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AN APPLICATION OF THE GENERALIZED BESSEL FUNCTION

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Abstract. We introduce and study some new subclasses of starlike, convex and close-to-convex functions defined by the generalized Bessel operator. Inclusion relations are established and integral operator in these subclasses is discussed.

Keywords: Bessel operator; starlike function; convex function; close-to-convex function

MSC 2010: 30C45

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqslant 0,\ n \in \mathbb{N}, \tag{1.1}$$

which are analytic in the unit disk $U = \{z\colon z \in \mathbb{C},\ |z| < 1\}$. A function $f \in A$ is said to be in the class $S^*(\alpha)$ of starlike functions of order α if it satisfies

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad 0 \leqslant \alpha < 1,\ z \in U. \tag{1.2}$$

We write $S^*(0) = S^*$ for the class of starlike functions in U. A function $f \in A$ is said to be in the class $C(\alpha)$ of convex functions of order α if it satisfies

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha, \quad 0 \leqslant \alpha < 1,\ z \in U. \tag{1.3}$$

The class of convex functions in U is denoted by $C = C(0)$. It follows from (1.2) and (1.3) that

$$f(z) \in C(\alpha) \Longleftrightarrow zf'(z) \in S^*(\alpha). \tag{1.4}$$

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A function $f \in A$ is said to be close-to-convex of order β and type γ in U if there exists a function $g \in S^*(\gamma)$ such that

$$\Re\Big(\frac{zf'(z)}{g(z)}\Big) > \beta, \quad 0 \leqslant \beta,\ \gamma < 1,\ z \in U. \tag{1.5}$$

We denote by $K(\beta,\gamma)$ the class of close-to-convex functions of order β and type γ. The class $K(\beta,\gamma)$ was studied by Libera [11]. For some recent investigation on starlikeness of analytic functions, one can refer to [1], [10], [16], [17], [18], [19], [21], [20].

For functions $f_j(z)$, $j = 1,2$ defined by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n, \quad n \in \mathbb{N} \tag{1.6}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$f_1(z) * f_2(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.7}$$

We recall here a generalized Bessel function w of the first kind of order γ, defined in [8] and given by

$$w_{\gamma,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\,\Gamma(\gamma + n + \frac{1}{2}(b+1))} \Big(\frac{z}{2}\Big)^{2n+\gamma}, \quad z \in U,$$

where $\Gamma(z)$ stands for Γ-Euler function. Here $w_{\gamma,b,c}(z)$ is the particular solution of the second-order homogenous differential equation (see [24])

$$z^2 w''(z) + bzw'(z) + (cz^2 - \gamma^2 + (1-b)\gamma)w(z) = 0,$$

where $z \in U$. Now we consider function $\varphi(z)$ defined by

$$\varphi_{\gamma,b,c}(z) = 2^{\gamma}\Gamma\Big(\gamma + \frac{b+1}{2}\Big) z^{1-\gamma/2} w_{\gamma,b,c}(\sqrt{z}).$$

By using the well-know Pochhammer symbol $(x)_\mu$ defined for $x, \mu \in \mathbb{C}$ and in terms of the Euler gamma function by

$$(x)_\mu = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \mu = 0, \\ x(x+1)...(x+n-1) & \mu \in \mathbb{N} = \{1,2,3,\ldots\}, \end{cases}$$

we can express $\varphi_{\gamma,b,c}(z) = \varphi_{k,c}(z)$ as

$$\varphi_{\gamma,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (k)_n n!} z^{n+1}, \quad k := \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^-,$$

where $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$.

Now, using the idea of Dziok and Srivastava [9], Baricz et. al. [4] introduced the B_k^c operator as:

$$B_k^c f(z) = \varphi_{k,c} * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1} a_n}{4^{n-1}(k)_{n-1}(n-1)!} z^n. \tag{1.8}$$

It is easy to verify from definition (1.8) that

$$z(B_{k+1}^c f(z))' = k B_k^c f(z) - (k-1) B_{k+1}^c f(z), \tag{1.9}$$

where $k = \gamma + \frac{1}{2}(b+1) \notin \mathbb{Z}_0^-$.

Some subclasses of starlike and convex functions defined using Bessel function were introduced by [3], [2], [5], [7], [22], [23].

Using the B_k^c operator we now introduce the following classes:

$$S_k^*(\alpha) = \{f \in A\colon\ B_k^c f(z) \in S^*(\alpha)\},$$
$$C_k(\alpha) = \{f \in A\colon\ B_k^c f(z) \in C(\alpha)\},$$
$$K_k(\beta, \gamma) = \{f \in A\colon\ B_k^c f(z) \in K(\beta, \gamma)\}.$$

In this paper, we shall establish inclusion relation for these classes and investigate integral operator in these classes.

2. Inclusion relation

In order to prove our main results, we shall require the following lemma.

Lemma 2.1 ([14], [15]). *Let $\varphi\colon D \longrightarrow \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane), and let $u = u_1 + \mathrm{i}u_2$, $v = v_1 + v_2$. Suppose that the function $\varphi(u, v)$ satisfies*

(a) *$\varphi(u, v)$ is continuous in D;*

(b) *$(1, 0) \in D$ and $\Re(\varphi(1, 0)) > 0$;*

(c) *for all $(\mathrm{i}u_2, v_1) \in D$ such that $v_1 < -\frac{1}{2}(1 + u_2^2)$, $\Re(\varphi(\mathrm{i}u_2, v_1)) \leqslant 0$.*

Let $p(z) = 1+p_1z+p_2z^2+\dots$ be regular in the unit disc U, and $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\Re(\varphi(p(z), zp'(z))) > 0$, $z \in U$, then $\Re(p(z)) > 0$, $z \in U$. Our first theorem is stated as:

Theorem 2.1. $S_k^*(\gamma) \subset S_{k+1}^*(\gamma)$, $k \geqslant 1-\gamma$.

Proof. Let $f(z) \in S_k^*(\gamma)$ and set

$$\frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c f(z)} = \gamma + (1-\gamma)h(z), \tag{2.1}$$

where $h(z) = 1 + c_1z + c_2z^2 + \dots$ Using identity (1.9) we have

$$\frac{B_k^c f(z)}{B_{k+1}^c f(z)} = \frac{1}{k}(k-1+\gamma+(1-\gamma)h(z)). \tag{2.2}$$

Differentiating (2.2), we obtain

$$\begin{aligned}\frac{z(B_k^c f(z))'}{B_k^c f(z)} &= \frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c f(z)} + \frac{(1-\gamma)zh'(z)}{k-1+\gamma+(1-\gamma)h(z)}\\ &= \gamma + (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{k-1+\gamma+(1-\gamma)h(z)}\end{aligned}$$

or

$$\frac{z(B_k^c f(z))'}{B_k^c f(z)} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{k-1+\gamma+(1-\gamma)h(z)}. \tag{2.3}$$

Now we form the function $\varphi(u, v)$ by taking $u = h(z)$, $v = zh'(z)$ in (2.3) as

$$\varphi(u, v) = (1-\gamma)u + \frac{(1-\gamma)v}{k-1+\gamma+(1-\gamma)u}. \tag{2.4}$$

It is easy to see that the function $\varphi(u, v)$ satisfies conditions (a) and (b) of Lemma 2.1 in $D = (\mathbb{C} - \{1 - k/(1-\gamma)\}) \times \mathbb{C}$. To verify condition (c), we calculate as:

$$\begin{aligned}\Re(\varphi(\mathrm{i}u_2, v_1)) = \Re\Big(\frac{(1-\gamma)v_1}{k-1+\gamma+(1-\gamma)\mathrm{i}u_2}\Big) &= \frac{(1-\gamma)(k-1+\gamma)v_1}{(k-1+\gamma)^2+(1-\gamma)^2u_2^2}\\ \leqslant -\frac{(1-\gamma)(k-1+\gamma)(1+u_2^2)}{2((k-1+\gamma)^2+(1-\gamma)^2u_2^2)} &\leqslant 0,\end{aligned}$$

where $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$ and $(\mathrm{i}u_2, v_1) \in D$. Therefore the function $\varphi(u, v)$ satisfies the conditions of Lemma 2.1. This shows that if $\Re(\varphi(h(z), zh'(z))) > 0$, $z \in U$, then $\Re(h(z)) > 0$, $z \in U$, that is if $f(z) \in S_k^*(\gamma)$, then $f(z) \in S_{k+1}^*(\gamma)$. The proof is complete. □

Theorem 2.2. $C_k(\gamma) \subset C_{k+1}(\gamma)$, $k \geqslant 1-\gamma$.

Proof. Let

$$\begin{aligned} f(z) \in C_k(\gamma) &\iff B_k^c f(z) \in C(\gamma) \iff z(B_k^c f(z))' \in S^*(\gamma) \\ &\iff B_k^c(zf'(z)) \in S^*(\gamma) \iff zf'(z) \in S_k^*(\gamma) \\ &\Rightarrow zf'(z) \in S_{k+1}^*(\gamma) \iff B_{k+1}^c(zf'(z)) \in S^*(\gamma) \\ &\iff z(B_{k+1}^c f(z))' \in S^*(\gamma) \iff B_{k+1}^c f(z) \in C(\gamma) \\ &\iff f(z) \in C_k(\gamma), \end{aligned}$$

which evidently proves Theorem 2.2. □

Theorem 2.3. $K_k(\beta,\gamma) \subset K_{k+1}(\beta,\gamma)$, $k \geqslant 1-\gamma$.

Proof. Let $f(z) \in K_k(\beta,\gamma)$. Then there exists a function $k(z) \in S^*(\gamma)$ such that

$$\Re\Big(\frac{z(B_k^c f(z))'}{k(z)}\Big) > \beta, \quad z \in U.$$

Taking the function $g(z)$ which satisfies $B_k^c g(z) = k(z)$, we have $g(z) \in S_k^*(\gamma)$ and

$$\Re\Big(\frac{z(B_k^c f(z))'}{B_k^c g(z)}\Big) > \beta, \quad z \in U.$$

Now put

$$\frac{z(B_{k+1}^c f(z))'}{B_{k+1}^c g(z)} = \beta + (1-\beta)h(z), \tag{2.5}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ From (2.5) we have

$$z(B_{k+1}^c f(z))' = B_{k+1}^c g(z)(\beta + (1-\beta)h(z)). \tag{2.6}$$

So from (2.6) and identity (1.9) we have

$$\begin{aligned} kz(B_k^c f(z))' = z(B_{k+1}^c g(z))'(\beta + (1-\beta)h(z)) + B_{k+1}^c g(z)((1-\beta)zh'(z)) \\ + (k-1)z(B_{k+1}^c f(z))'. \end{aligned} \tag{2.7}$$

Now apply (1.9) to the function $g(z)$ and use (2.7) to obtain

$$\frac{z(B_k^c f(z))'}{B_k^c g(z)} = \beta + (1-\beta)h(z) + \frac{B_{k+1}^c g(z)}{B_k^c g(z)} \frac{(1-\beta)zh'(z)}{k}. \tag{2.8}$$

Since $g(z) \in S_k^*(\gamma)$ and $S_k^*(\gamma) \subset S_{k+1}^*(\gamma)$, we let

$$\frac{z(B_{k+1}^c g(z))'}{B_{k+1}^c g(z)} = \gamma + (1-\gamma)H(z),$$

where $\Re(H(z)) > 0$, $z \in U$. Thus (2.8) can be written as

$$\frac{z(B_k^c f(z))'}{B_k^c g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{k-1+\gamma+(1-\gamma)H(z)}. \tag{2.9}$$

Now we form the function $\varphi(u,v)$ by taking $u = h(z)$, $v = zh'(z)$ in (2.9) as:

$$\varphi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{k-1+\gamma+(1-\gamma)H(z)}.$$

It is easy to see that the function $\varphi(u,v)$ satisfies conditions (a) and (b) of Lemma 2.1, in $D = \mathbb{C} \times \mathbb{C}$. To verify condition (c), we proceed as:

$$\Re(\varphi(\mathrm{i}u_2, v_1)) = \frac{(1-\beta)v_1(k-1+\gamma+(1-\gamma)h_1(x,y))}{(k-1+\gamma+(1-\gamma)h_1(x,y))^2 + ((1-\gamma)h_2(x,y))^2},$$

where $H(z) = h_1(x,y) + \mathrm{i}h_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being functions of x and y and $\Re(H(z)) = h_1(x,y) > 0$. By putting $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$, we have

$$\Re(\varphi(\mathrm{i}u_2, v_1)) = -\frac{(1-\beta)(1+u_2^2)(k-1+\gamma+(1-\gamma)h_1(x,y))}{2(k-1+\gamma+(1-\gamma)h_1(x,y))^2 + ((1-\gamma)h_2(x,y))^2} < 0.$$

Hence $\Re(h(z)) > 0$, $z \in U$, and $f(z) \in K_{k+1}(\beta,\gamma)$. This completes the proof of Theorem 2.3. □

3. Integral operator

For $c > -1$ and $f(z) \in A$ we define the integral operator $J_c(f(z))$ as

$$J_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,\mathrm{d}t. \tag{3.1}$$

The operator J_c ($c \in \mathbb{N}$) was introduced by Bernardi [6]. In particular, the operator J_1 was studied earlier by Libera [12] and Livingston [13].

Theorem 3.1. *Let $c > -\gamma$. If $f(z) \in S_k^*(\gamma)$, then $J_c(f(z)) \in S_k^*(\gamma)$.*

P r o o f. Let $f(z) \in S_k^*(\gamma)$. From (3.1) we have

$$z(B_k^c J_c(f(z)))' = (c+1)B_k^c f(z) - cB_k^c J_c(f(z)). \tag{3.2}$$

Set

$$\frac{z(B_k^c J_c(f(z)))'}{B_k^c J_c(f(z))} = \gamma + (1-\gamma)h(z), \tag{3.3}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$, using the identity (3.2) we have

$$\frac{B_k^c(f(z))}{B_k^c J_c(f(z))} = \frac{1}{c+1}(c + \gamma + (1-\gamma)h(z)). \tag{3.4}$$

Differentiating (3.4), we obtain

$$\frac{z(B_k^c(f(z)))'}{B_k^c(f(z))} - \gamma = (1-\gamma)h(z) + \frac{(1-\gamma)zh'(z)}{c+\gamma+(1-\gamma)h(z)}. \tag{3.5}$$

Now we form the function $\varphi(u,v)$ by taking $u = h(z)$, $v = zh'(z)$ in (2.3) as:

$$\varphi(u,v) = (1-\gamma)u + \frac{(1-\gamma)v}{c+\gamma+(1-\gamma)u}.$$

It is easy to see that the function $\varphi(u,v)$ satisfies conditions (a) and (b) of Lemma 2.1, in $D = (\mathbb{C} - \{(c+\gamma)/(\gamma-1)\}) \times \mathbb{C}$. To verify condition (c), we calculate as:

$$\begin{aligned}\Re(\varphi(\mathrm{i}u_2, v_1)) = \Re\Big(\frac{(1-\gamma)v_1}{c+\gamma+(1-\gamma)\mathrm{i}u_2}\Big) &= \frac{(1-\gamma)(c+\gamma)v_1}{(c+\gamma)^2 + (1-\gamma)^2 u_2^2} \\ &\leqslant -\frac{(1-\gamma)(c+\gamma)(1+u_2^2)}{2((c+\gamma)^2+(1-\gamma)^2u_2^2)} < 0,\end{aligned}$$

where $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$ and $(\mathrm{i}u_2, v_1) \in D$. Therefore the function $\varphi(u,v)$ satisfies the conditions of Lemma 2.1. This shows that if $\Re(\varphi(h(z), zh'(z))) > 0$, $z \in U$, then $\Re(h(z)) > 0$, $z \in U$, that is if $f(z) \in S_k^*(\gamma)$, then $J_c(f(z)) \in S_k^*(\gamma)$. The proof is complete. □

Theorem 3.2. *Let $c > -\gamma$. If $f(z) \in C_k(\gamma)$, then $J_c(f(z)) \in C_k(\gamma)$.*

P r o o f. Let

$$\begin{aligned} f(z) \in C_k(\gamma) &\Longrightarrow zf'(z) \in S_k^*(\gamma) \Longrightarrow J_c(zf'(z)) \in S_k^*(\gamma) \\ &\Longleftrightarrow z(J_c f(z))' \in S_k^*(\gamma) \Longleftrightarrow J_c(f(z)) \in C_k(\gamma).\end{aligned}$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $c > -\gamma$. If $f(z) \in K_k(\beta,\gamma)$, then $J_c(f(z)) \in K_k(\beta,\gamma)$.*

P r o o f. Let $f(z) \in K_k(\beta,\gamma)$. Then by the definition there exists a function $g(z) \in S_k^*(\gamma)$ such that

$$\Re\Big(\frac{z(B_k^c f(z))'}{B_k^c g(z)}\Big) > \beta, \quad z \in U.$$

Put

$$\frac{z(B_k^c J_c f(z))'}{B_k^c J_c g(z)} = \beta + (1-\beta)h(z), \tag{3.6}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ From (3.2) we have

$$\begin{aligned}(c+1)z(B_k^c f(z))' &= z(B_k^c J_c(g(z)))'(\beta + (1-\beta)h(z)) \\ &\quad + B_k^c J_c(g(z))((1-\beta)zh'(z)) + cz(B_k^c J_c(f(z)))'.\end{aligned} \tag{3.7}$$

Now apply (3.2) to the function $g(z)$ and use (3.7) to obtain

$$\frac{z(B_k^c f(z))'}{B_k^c g(z)} = \beta + (1-\beta)h(z) + \frac{B_k^c J_c(g(z))}{B_k^c g(z)}\frac{(1-\beta)zh'(z)}{c+1}. \tag{3.8}$$

Since $g(z) \in S_k^*(\gamma)$, then from Theorem 3.1 $J_c(f(z)) \in S_k^*(\gamma)$, we let

$$\frac{z(B_k^c J_c(g(z)))'}{B_k^c J_c(g(z))} = \gamma + (1-\gamma)H(z),$$

where $\Re(H(z)) > 0$, $z \in U$. Thus (3.9) can be written as

$$\frac{z(B_k^c f(z))'}{B_k^c g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{c+\gamma+(1-\gamma)H(z)}. \tag{3.9}$$

Now we form the function $\varphi(u,v)$ by taking $u = h(z)$, $v = zh'(z)$ in (3.10) as:

$$\varphi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{c+\gamma+(1-\gamma)H(z)}.$$

It is easy to see that the function $\varphi(u,v)$ satisfies conditions (a) and (b) of Lemma 2.1, in $D = \mathbb{C}\times\mathbb{C}$. To verify condition (c), we proceed as:

$$\Re(\varphi(\mathrm{i}u_2, v_1)) = \frac{(1-\beta)v_1(c+\gamma+(1-\gamma)h_1(x,y))}{(c+\gamma+(1-\gamma)h_1(x,y))^2 + ((1-\gamma)h_2(x,y))^2},$$

where $H(z) = h_1(x,y) + \mathrm{i}h_2(x,y)$, $h_1(x,y)$ and $h_2(x,y)$ being functions of x and y, respectively, and $\Re(H(Z)) = h_1(x,y) > 0$. By putting $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$, we have

$$\Re(\varphi(\mathrm{i}u_2, v_1)) = \frac{(1-\beta)(1+u_2^2)(c+\gamma+(1-\gamma)h_1(x,y))}{2((c+\gamma+(1-\gamma)h_1(x,y))^2 + ((1-\gamma)h_2(x,y))^2)} \leqslant 0.$$

Hence $\Re(h(z)) > 0$, $z \in U$, and $J_c(f(z)) \in K_k(\beta,\gamma)$. This completes the proof of Theorem 3.3. □

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