Hanan Darwish; Abdel Moneim Lashin; Bashar Hassan
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AN APPLICATION OF THE GENERALIZED BESSEL FUNCTION

HANAN DARWISH, ABDEL MONEIM LASHIN, BASHAR HASSAN, Mansoura

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Abstract. We introduce and study some new subclasses of starlike, convex and close-to-convex functions defined by the generalized Bessel operator. Inclusion relations are established and integral operator in these subclasses is discussed.

Keywords: Bessel operator; starlike function; convex function; close-to-convex function

MSC 2010: 30C45

1. Introduction

Let $A$ denote the class of functions of the form:

\begin{equation}
  f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n \in \mathbb{N},
\end{equation}

which are analytic in the unit disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$. A function $f \in A$ is said to be in the class $S^\ast(\alpha)$ of starlike functions of order $\alpha$ if it satisfies

\begin{equation}
  \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad 0 \leq \alpha < 1, \quad z \in U.
\end{equation}

We write $S^\ast(0) = S^\ast$ for the class of starlike functions in $U$. A function $f \in A$ is said to be in the class $C(\alpha)$ of convex functions of order $\alpha$ if it satisfies

\begin{equation}
  \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad 0 \leq \alpha < 1, \quad z \in U.
\end{equation}

The class of convex functions in $U$ is denoted by $C = C(0)$. It follows from (1.2) and (1.3) that

\begin{equation}
  f(z) \in C(\alpha) \iff zf'(z) \in S^\ast(\alpha).
\end{equation}
A function \( f \in A \) is said to be close-to-convex of order \( \beta \) and type \( \gamma \) in \( U \) if there exists a function \( g \in S^*(\gamma) \) such that

\[
\Re\left(\frac{zf'(z)}{g(z)}\right) > \beta, \quad 0 \leq \beta, \gamma < 1, \quad z \in U.
\]

We denote by \( K(\beta, \gamma) \) the class of close-to-convex functions of order \( \beta \) and type \( \gamma \). The class \( K(\beta, \gamma) \) was studied by Libera [11]. For some recent investigation on starlikeness of analytic functions, one can refer to [1], [10], [16], [17], [18], [19], [21], [20].

For functions \( f_j(z), j = 1, 2 \) defined by

\[
f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n, \quad n \in \mathbb{N}
\]

we denote the Hadamard product (or convolution) of \( f_1(z) \) and \( f_2(z) \) by

\[
f_1(z) \ast f_2(z) = z + \sum_{n=2}^{\infty} a_{n,1}a_{n,2} z^n.
\]

We recall here a generalized Bessel function \( w \) of the first kind of order \( \gamma \), defined in [8] and given by

\[
w_{\gamma,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma(\gamma + n + \frac{1}{2}(b + 1))} \left( \frac{z}{2} \right)^{2n+\gamma}, \quad z \in U,
\]

where \( \Gamma(z) \) stands for \( \Gamma \)-Euler function. Here \( w_{\gamma,b,c}(z) \) is the particular solution of the second-order homogenous differential equation (see [24])

\[
z^2 w''(z) + bw'(z) + (cz^2 - \gamma^2 + (1 - b)\gamma)w(z) = 0,
\]

where \( z \in U \). Now we consider function \( \varphi(z) \) defined by

\[
\varphi_{\gamma,b,c}(z) = 2^\gamma \Gamma\left(\gamma + \frac{b + 1}{2}\right) z^{1-\gamma/2} w_{\gamma,b,c}(\sqrt{z}).
\]

By using the well-know Pochhammer symbol \( (x)_\mu \) defined for \( x, \mu \in \mathbb{C} \) and in terms of the Euler gamma function by

\[
(x)_\mu = \frac{\Gamma(x + n)}{\Gamma(x)} = \begin{cases} 1 & \mu = 0, \\
x(x + 1)...(x + n - 1) & \mu \in \mathbb{N} = \{1, 2, 3, \ldots\}, \end{cases}
\]
we can express \( \varphi_{\gamma,b,c}(z) = \varphi_{k,c}(z) \) as

\[
\varphi_{\gamma,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n(k)n!} z^{n+1}, \quad k := \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^-,
\]

where \( \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} \).

Now, using the idea of Dziok and Srivastava [9], Baricz et. al. [4] introduced the \( B_k^c \) operator as:

\[
(1.8) \quad B_k^c f(z) = \varphi_{k,c} \ast f(z) = z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}a_n}{4^{n-1}(k)n-1(n-1)!} z^n.
\]

It is easy to verify from definition (1.8) that

\[
(1.9) \quad z(B_{k+1}^c f(z))' = kB_k^c f(z) - (k - 1)B_{k+1}^c f(z),
\]

where \( k = \gamma + \frac{1}{2}(b+1) \notin \mathbb{Z}_0^- \).

Some subclasses of starlike and convex functions defined using Bessel function were introduced by [3], [2], [5], [7], [22], [23].

Using the \( B_k^c \) operator we now introduce the following classes:

\[
S_k^*(\alpha) = \{ f \in A : B_k^c f(z) \in S^*(\alpha) \},
\]
\[
C_k(\alpha) = \{ f \in A : B_k^c f(z) \in C(\alpha) \},
\]
\[
K_k(\beta, \gamma) = \{ f \in A : B_k^c f(z) \in K(\beta, \gamma) \}.
\]

In this paper, we shall establish inclusion relation for these classes and investigate integral operator in these classes.

### 2. Inclusion relation

In order to prove our main results, we shall require the following lemma.

**Lemma 2.1** ([14], [15]). Let \( \varphi : D \rightarrow \mathbb{C}, D \subset \mathbb{C} \times \mathbb{C} \) (\( \mathbb{C} \) is the complex plane), and let \( u = u_1 + iu_2, v = v_1 + v_2 \). Suppose that the function \( \varphi(u,v) \) satisfies

(a) \( \varphi(u,v) \) is continuous in \( D \);

(b) \( (1,0) \in D \) and \( \Re(\varphi(1,0)) > 0 \);

(c) for all \( (iu_2,v_1) \in D \) such that \( v_1 < -\frac{1}{2}(1 + u_2^2) \), \( \Re(\varphi(iu_2,v_1)) \leq 0 \).
Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc $U$, and $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\Re(\varphi(p(z), zp'(z))) > 0$, $z \in U$, then $\Re(p(z)) > 0$, $z \in U$. Our first theorem is stated as:

**Theorem 2.1.** $S_k^*(\gamma) \subset S_{k+1}^*(\gamma)$, $k \geq 1 - \gamma$.

**Proof.** Let $f(z) \in S_k^*(\gamma)$ and set

\[
(2.1) \quad \frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c f(z)} = \gamma + (1 - \gamma)h(z),
\]

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ Using identity (1.9) we have

\[
(2.2) \quad \frac{B_k^c f(z)}{B_{k+1}^c f(z)} = \frac{1}{k}(k - 1 + \gamma + (1 - \gamma)h(z)).
\]

Differentiating (2.2), we obtain

\[
\frac{z(B_k^c f(z))'}{B_k^c f(z)} = \frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c f(z)} + \frac{(1 - \gamma)zh'(z)}{k - 1 + \gamma + (1 - \gamma)h(z)} = \gamma + (1 - \gamma)h(z) + \frac{(1 - \gamma)zh'(z)}{k - 1 + \gamma + (1 - \gamma)h(z)}
\]

or

\[
(2.3) \quad \frac{z(B_k^c f(z))'}{B_k^c f(z)} - \gamma = (1 - \gamma)h(z) + \frac{(1 - \gamma)zh'(z)}{k - 1 + \gamma + (1 - \gamma)h(z)}.
\]

Now we form the function $\varphi(u, v)$ by taking $u = h(z)$, $v = zh'(z)$ in (2.3) as

\[
(2.4) \quad \varphi(u, v) = (1 - \gamma)u + \frac{(1 - \gamma)v}{k - 1 + \gamma + (1 - \gamma)u}.
\]

It is easy to see that the function $\varphi(u, v)$ satisfies conditions (a) and (b) of Lemma 2.1 in $D = (\mathbb{C} - \{1 - k/(1 - \gamma)\}) \times \mathbb{C}$. To verify condition (c), we calculate as:

\[
\Re(\varphi(iu_2, v_1)) = \Re\left(\frac{(1 - \gamma)v_1}{k - 1 + \gamma + (1 - \gamma)iu_2}\right) = \frac{(1 - \gamma)(k - 1 + \gamma)v_1}{(k - 1 + \gamma)^2 + (1 - \gamma)^2u_2^2},
\]

\[
\leq -\frac{(1 - \gamma)(k - 1 + \gamma)(1 + u_2^2)}{2((k - 1 + \gamma)^2 + (1 - \gamma)^2u_2^2)} \leq 0,
\]

where $v_1 \leq -\frac{1}{2}(1 + u_2^2)$ and $(iu_2, v_1) \in D$. Therefore the function $\varphi(u, v)$ satisfies the conditions of Lemma 2.1. This shows that if $\Re(\varphi(h(z), zh'(z))) > 0$, $z \in U$, then $\Re(h(z)) > 0$, $z \in U$, that is if $f(z) \in S_k^*(\gamma)$, then $f(z) \in S_{k+1}^*(\gamma)$. The proof is complete. \[\square\]

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Theorem 2.2. \( C_k(\gamma) \subset C_{k+1}(\gamma), k \geq 1 - \gamma. \)

Proof. Let

\[
f(z) \in C_k(\gamma) \iff B_k^c f(z) \in C(\gamma) \iff z(B_k^c f(z))' \in S^*(\gamma) \\
\iff B_k^c (z f'(z)) \in S^*(\gamma) \iff z f'(z) \in S_k^*(\gamma) \\
\Rightarrow z f'(z) \in S_{k+1}^*(\gamma) \iff B_{k+1}^c (z f'(z)) \in S^*(\gamma) \\
\iff z(B_{k+1}^c f(z))' \in S^*(\gamma) \iff B_{k+1}^c f(z) \in C(\gamma) \\
\iff f(z) \in C_k(\gamma),
\]

which evidently proves Theorem 2.2. \(\square\)

Theorem 2.3. \( K_k(\beta, \gamma) \subset K_{k+1}(\beta, \gamma), k \geq 1 - \gamma. \)

Proof. Let \( f(z) \in K_k(\beta, \gamma). \) Then there exists a function \( k(z) \in S^*(\gamma) \) such that

\[
\Re\left(\frac{z(B_k^c f(z))'}{k(z)}\right) > \beta, \quad z \in U.
\]

Taking the function \( g(z) \) which satisfies \( B_k^c g(z) = k(z) \), we have \( g(z) \in S_k^*(\gamma) \) and

\[
\Re\left(\frac{z(B_k^c f(z))'}{B_k^c g(z)}\right) > \beta, \quad z \in U.
\]

Now put

\[
(2.5) \quad \frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c g(z)} = \beta + (1 - \beta)h(z),
\]

where \( h(z) = 1 + c_1 z + c_2 z^2 + \ldots \) From (2.5) we have

\[
(2.6) \quad z(B_{k+1}^c f(z))' = B_{k+1}^c g(z)(\beta + (1 - \beta)h(z)).
\]

So from (2.6) and identity (1.9) we have

\[
(2.7) \quad k z(B_k^c f(z))' = z(B_{k+1}^c g(z))' (\beta + (1 - \beta)h(z)) + B_{k+1}^c g(z)((1 - \beta)zh'(z)) \\
+ (k - 1) z(B_{k+1}^c f(z))'.
\]

Now apply (1.9) to the function \( g(z) \) and use (2.7) to obtain

\[
(2.8) \quad \frac{z(B_k^c f(z))'}{B_k^c g(z)} = \beta + (1 - \beta)h(z) + \frac{B_{k+1}^c g(z)(1 - \beta)zh'(z)}{B_k^c g(z)}, \quad k
\]
Since \( g(z) \in S_k^*(\gamma) \) and \( S_k^*(\gamma) \subset S_{k+1}^*(\gamma) \), we let

\[
\frac{z(B_{k+1}^c g(z))'}{B_{k+1}^c g(z)} = \gamma + (1 - \gamma)H(z),
\]

where \( \Re(H(z)) > 0, \, z \in U \). Thus (2.8) can be written as

\[
(2.9) \quad \frac{z(B_{k}^c f(z))'}{B_{k}^c g(z)} = (1 - \beta)h(z) + \frac{(1 - \beta)zh'(z)}{k - 1 + \gamma + (1 - \gamma)H(z)}.
\]

Now we form the function \( \varphi(u, v) \) by taking \( u = h(z), \, v = zh'(z) \) in (2.9) as:

\[
\varphi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{k - 1 + \gamma + (1 - \gamma)H(z)}.
\]

It is easy to see that the function \( \varphi(u, v) \) satisfies conditions (a) and (b) of Lemma 2.1, in \( D = \mathbb{C} \times \mathbb{C} \). To verify condition (c), we proceed as:

\[
\Re(\varphi(iu_2, v_1)) = \frac{(1 - \beta)v_1(k - 1 + \gamma + (1 - \gamma)h_1(x, y))}{(k - 1 + \gamma + (1 - \gamma)h_1(x, y))^2 + ((1 - \gamma)h_2(x, y))^2},
\]

where \( H(z) = h_1(x, y) + ih_2(x, y), \, h_1(x, y) \) and \( h_2(x, y) \) being functions of \( x \) and \( y \) and \( \Re(H(z)) = h_1(x, y) > 0 \). By putting \( v_1 \leq -\frac{1}{2}(1 + u_2^2) \), we have

\[
\Re(\varphi(iu_2, v_1)) = -\frac{(1 - \beta)(1 + u_2^2)(k - 1 + \gamma + (1 - \gamma)h_1(x, y))}{2(k - 1 + \gamma + (1 - \gamma)h_1(x, y))^2 + ((1 - \gamma)h_2(x, y))^2} < 0.
\]

Hence \( \Re(h(z)) > 0, \, z \in U, \) and \( f(z) \in K_{k+1}(\beta, \gamma) \). This completes the proof of Theorem 2.3. \( \square \)

### 3. Integral Operator

For \( c > -1 \) and \( f(z) \in A \) we define the integral operator \( J_c(f(z)) \) as

\[
(3.1) \quad J_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \, dt.
\]

The operator \( J_c \) \( (c \in \mathbb{N}) \) was introduced by Bernardi [6]. In particular, the operator \( J_1 \) was studied earlier by Libera [12] and Livingston [13].

**Theorem 3.1.** Let \( c > -\gamma \). If \( f(z) \in S_k^*(\gamma) \), then \( J_c(f(z)) \in S_k^*(\gamma) \).
Proof. Let \( f(z) \in S^*_k(\gamma) \). From (3.1) we have

\[
z(B_k^c J_c(f(z)))' = (c+1)B_k^c f(z) - cB_k^c J_c(f(z)).
\] (3.2)

Set

\[
\frac{z(B_k^c J_c(f(z)))'}{B_k^c J_c(f(z))} = \gamma + (1-\gamma)h(z),
\] (3.3)

where \( h(z) = 1 + c_1 z + c_2 z^2 + \ldots \), using the identity (3.2) we have

\[
\frac{B_k^c(f(z))}{B_k^c J_c(f(z))} = \frac{1}{c+1} (c + \gamma + (1-\gamma)h(z)).
\] (3.4)

Differentiating (3.4), we obtain

\[
\frac{z(B_k^c(f(z)))'}{B_k^c(f(z))} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{c + \gamma + (1-\gamma)h(z)}.
\] (3.5)

Now we form the function \( \varphi(u, v) \) by taking \( u = h(z) \), \( v = zh'(z) \) in (2.3) as:

\[
\varphi(u, v) = (1-\gamma)u + \frac{(1-\gamma)v}{c + \gamma + (1-\gamma)u}.
\]

It is easy to see that the function \( \varphi(u, v) \) satisfies conditions (a) and (b) of Lemma 2.1, in \( D = (C - \{(c + \gamma)/(\gamma - 1)\}) \times C \). To verify condition (c), we calculate as:

\[
\Re(\varphi(\alpha u_2, v_1)) = \Re\left(\frac{(1-\gamma)v_1}{c + \gamma + (1-\gamma)\alpha u_2} \right) = \frac{(1-\gamma)(c + \gamma)v_1}{(c + \gamma)^2 + (1-\gamma)^2u_2^2} \\
\leq -\frac{(1-\gamma)(c + \gamma)(1 + u_2^2)}{2((c + \gamma)^2 + (1-\gamma)^2u_2^2)} < 0,
\]

where \( v_1 \leq -\frac{1}{\beta}(1 + u_2^2) \) and \( (\alpha u_2, v_1) \in D \). Therefore the function \( \varphi(u, v) \) satisfies the conditions of Lemma 2.1. This shows that if \( \Re(\varphi(h(z), zh'(z))) > 0 \), \( z \in U \), then \( \Re(h(z)) > 0 \), \( z \in U \), that is if \( f(z) \in S^*_k(\gamma) \), then \( J_c(f(z)) \in S^*_k(\gamma) \). The proof is complete. \( \square \)

**Theorem 3.2.** Let \( c > -\gamma \). If \( f(z) \in C_k(\gamma) \), then \( J_c(f(z)) \in C_k(\gamma) \).

Proof. Let

\[
f(z) \in C_k(\gamma) \implies zf'(z) \in S^*_k(\gamma) \implies J_c(zf'(z)) \in S^*_k(\gamma) \\
\leftrightarrow z(J_c f(z))' \in S^*_k(\gamma) \iff J_c(f(z)) \in C_k(\gamma).
\]

This completes the proof of Theorem 3.2. \( \square \)
Theorem 3.3. Let $c > -\gamma$. If $f(z) \in K_k(\beta, \gamma)$, then $J_c(f(z)) \in K_k(\beta, \gamma)$.

Proof. Let $f(z) \in K_k(\beta, \gamma)$. Then by the definition there exists a function $g(z) \in S_k^*(\gamma)$ such that
\[
\Re\left(\frac{z(B_k^c f(z))'}{B_k^c g(z)}\right) > \beta, \quad z \in U.
\]
Put
\[
(3.6) \quad \frac{z(B_k^c f(z))'}{B_k^c g(z)} = \beta + (1 - \beta)h(z),
\]
where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ From (3.2) we have
\[
(3.7) \quad (c + 1)z(B_k^c f(z))' = z(B_k^c J_c(g(z)))'((\beta + (1 - \beta)h(z)) + B_k^c J_c(g(z))((1 - \beta)zh'(z)) + cz(B_k^c J_c(f(z)))'.
\]
Now apply (3.2) to the function $g(z)$ and use (3.7) to obtain
\[
(3.8) \quad \frac{z(B_k^c f(z))'}{B_k^c g(z)} = \beta + (1 - \beta)h(z) + \frac{B_k^c J_c(g(z))((1 - \beta)zh'(z)) + cz(B_k^c J_c(f(z))'}{c + 1}.
\]
Since $g(z) \in S_k^*(\gamma)$, then from Theorem 3.1 $J_c(f(z)) \in S_k^*(\gamma)$, we let
\[
\frac{z(B_k^c J_c(g(z))'}{B_k^c J_c(g(z))} = \gamma + (1 - \gamma)H(z),
\]
where $\Re(H(z)) > 0, z \in U$. Thus (3.9) can be written as
\[
(3.9) \quad \frac{z(B_k^c f(z))'}{B_k^c g(z)} - \beta = (1 - \beta)h(z) + \frac{(1 - \beta)zh'(z)}{c + \gamma + (1 - \gamma)H(z)}.
\]
Now we form the function $\varphi(u, v)$ by taking $u = h(z)$, $v = zh'(z)$ in (3.10) as:
\[
\varphi(u, v) = (1 - \beta)u + \frac{(1 - \beta)v}{c + \gamma + (1 - \gamma)H(z)}.
\]
It is easy to see that the function $\varphi(u, v)$ satisfies conditions (a) and (b) of Lemma 2.1, in $D = \mathbb{C} \times \mathbb{C}$. To verify condition (c), we proceed as:
\[
\Re(\varphi(iu_2, v_1)) = \frac{(1 - \beta)v_1(c + \gamma + (1 - \gamma)h_1(x, y))}{(c + \gamma + (1 - \gamma)h_1(x, y))^2 + ((1 - \gamma)h_2(x, y))^2},
\]
where $H(z) = h_1(x, y) + ih_2(x, y), h_1(x, y)$ and $h_2(x, y)$ being functions of $x$ and $y$, respectively, and $\Re(H(Z)) = h_1(x, y) > 0$. By putting $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we have
\[
\Re(\varphi(iu_2, v_1)) = \frac{(1 - \beta)(1 + u_2^2)(c + \gamma + (1 - \gamma)h_1(x, y))}{2((c + \gamma + (1 - \gamma)h_1(x, y))^2 + ((1 - \gamma)h_2(x, y))^2)} \leq 0.
\]
Hence $\Re(h(z)) > 0, z \in U$, and $J_c(f(z)) \in K_k(\beta, \gamma)$. This completes the proof of Theorem 3.3. \qed
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References


Authors’ address: Hanan Darwish, Abdel Moneim Lashin, Bashar Hassan, Department of Mathematics, Faculty of Science, Mansoura University, El Gomhouria St., 35516 Mansoura, Egypt, e-mail: darwish333@yahoo.com, aylashin@mans.edu.eg, basharfalh@yahoo.com.