}}(G_{e,t}) > \gamma_{\mathcal{P}}(G)\}$;

▷ $\text{msd}^{-}_{\mathcal{P}}(e) = \min\{t\colon\ \gamma_{\mathcal{P}}(G_{e,t}) < \gamma_{\mathcal{P}}(G)\}$.

If $\gamma_{\mathcal{P}}(G_{e,t}) \geqslant \gamma_{\mathcal{P}}(G)$ for every $t \geqslant 1$, then we write $\text{msd}^{-}_{\mathcal{P}}(e) = \infty$. If $\gamma_{\mathcal{P}}(G_{e,t}) \leqslant \gamma_{\mathcal{P}}(G)$ for every $t \geqslant 1$, then we write $\text{msd}^{+}_{\mathcal{P}}(e) = \infty$.

Definition 1.1. For every graph G with at least one edge and every nondegenerate property $\mathcal{P}$, we define the domination multisubdivision (plus domination multisubdivision, minus domination multisubdivision) number with respect to the property $\mathcal{P}$, denoted $\text{msd}_{\mathcal{P}}(G)$ ($\text{msd}^{+}_{\mathcal{P}}$, $\text{msd}^{-}_{\mathcal{P}}(G)$, respectively) to be

▷ $\text{msd}_{\mathcal{P}}(G) = \min\{\text{msd}_{\mathcal{P}}(e)\colon\ e \in E(G)\}$,

▷ $\text{msd}^{+}_{\mathcal{P}}(G) = \min\{\text{msd}^{+}_{\mathcal{P}}(e)\colon\ e \in E(G)\}$,

▷ $\text{msd}^{-}_{\mathcal{P}}(G) = \min\{\text{msd}^{-}_{\mathcal{P}}(e)\colon\ e \in E(G)\}$,

respectively. If $\gamma_{\mathcal{P}}(G_{e,t}) \geqslant \gamma_{\mathcal{P}}(G)$ for every t and every edge $e \in E(G)$, then we write $\text{msd}^{-}_{\mathcal{P}}(G) = \infty$.

The parameters $\text{msd}^{+}(G)$ and $\text{msd}^{+}_{\mathcal{T}}(G)$ (in our designation) were introduced by Dettlaff, Raczek and Topp in [6] and by Avella-Alaminos, Dettlaff, Lemańska and Zuazua in [1], respectively. Note that in the case when $\mathcal{P} = \mathcal{I}$, clearly, $\text{msd}(G) = \text{msd}^{+}(G)$, and $\text{msd}^{-}(G) = \infty$. In Section 3 we prove that for every edge of a graph G we have $\gamma_{\mathcal{P}}(G-e) \leqslant \gamma_{\mathcal{P}}(G_{e,3}) \leqslant \gamma_{\mathcal{P}}(G-e)+1$ and we present necessary and sufficient conditions for the validity of $\gamma_{\mathcal{P}}(G-e) = \gamma_{\mathcal{P}}(G_{e,3})$. Our main result in that section is that $\text{msd}_{\mathcal{P}}(G) \leqslant 3$ for any graph G and any graph property $\mathcal{P}$ which is hereditary and closed under union with K_1.

2. Single subdivision: critical edges

We begin this section with a characterization of $\gamma_{\mathcal{P}}$-S^{+}-critical edges of a graph. Note that if a property $\mathcal{P}$ is induced-hereditary and closed under union with K_1 then $\mathcal{P}$ is nondegenerate.

Theorem 2.1. *Let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with K_1. Let G be a graph and $e = uv \in E(G)$. Then $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G) + 1$. If e is a $\gamma_{\mathcal{H}}$-S^{+}-critical edge of G then $\gamma_{\mathcal{H}}(G_e) = \gamma_{\mathcal{H}}(G)+1$ and for each $\gamma_{\mathcal{H}}$-set M of G one of the following conditions holds:*

(i) *$u, v \in V(G) - M$;*

(ii) $u \in M$, $v \in pn_G[u, M]$ *and* $pn_G[u, M]$ *is not a subset of* $\{u, v\}$;

(iii) $v \in M$, $u \in pn_G[v, M]$ *and* $pn_G[u, M]$ *is not a subset of* $\{u, v\}$.

If e *is not* $\gamma_{\mathcal{P}}$-S^+*-critical and for each* $\gamma_{\mathcal{H}}$*-set* M *of* G *one of* (i), (ii) *and* (iii) *holds then there is a dominating* $\mathcal{H}$*-set* R *of* $G - uv$ *with* $u, v \in R$ *and* $|R| \leqslant \gamma_{\mathcal{H}}(G)$.

Proof. Let $x \in V(G_e)$ be the subdivision vertex and let M be a $\gamma_{\mathcal{H}}$-set of G. If $u, v \notin M$ then $M \cup \{x\}$ is a dominating $\mathcal{H}$-set of G_e ($\mathcal{H}$ is closed under union with K_1) and we have $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G) + 1$. If both u and v are in M then M is a dominating $\mathcal{H}$-set of G_e($\mathcal{H}$ is hereditary), which implies $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. If $u \in M$, $v \notin M$ and $v \notin pn_G[u, M]$ then again M is a dominating $\mathcal{H}$-set of G_e and hence $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. So, let $u \in M$, $v \notin M$ and $v \in pn_G[u, M]$. If either $\{v\}$ or $\{u, v\}$ coincides with $pn_G[u, M]$ then $(M - \{u\}) \cup \{x\}$ is a dominating $\mathcal{H}$-set of G_e; hence $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. If neither $pn_G[u, M] = \{v\}$ nor $pn_G[u, M] = \{u, v\}$ then $M \cup \{v\}$ is a dominating $\mathcal{H}$-set of G_e and we have $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G) + 1$. Thus $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G) + 1$ and if the equality is fulfilled then one of (i), (ii) and (iii) holds.

Now, let for each $\gamma_{\mathcal{H}}$-set M of G one of (i), (ii) and (iii) holds. Assume $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$ and let R be a $\gamma_{\mathcal{H}}$-set of G_e.

Case 1: $u, v \notin R$. Hence $x \in R$. If $u, v \notin pn_{G_e}[x, R]$ then $R - \{x\}$ is a dominating $\mathcal{H}$-set of G, a contradiction with $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. If $u \in pn_{G_e}[x, R]$ and $v \notin pn_{G_e}[x, R]$ then $R_1 = (R - \{x\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of G of cardinality $|R_1| = |R| = \gamma_{\mathcal{H}}(G_e)$. Since $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$, we have that R_1 is a $\gamma_{\mathcal{H}}$-set of G. But then $u \in R_1$, $v \notin R_1$ and $v \notin pn_G[u, R_1]$, contradicting (ii). If $u, v \in pn_G[x, R]$ then as above R_1 is a $\gamma_{\mathcal{H}}$-set of G and since $u \in R_1$ and $\{u, v\} = pn_G[u, R_1]$, again we arrive at a contradiction with (ii).

Case 2: $u \in R$ and $v \notin R$. Hence $x \notin R$, otherwise $R - \{x\}$ is a dominating $\mathcal{H}$-set of G, contradicting $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. This implies that R is a $\gamma_{\mathcal{H}}$-set of G, $u \in R$ and $v \notin pn_G[u, R]$, a contradiction with (ii).

Case 3: $u, v \in R$. Hence R is a dominating $\mathcal{H}$-set of $G - uv$ and $|R| = \gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G)$. □

When we restrict our attention to the case where $\mathcal{H} = \mathcal{I}$, we can describe more precisely when an edge of a graph G is γ-S^+-critical.

Corollary 2.2. *Let* G *be a graph and* $e = uv \in E(G)$. *Then* e *is a* γ-S^+*-critical edge of* G *if and only if for each* γ*-set* M *of* G *one of* (i), (ii) *and* (iii) *stated in* Theorem 2.1 *holds.*

Proof. *Necessity*: The result immediately follows by Theorem 2.1.

Sufficiency: Assume $\gamma(G_e) \leqslant \gamma(G)$. Then by Theorem 2.1, there is a dominating set R of $G - uv$ with $u, v \in R$ and $|R| \leqslant \gamma(G)$. But it is a well known fact that if f

is an edge of a graph G then always $\gamma(G-f) \geqslant \gamma(G)$. Hence R is a γ-set of both G and $G-e$ and $u, v \in R$, contradicting all (i), (ii) and (iii). □

Theorem 2.3. *Let $\mathcal{H} \subseteq \mathcal{I}$ be induced-hereditary and closed under union with K_1. An edge e of a graph G is $\gamma_{\mathcal{H}}$-S^--critical if and only if e is $\gamma_{\mathcal{H}}$-ER^--critical.*

Proof. As we have already shown, $\mathcal{H}$ is nondegenerate and then all $\gamma_{\mathcal{H}}(G-e)$, $\gamma_{\mathcal{H}}(G_e)$ and $\gamma_{\mathcal{H}}(G)$ exist. Let v be the subdivision vertex of G_e.

Sufficiency: Let $e = xy$ be a $\gamma_{\mathcal{H}}$-ER^--critical edge of G and M a $\gamma_{\mathcal{H}}$-set of $G-e$. Hence $\gamma_{\mathcal{H}}(G-e) < \gamma_{\mathcal{H}}(G)$ and $x, y \in M$. But then M is a dominating $\mathcal{H}$-set of G_e, which leads to $\gamma_{\mathcal{H}}(G_e) \leqslant \gamma_{\mathcal{H}}(G-e) < \gamma_{\mathcal{H}}(G)$.

Necessity: Let $e = xy$ be a $\gamma_{\mathcal{H}}$-S$^-$-critical edge of G and M a $\gamma_{\mathcal{H}}$-set of G_e. Hence $\gamma_{\mathcal{H}}(G_e) < \gamma_{\mathcal{H}}(G)$. Assume $v \notin M$. Hence at least one of x and y is in M. If both $x, y \in M$ then M is a dominating $\mathcal{H}$-set of $G-e$ and the result follows. If $x \notin M$ and $y \in M$ then M is a dominating $\mathcal{H}$-set of G, a contradiction. Thus we may assume v is in all $\gamma_{\mathcal{H}}$-sets of G_e. Since $\mathcal{H}$ is induced-hereditary, at least one of x and y is not in M. First let $x \in M$ and $y \notin M$. Then $y \in pn_{G_e}[v, M]$, which implies $M - \{v\}$ is a dominating $\mathcal{H}$-set of G, a contradiction. Hence neither x nor y are in M. If $x, y \notin pn_{G_e}[v, M]$ then $M - \{v\}$ is a dominating $\mathcal{H}$-set of G, a contradiction. Hence at least one of x and y, say y, is in $pn_{G_e}[v, M]$. But then $(M - \{v\}) \cup \{y\}$ is a dominating $\mathcal{H}$-set of G, a contradiction. □

Note that

(a) there do not exist γ-ER^--critical edges (see [13]), and

(b) necessary and sufficient conditions for an edge of a graph G to be $\gamma_{\mathcal{H}}$-ER^--critical may be found in [20].

Now we define the following classes of graphs:

- ▷ $(CS^-_{\mathcal{P}})$ $\gamma_{\mathcal{P}}(G) > \gamma_{\mathcal{P}}(G_e)$ for every edge e of G, and
- ▷ $(CER^-_{\mathcal{P}})$ $\gamma_{\mathcal{P}}(G) > \gamma_{\mathcal{P}}(G-e)$ for every edge e of G.

As an immediate consequence of Theorem 2.3 we obtain the next result.

Corollary 2.4. *If $\mathcal{H} \subseteq \mathcal{I}$ is induced-hereditary and closed under union with K_1 then the classes of graphs $CS^-_{\mathcal{P}}$ and $CER^-_{\mathcal{P}}$ coincide.*

Note that the class $CER^-_{\mathcal{P}}$ in the case when $\mathcal{P} = \mathcal{O}$ was introduced by Grobler [11] and also considered in [12], [13], [4].

We first state our theorems, then we pose a problem they generate, and after that we give the proofs.

Recall that $G_{e,t}$ denotes the graph obtained from a graph G by the subdivision of the edge $e \in E(G)$ with t vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For any graph G and any nondegenerate property $\mathcal{P}$ let us denote by $V^-_{\mathcal{P}}(G)$ the set $\{v \in V(G)\colon \gamma_{\mathcal{P}}(G - v) < \gamma_{\mathcal{P}}(G)\}$. Our first result shows that the value of the difference $\gamma_{\mathcal{P}}(G_{e,3}) - \gamma_{\mathcal{P}}(G - e)$ is either 0 or 1.

Theorem 3.1. *Let $\mathcal{H} \subseteq \mathcal{I}$ be induced-hereditary and closed under union with K_1. If $e = uv$ is an edge of a graph G then $\gamma_{\mathcal{H}}(G - e) \leqslant \gamma_{\mathcal{H}}(G_{e,3}) \leqslant \gamma_{\mathcal{H}}(G - e) + 1$. Moreover, the following conditions are equivalent:*

($\mathbb{A}_1$) $\gamma_{\mathcal{H}}(G - e) = \gamma_{\mathcal{H}}(G_{e,3})$;

($\mathbb{A}_2$) *at least one of the following holds:*

- (i) $u \in V^-_{\mathcal{H}}(G - e)$ *and* v *belongs to some* $\gamma_{\mathcal{H}}$*-set of* $G - u$;
- (ii) $v \in V^-_{\mathcal{H}}(G - e)$ *and* u *belongs to some* $\gamma_{\mathcal{H}}$*-set of* $G - v$.

If in addition $\mathcal{H}$ is hereditary then ($\mathbb{A}_1$) *and* ($\mathbb{A}_2$) *are equivalent to*

($\mathbb{A}_3$) $\gamma_{\mathcal{H}}(G - e) = 1 + \gamma_{\mathcal{H}}(G)$.

The main result in this section is the following.

Theorem 3.2. *Let e be an edge of a graph G and let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with K_1.*

- (i) *Then* $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3})$ *if and only if* $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) + 1$.
- (ii) *If* $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) + 1$ *then* $\mathrm{msd}_{\mathcal{H}}(e) = \mathrm{msd}^-_{\mathcal{H}}(e) = 1$, $\mathrm{msd}^+_{\mathcal{H}}(e) = 6$ *and* $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,1}) + 1 = \gamma_{\mathcal{H}}(G_{e,2}) + 1 = \gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G_{e,4}) = \gamma_{\mathcal{H}}(G_{e,5}) = \gamma_{\mathcal{H}}(G_{e,6}) - 1$.
- (iii) *Then* $\mathrm{msd}_{\mathcal{H}}(e) \leqslant 3$. *In particular* (*Dettlaff, Raczek and Topp* [6] *when* $\mathcal{H} = \mathcal{I}$), $\mathrm{msd}_{\mathcal{H}}(G) \leqslant 3$.

Example 3.3. It is easy to see that if $G = K_{3n_2\ldots n_m}$, where $m \geqslant 2$ and $n_i \geqslant 3$ for $2 \leqslant i \leqslant m$, then $\gamma_{\mathcal{O}}(G) = \gamma_{\mathcal{O}}(G_{e,3}) = \gamma_{\mathcal{O}}(G - e) + 1 = 3$ for every edge e of G. Hence by Theorem 3.2, $\mathrm{msd}_{\mathcal{O}}(G) = \mathrm{msd}^-_{\mathcal{O}}(G) = 1$ and $\mathrm{msd}^+_{\mathcal{O}}(G) = 6$.

In view of Theorem 3.2 (iii), we can split the family of all graphs G into three classes with respect to the value of $\mathrm{msd}_{\mathcal{P}}(G)$, where $\mathcal{P} \subseteq \mathcal{I}$ is hereditary and closed under union with K_1. We define that a graph G belongs to the class $S^i_{\mathcal{P}}$ whenever $\mathrm{msd}_{\mathcal{P}}(G) = i$, $i \in \{1, 2, 3\}$. It is straightforward to verify that if $k \geqslant 1$ and $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I}$ then

▷ $P_{3k}, C_{3k} \in S^1_{\mathcal{P}}$; $P_{3k+2}, C_{3k+2} \in S^2_{\mathcal{P}}$; and $P_{3k+1}, C_{3k+1} \in S^3_{\mathcal{P}}$.

Thus, none of $S^1_{\mathcal{P}}, S^2_{\mathcal{P}}$ and $S^3_{\mathcal{P}}$ is empty.

We conclude this part with an open problem.

P r o b l e m 3.4. Characterize the graphs belonging to $S^i_{\mathcal{P}}$, or find further properties of such graphs.

Remark that Dettlaff, Raczek and Topp recently characterized all trees belonging to S^1 and S^3 (see [6]).

3.1. Proofs. For the proofs of Theorems 3.1 and 3.2, we need the following results.

Theorem A ([20]). *Let $\mathcal{H} \subseteq \mathcal{I}$ be nondegenerate and closed under union with K_1. Let G be a graph and $v \in V(G)$.*

(i) *If v belongs to no $\gamma_{\mathcal{H}}$-set of G then $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G)$.*

(ii) *If $\gamma_{\mathcal{H}}(G - v) < \gamma_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$. Moreover, if M is a $\gamma_{\mathcal{H}}$-set of $G - v$ then $M \cup \{v\}$ is a $\gamma_{\mathcal{H}}$-set of G and $\{v\} = pn_G[v, M \cup \{v\}]$.*

Theorem B ([20]). *Let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with K_1. Let $e = uv$ be an edge of a graph G. If $\gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G - e)$ then $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1$. Moreover, $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1$ if and only if at least one of the conditions* (i) *and* (ii) *stated in Theorem* 3.1 *holds.*

Theorem C ([20]). *Let $e = xy$ be an edge of a graph G and let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with K_1. If $\gamma_{\mathcal{H}}(G) > \gamma_{\mathcal{H}}(G - e)$ then:*

(i) *no $\gamma_{\mathcal{H}}$-set of $G - e$ is an $\mathcal{H}$-set of G;*

(ii) *both x and y are in all $\gamma_{\mathcal{H}}$-sets of $G - e$;*

(iii) *$\gamma_{\mathcal{H}}(G - x) \geqslant \gamma_{\mathcal{H}}(G - e)$ and $\gamma_{\mathcal{H}}(G - y) \geqslant \gamma_{\mathcal{H}}(G - e)$;*

(iv) *if $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G - e)$ then y belongs to no $\gamma_{\mathcal{H}}$-set of $G - x$;*

(v) *if $\gamma_{\mathcal{H}}(G - y) = \gamma_{\mathcal{H}}(G - e)$ then x belongs to no $\gamma_{\mathcal{H}}$-set of $G - y$.*

P r o o f of Theorem 3.1. Let D be a $\gamma_{\mathcal{H}}$-set of $G - e$. Then since $\mathcal{H}$ is closed under union with K_1, $D \cup \{x_2\}$ is a dominating $\mathcal{H}$-set of $G_{e,3}$. Hence $\gamma_{\mathcal{H}}(G_{e,3}) \leqslant |D \cup \{y\}| \leqslant \gamma_{\mathcal{H}}(G - e) + 1$.

For the left-hand side inequality, let $\widetilde{D}$ be a $\gamma_{\mathcal{H}}$-set of $G_{e,3}$ and $S = \widetilde{D} \cap \{x_1, x_2, x_3\}$. If $S = \{x_2\}$ then $\widetilde{D} - \{x_2\}$ is a dominating $\mathcal{H}$-set of $G - e$ and $\gamma_{\mathcal{H}}(G - e) \leqslant |\widetilde{D} - \{x_2\}| = \gamma_{\mathcal{H}}(G_{e,3}) - 1$. If $S = \{x_1, x_2\}$ then $pn_{G_{e,3}}[x_1, \widetilde{D}] = \{u\}$ and hence $\widetilde{D}_1 = (\widetilde{D} - \{x_1, x_2\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$, which implies $\gamma_{\mathcal{H}}(G - e) \leqslant |\widetilde{D}_1| < |\widetilde{D}| = \gamma_{\mathcal{H}}(G_{e,3})$.

Let $S = \{x_1\}$. If $u \in pn[x_1, \widetilde{D}]$ then $\widetilde{D}_2 = (\widetilde{D} - \{x_1\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$ and hence $\gamma_{\mathcal{H}}(G - e) \leqslant |\widetilde{D}_2| = |\widetilde{D}| = \gamma_{\mathcal{H}}(G_{e,3})$. If $u \notin pn[x_1, \widetilde{D}]$ then $\widetilde{D} - \{x_1\}$ is a dominating $\mathcal{H}$-set of $G - e$ and $\gamma_{\mathcal{H}}(G - e) \leqslant |\widetilde{D}| - 1 = \gamma_{\mathcal{H}}(G_{e,3}) - 1$.

If $S = \{x_1, x_3\}$ then at least one of $pn_{G_{e,3}}[x_1, \widetilde{D}] = \{x_1, u\}$ and $pn_{G_{e,3}}[x_3, \widetilde{D}] = \{x_3, v\}$ holds, otherwise $(\widetilde{D} - \{x_1, x_3\}) \cup \{x_2\}$ would be a dominating $\mathcal{H}$-set of $G_{e,3}$, contradicting the choice of $\widetilde{D}$. Say, without loss of generality, $pn_{G_{e,3}}[x_3, \widetilde{D}] = \{x_3, v\}$. Then $\widetilde{D}_3 = (\widetilde{D} - \{x_3\}) \cup \{v\}$ is a $\gamma_{\mathcal{H}}$-set of $G_{e,3}$ and $\widetilde{D}_3 \cap \{x_1, x_2, x_3\} = \{x_1\}$. As above we obtain $\gamma_{\mathcal{H}}(G - e) < \gamma_{\mathcal{H}}(G_{e,3})$. By reason of symmetry, the left-hand side inequality is proved.

$(\mathbb{A}_2) \Rightarrow (\mathbb{A}_1)$ Let us assume without loss of generality that (i) holds. Let D be a $\gamma_{\mathcal{H}}(G - u)$-set and $v \in D$. By Theorem A, $D \cup \{u\}$ is a $\gamma_{\mathcal{H}}$-set of $G - e$ and $pn_{G-e}[u, D \cup \{u\}] = \{u\}$. Hence $D \cup \{x_1\}$ is a dominating $\mathcal{H}$-set of $G_{e,3}$ and $\gamma_{\mathcal{H}}(G_{e,3}) \leqslant |D \cup \{x_1\}| = \gamma_{\mathcal{H}}(G - e)$. But we have already shown that $\gamma_{\mathcal{H}}(G_{e,3}) \geqslant \gamma_{\mathcal{H}}(G - e)$. Therefore $\gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G - e)$.

$(\mathbb{A}_1) \Rightarrow (\mathbb{A}_2)$ Suppose $\gamma_{\mathcal{H}}(G - e) = \gamma_{\mathcal{H}}(G_{e,3})$. Let $\widetilde{D}$ be a $\gamma_{\mathcal{H}}(G_{e,3})$-set and $S = \widetilde{D} \cap \{x_1, x_2, x_3\}$. If $S = \{x_2\}$ then $\widetilde{D} - \{x_2\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $S = \{x_1, x_2\}$ then clearly $pn_{G_{e,3}}[x_1, \widetilde{D}] = \{u\}$, which implies that $(\widetilde{D} - \{x_1, x_2\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction.

Let $S = \{x_1\}$. Hence $v \in \widetilde{D}$. If $u \notin pn_{G_{e,3}}[x_1, \widetilde{D}]$ then $\widetilde{D} - \{x_1\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $u \in pn_{G_{e,3}}[x_1, \widetilde{D}]$ then $D_1 = (\widetilde{D} - \{x_1\}) \cup \{u\}$ is a $\gamma_{\mathcal{H}}$-set of $G - e$, $u, v \in D_1$, $D_1 - \{u\}$ is a $\gamma_{\mathcal{H}}$-set of $G - u$ (by Theorem A) and $v \in D_1 - \{u\}$. In addition it follows that $u \in V_{\mathcal{H}}^-(G - e)$. Thus, (i) holds.

By symmetry we still have the case when $S = \{x_1, x_3\}$. If $u \notin pn_{G_{e,3}}[x_1, \widetilde{D}]$ and $v \notin pn_{G_{e,3}}[x_3, \widetilde{D}]$ then $\widetilde{D} - \{x_1, x_3\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $u \in pn_{G_{e,3}}[x_1, \widetilde{D}]$ and $v \notin pn_{G_{e,3}}[x_3, \widetilde{D}]$ then $(\widetilde{D} - \{x_1, x_3\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. So, $u \in pn_{G_{e,3}}[x_1, \widetilde{D}]$ and $v \in pn_{G_{e,3}}[x_3, \widetilde{D}]$. Then $D_2 = (\widetilde{D} - \{x_1, x_3\}) \cup \{u, v\}$ is a $\gamma_{\mathcal{H}}$-set of $G - e$ and both $\{u\} = pn_{G-e}[x_1, D_2]$ and $\{v\} = pn_{G-e}[x_3, D_2]$ hold. Thus both (i) and (ii) are fulfilled.

$(\mathbb{A}_2) \Leftrightarrow (\mathbb{A}_3)$ By Theorem B. □

P r o o f of Theorem 3.2. (i) *Necessity*: Let $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3})$. By Theorem 3.1 we know that $\gamma_{\mathcal{H}}(G - e) \leqslant \gamma_{\mathcal{H}}(G_{e,3}) \leqslant \gamma_{\mathcal{H}}(G - e) + 1$ and if $\gamma_{\mathcal{H}}(G - e) = \gamma_{\mathcal{H}}(G_{e,3})$ then $\gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G) + 1$. Thus $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G - e) + 1$.

Sufficiency: Let $\gamma_{\mathcal{H}}(G - e) + 1 = \gamma_{\mathcal{H}}(G)$. Assume $\gamma_{\mathcal{H}}(G) \neq \gamma_{\mathcal{H}}(G_{e,3})$. Now by Theorem 3.1, $\gamma_{\mathcal{H}}(G_{e,3}) = \gamma_{\mathcal{H}}(G - e)$. Applying again Theorem 3.1 we obtain $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1$, a contradiction. Thus, $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3})$.

(ii) By (i), $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,3})$. Let M be a $\gamma_{\mathcal{H}}$-set of $G - e$ and $e = uv$. By Theorem C (ii), both u and v are in M. Then

(a) M is a dominating $\mathcal{H}$-set of $G_{e,1}$ and $G_{e,2}$,

(b) $M \cup \{x_3\}$ is a dominating $\mathcal{H}$-set of $G_{e,4}$ and $G_{e,5}$, and

(c) $M \cup \{x_3, x_5\}$ is a dominating $\mathcal{H}$-set of $G_{e,6}$. Hence

(A) $\gamma_{\mathcal{H}}(G_{e,i}) \leqslant \gamma_{\mathcal{H}}(G-e) = \gamma_{\mathcal{H}}(G) - 1$ for $i = 1, 2$; $\gamma_{\mathcal{H}}(G_{e,j}) \leqslant \gamma_{\mathcal{H}}(G-e) + 1 = \gamma_{\mathcal{H}}(G)$ for $i = 4, 5$; $\gamma_{\mathcal{H}}(G_{e,6}) \leqslant \gamma_{\mathcal{H}}(G-e) + 2 = \gamma_{\mathcal{H}}(G) + 1$.

By Theorem C, $\min\{\gamma_{\mathcal{H}}(G-u), \gamma_{\mathcal{H}}(G-v)\} \geqslant \gamma_{\mathcal{H}}(G-e)$ and by Theorem A we have $\gamma_{\mathcal{H}}(G - \{u,v\}) = \gamma_{\mathcal{H}}((G-u)-v) \geqslant \gamma_{\mathcal{H}}(G-u) - 1 \geqslant \gamma_{\mathcal{H}}(G-e) - 1$. Suppose that $\gamma_{\mathcal{H}}(G - \{u,v\}) = \gamma_{\mathcal{H}}(G-e) - 1$. Then both $\gamma_{\mathcal{H}}(G-u) = \gamma_{\mathcal{H}}(G-e)$ and $\gamma_{\mathcal{H}}((G-u)-v) = \gamma_{\mathcal{H}}(G-u) - 1$ hold. By the second equality and Theorem A we deduce that v belongs to some $\gamma_{\mathcal{H}}$-set of $G-u$. On the other hand, since $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G-e) + 1 > \gamma_{\mathcal{H}}(G-u)$, v belongs to no $\gamma_{\mathcal{H}}$-set of $G-u$, a contradiction. Thus,

(B) $\min\{\gamma_{\mathcal{H}}(G-u), \gamma_{\mathcal{H}}(G-v), \gamma_{\mathcal{H}}(G-\{u,v\})\} \geqslant \gamma_{\mathcal{H}}(G-e)$.

Let D_t be a $\gamma_{\mathcal{H}}$-set of $G_{e,t}$ and $U_t = D_t \cap \{x_1, \ldots, x_t\}$, where $t = 1, \ldots, 6$.

Case 1: $t \in \{1, 2\}$. Assume $U_t \neq \emptyset$. Then $D_t - U_t$ is a dominating $\mathcal{H}$-set for at least one of the graphs $G-e$, $G-u$, $G-v$ and $G-\{u,v\}$. Using (B) we have

$$\begin{aligned}\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G-e) + 1 \leqslant |D_t - U_t| + 1 &= \gamma_{\mathcal{H}}(G_{e,t}) - |U_t| + 1 \\ &\leqslant \gamma_{\mathcal{H}}(G_{e,t}),\end{aligned}$$

contradicting (A). Thus U_t is empty. But then D_t is a dominating $\mathcal{H}$-set of $G-e$, which leads to $\gamma_{\mathcal{H}}(G_{e,t}) \geqslant \gamma_{\mathcal{H}}(G-e)$. Now by (A) the equality $\gamma_{\mathcal{H}}(G_{e,t}) = \gamma_{\mathcal{H}}(G-e)$ follows.

Case 2: $t \in \{4, 5\}$. Obviously $U_t \neq \emptyset$. As in Case 1 we obtain $\gamma_{\mathcal{H}}(G) \leqslant \gamma_{\mathcal{H}}(G_{e,t})$. Since by (A) $\gamma_{\mathcal{H}}(G_{e,t}) \leqslant \gamma_{\mathcal{H}}(G)$, we have $\gamma_{\mathcal{H}}(G_{e,t}) = \gamma_{\mathcal{H}}(G)$.

Case 3: $t = 6$. Clearly $|U_6| \geqslant 2$. As in Case 1 we obtain $\gamma_{\mathcal{H}}(G) \leqslant \gamma_{\mathcal{H}}(G_{e,6}) - |U_6| + 1$. Since $|U_6| \geqslant 2$, we have $\gamma_{\mathcal{H}}(G) \leqslant \gamma_{\mathcal{H}}(G_{e,6}) - 1$. Now by (A) we deduce that $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_{e,6}) - 1$.

(iii) Immediately by (i) and (ii). □

A c k n o w l e d g e m e n t. The author thanks the anonymous referee for carefully reading the manuscript and making suggestions that improved the content and presentation of the paper.

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Author's address: *Vladimir Samodivkin*, Departement of Mathematics, Faculty of Transportation Engineering, Civil Engineering and Geodesy, University of Architecture, 1 Hristo Smirnenski Blvd., 1046 Sofia, Bulgaria, e-mail: vl.samodivkin@gmail.com.