Abstract. In this paper, we are concerned with G-rings. We generalize the Kaplansky’s theorem to rings with zero-divisors. Also, we assert that if $R \subseteq T$ is a ring extension such that $mT \subseteq R$ for some regular element $m$ of $T$, then $T$ is a G-ring if and only if so is $R$. Also, we examine the transfer of the G-ring property to trivial ring extensions. Finally, we conclude the paper with illustrative examples discussing the utility and limits of our results.

Keywords: G-ring; pullback; trivial extension

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1. Introduction

All rings considered below are commutative with unit and all modules are unital. Let $R$ be a commutative ring and let $Q(R)$ denote the total quotient ring of $R$. We call $R$ a G-ring if $Q(R) = R[u^{-1}]$ for some regular element $u \in R$ (equivalently, if $Q(R)$ is finitely generated as a ring over $R$) [1]. This generalizes Kaplansky’s definition of G-domain [12]. Also, he shows that if $R \subseteq T$ are domains and if $T$ is algebraic over $R$ and finitely generated as a ring over $R$, then $R$ is a G-domain if and only if so is $T$ [12, Theorem 22].

In this paper, we are concerned with G-rings. Our main result of Section 2 is to generalize the above Kaplansky’s theorem to rings with zero-divisors. Also, we assert that if $R \subseteq T$ is a ring extension such that $mT \subseteq R$ for some regular element $m$ of $T$, then $T$ is a G-ring if and only if so is $R$. As an immediate consequence, we get a corollary on the transfer of the G-ring property to pullbacks issued from domains. Our main result of Section 3 examines the transfer of the G-ring property to trivial ring extensions; precisely, it states that if $A$ is a ring, $E$ is an $A$-module such that $Z(E) \subseteq Z(A)$ (where $Z(E) := \{a \in A; ae = 0$ for some $e \in E - \{0\}\}$ is the set of zero-divisors on $E$), then the trivial extension of $A$ by $E$ is a G-ring if and only if $A$ is a G-ring. In Section 4, we conclude the paper with illustrative examples discussing the utility and limits of our results.

2. The G-ring property in a pullback

Let $R$ be a ring and $R_u := R[1/u]$, where $u$ is regular in $R$. We first give a zero-divisor extension of Kaplansky’s theorem [12, Theorem 22].
Theorem 2.1. Let $R$ be a subring of $T$ such that each regular element of $R$ is regular in $T$ (consequently, $K := Q(R) \subseteq L := Q(T)$). Assume that $L$ is integral over $K$. Then:

(1) if $R$ is a G-ring, then $T$ is a G-ring;
(2) if $T$ is a finitely generated $R$-algebra, then $T$ is a G-ring if and only if $R$ is a G-ring.

Proof: (1) Assume that $R$ is a G-ring. Hence, $K := Q(R) = R_u$ for some regular element $u \in R$. But, $K := R_u \subseteq T_u \subseteq L := Q(T)$. Hence, $L$ is integral over $T_u$ since $L$ is integral over $K$. Therefore, $L = T_u$ since $L$ is a fraction ring of $T_u$ and so $T$ is a G-ring.

(2) If $R$ is a G-ring, then $T$ is a G-ring by (1). Conversely, assume that $T$ is a G-ring. Hence, $L = T_u$ for some regular element $v \in T$ and $T = R[w_1, \ldots, w_k]$ for some $w_i \in T$ and for a positive integer $k$ (since $T$ is a finitely generated $R$-algebra). Then, the elements $v^{-1}, w_1, \ldots, w_k$ are integral over $K$. So, we get Kaplansky’s equations (see proof of [12, Theorem 22]) with $a$, $b_i$ being regular elements of $R$. Let $R_1 := R[a^{-1}, b_1^{-1}, \ldots, b_k^{-1}]$. As argued by [12, Theorem 22], $L = R_1[w_1, \ldots, w_k, v^{-1}]$ and $L$ is integral over $R_1$. Then, $K$ is integral over $R_1$ and so $K = R_1$ since $K$ is a fraction ring of $R_1$. Hence, $R$ is a G-ring and this completes the proof of Theorem 2.1. □

Now, we provide a somewhat analogue of a zero-divisor extension of Kaplansky’s result mentioned above. Precisely, we have:

Theorem 2.2. Let $R \subseteq T$ be a ring extension such that $mT \subseteq R$, for some regular element $m \in T$. Then $T$ is a G-ring if and only if $R$ is a G-ring.

The proof of this theorem requires the following lemma.

Lemma 2.3. Let $R$ be a ring and $R_f = R[1/f]$, where $f$ is regular in $R$. Then $R$ is a G-ring if and only if $R_f$ is a G-ring.

Proof: It is clear that $R_f = \{af^{-n}; a \in R$ and $n \in \mathbb{N}\}$. Hence, $Q(R_f) = Q(R)$ since $af^{-n}$ is regular in $R_f$ if and only if $a$ is regular in $R$ (because $f$ is invertible in $R_f$).

Assume that $R$ is a G-ring. Hence, $Q(R) = R_u$ for some regular element $u \in R$. But, $Q(R_f) = Q(R) = R_u \subseteq (R_f)_u \subseteq Q(R_f)$. Therefore, $Q(R_f) = (R_f)_u$ and so $R_f$ is a G-ring.

Conversely, assume that $R_f$ is a G-ring, that is, $Q(R_f) = (R_f)_u$ for some regular element $u \in R_f$. We may assume that $u \in R$ since $u = af^{-n}$ for some regular element $a \in R$ and $n \in \mathbb{N}$, and since $f^{-n}$ is invertible in $R_f$. It is well-known and easy to see that $(R_f)_u = R_{fu}$. Therefore, $Q(R) = Q(R_f) = (R_f)_u \subseteq R_{fu} \subseteq Q(R)$ and so $Q(R) = R_{fu}$ and this completes the proof of Lemma 2.3. □
Proof of Theorem 2.2: Let \( R \subseteq T \) be a ring extension such that \( mT \subseteq R \), for some regular element \( m \) of \( T \). Clearly, \( m \in R \) and \( m \) is regular element of \( R \). But \( R_m = T_m \) since \( R_m \subseteq T_m = \{ am^{-n}; a \in T \text{ and } n \in \mathbb{N} \} = \{ (am)m^{-(n+1)}; (am) \in R \text{ and } n \in \mathbb{N} \} \subseteq R_m \). Therefore, \( R \) is a G-ring if and only if \( T \) is a G-ring by Lemma 2.3 since \( T_m = R_m \).

The above result generates new families of examples of G-domains not covered by Kaplansky’s result [12, Theorem 22] mentioned above. It also denies any similitude with this result as shown by the following corollary.

Corollary 2.4. Let \( D \) be a domain which is not a G-domain, \( K = Q(D) \) and \( T \) a domain such that \( T/M = K \) for some nonzero maximal ideal \( M \) of \( T \). Let \( f : T \rightarrow K \) be the canonical surjection and \( R = f^{-1}(D) \). Then:

1. \( T \) is a G-domain if and only if \( R \) is a G-domain;
2. \( T \) is not finitely generated as a ring over \( R \).

Proof: (1) Results by Theorem 2.2 because \( mT \subseteq M \subseteq kerf \subseteq R \) and \( R_m = T_m \) for each nonzero \( m \) in \( M \).

(2) Assume that \( T \) is finitely generated as a ring over \( R \). Then \( T = R[x_1, \ldots, x_n] \), for some \( x_i \in T \), where \( n \) is a positive integer. Hence, \( K = T/M = (R/M)[x_1, \ldots, x_n] = D[x_1, \ldots, x_n] \), a contradiction since \( D \) is not a G-domain. Therefore, \( T \) is not finitely generated as a ring over \( R \). □

Remark 2.5. Part (1) of Corollary 2.4 generalizes [9, Theorem 2.7 (a), p. 341].

A pair of rings \( A \subseteq B \) is called a G-ring pair if \( D \) is a G-ring for each ring \( D \) such that \( A \subseteq D \subseteq B \). In [6, Theorem 2.1], Dobbs gives necessary and sufficient conditions to have a G-domain pair. In the context of Theorem 2.2, we obtain:

Corollary 2.6. Let \( T, R, \) and \( m \) be as in Theorem 2.2. Then \((R,T)\) is a G-ring pair if and only if \( T \) (resp., \( R \)) is a G-ring.

Proof: Let \( S \) be a ring such that \( R \subseteq S \subseteq T \). Hence, \( mS \subseteq mT \subseteq R \) and \( m \) is regular in \( S \). Therefore, Theorem 2.2 completes the proof of Corollary 2.6. □

Remark 2.7. In Theorem 2.2, the hypothesis “\( m \) is a regular element of \( T \)” is necessary (see Example 4.4).

3. G-ring property in trivial extension

Let \( A \) be a ring, \( E \) be an \( A \)-module and \( R = A \otimes E \) be the set of pairs \((a,e)\) with pairwise addition and multiplication given by: \((a,e)(b,f) = (ab,af + be)\). \( R \) is called the trivial ring extension of \( A \) by \( E \). Recall that a maximal ideal of \( R \) has always the form \( M \otimes E \), where \( M \) is a maximal ideal of \( A \) [11, Theorem 25.1(3)]. The author of [11] also confirms by a private communication that [11, Theorem 25.1] is not true, that is, an ideal \( J \) of \( R \) has not always the form: \( J = I \otimes E' \), where \( I = \{ a \in A | (a,e) \in J \text{ for some } e \in E \} \) and \( E' = \{ e \in E | (a,e) \in J \text{ for some } a \in A \} \). We only have that \( J \subseteq I \otimes E' \).
(see [14]). Nevertheless, it is easily seen that \( J = I \preceq E' \) if and only if \( 0 \preceq E' \subseteq J \) if and only if \( I \preceq 0 \subseteq J \).

In this section, we study the possible transfer of the G-ring property for various trivial extension contexts.

**Theorem 3.1.** Let \( A \) be a ring, \( E \) be an \( A \)-module such that \( Z(E) \subseteq Z(A) \) (where \( Z(E) \) denotes the set of zero-divisors on \( E \)), and \( R := A \preceq E \) be the trivial ring extension of \( A \) by \( E \). Then \( R \) is a G-ring if and only if \( A \) is a G-ring.

**Proof:** Set \( S = A - Z(A) \). Then \( Z(R) = Z(A) \preceq E \) and \( Q(R) = Q(A) \preceq E_S \) by [11, p. 164–165]. Assume that \( A \) is a G-ring. Hence, \( Q(A) = A_\alpha \) for some \( \alpha \in S \). Then, \( (a, 0) \notin Z(R) \) and \( E_a := E \otimes_A A_\alpha = E \otimes_A Q(A) = E_S \). So, \( Q(R) = Q(A) \preceq E_S = A_\alpha \preceq E_a = \{ (xa^{-n}, ea^{-m}) ; (x, e) \in R \) and \( n, m \in \mathbb{N} \} = \{ (xa^{-n}, ea^{-m})(a, 0)^{-p}; (x, e) \in R, n, m \in \mathbb{N} \) and \( p = \sup(n, m) \} \subseteq R_{(a,0)} \subseteq Q(R) \). Therefore, \( Q(R) = R_{(a,0)} \) and then \( R \) is a G-ring.

Conversely, assume that \( R \) is a G-ring. Hence, \( Q(R) = R_{(a,0)} \) for some \( (a, e) \notin Z(R) \). If \( Q(R) := Q(A) \preceq E_S \) and \( p : Q(R) \rightarrow Q(A) \) is the map \( p(x, y) = x \), we claim that \( Q(A) = p(R_{(a,0)}) = A_\alpha \). Indeed, let \( (x, y)(a, e)^{-n} \in R_{(a,0)} \), where \( (x, y) \in R \) and \( n \in \mathbb{N} \). Hence, \( a^n p((x, y)(a, e)^{-n}) = p((a, 0)^n(x, y)(a, e)^{-n}) = p((x, y)((a^{-n}, 0)(a, e)^n)^{-1}) = p((x, y)((a^{-n}, 0)(a, e_n)^{-1}) = p((x, y)(1, a^{-n}e_n)^{-1}) = p((x, y)(1, a^{-n}e_n)) = p(x, y-xa^{-n}e_n) = x \in A \), so \( p((x, y)(a, e)^{-n}) = xa^{-n} \in A_\alpha \). Therefore, \( Q(A) = A_\alpha \) and then \( A \) is a G-ring.

If \( A \) is a domain and \( E \) is a torsion-free \( A \)-module, we obtain by Theorem 3.1:

**Corollary 3.2.** Let \( A \) be a domain, \( E \) be a torsion-free \( A \)-module, and \( R := A \preceq E \) be the trivial ring extension of \( A \) by \( E \). Then \( R \) is a G-ring if and only if \( A \) is a G-domain.

If \( R := A \preceq E \) is a trivial extension of a ring \( A \) by an \( A \)-module \( E \), we do not have in general that \( R \) is a G-ring if and only if \( A \) is a G-ring, as shown by the following result.

**Proposition 3.3.** Let \( (A, M) \) be a local ring and \( E \) an \( A \)-module such that \( ME = 0 \). Then the trivial ring extension of \( A \) by \( E \) is a G-ring.

**Proof:** The result holds since the trivial ring extension of \( A \) by \( E \) is a total ring (since \( (M \preceq E)(0, 1) = (0, 0) \) and \( M \preceq E \) is a maximal ideal of a local ring \( A \preceq E \)).

4. Examples

In this section, we exhibit a non-Noetherian coherent G-domain (Example 4.1). Then, we give non-coherent G-rings (Examples 4.2 and 4.3). We also show that if \( f : R \rightarrow S \) is a faithfully flat ring extension such that \( S \) is a G-ring, then \( R \) is not a G-ring, in general (Examples 4.1(4) and 4.2(3)). Finally, we give a counter-example showing that the hypothesis “\( m \) is a regular element of \( T \)” is necessary in Theorem 2.1 (Example 4.4).
Example 4.1. Let $T = \mathbb{Q}[X] = \mathbb{Q} +XT$ be the formal power series ring over the field $\mathbb{Q}$ and let $R = \mathbb{Z} + XT$. Then:

1. $R$ is a G-domain by Theorem 2.2 since $T$ is a local G-domain and $XT \subseteq R$;
2. $R$ is a coherent domain by [8, Theorem 3] and is not Noetherian by [4, Theorem 4];
3. $T$ is not finitely generated as a ring over $R$ by Corollary 2.4;
4. $\mathbb{Z} \to R$ is a faithfully flat ring extension and $\mathbb{Z}$ is not a G-domain.

Example 4.2. Let $T = \mathbb{R}[X](X) = \mathbb{R} + XT$, where $X$ is an indeterminate over $\mathbb{R}$, and let $R = \mathbb{Z} + XT$. Then:

1. $R$ is a G-domain by Theorem 2.2 since $T$ is a local G-domain and $XT \subseteq R$;
2. $R$ is not a coherent domain ([8, Theorem 3]);
3. $\mathbb{Z} \to R$ is a faithfully flat ring extension and $\mathbb{Z}$ is not a G-domain.

Example 4.3. Let $A$ be a G-domain which is not a field, $K = qf(A)$, and let $R := A \times K$ be the trivial ring extension of $A$ by $K$. Then:

1. $R$ is a G-ring by Corollary 3.2 since $A$ is a G-domain;
2. $R$ is not a coherent ring since $R(0,1)$ is a finitely generated ideal which is not finitely presented as shown by the exact sequence of $R$-modules:

$$0 \to 0 \times K \to R \xrightarrow{u} R(0,1) \to 0$$

where $u(a,e) = (a,e)(0,1) = (0,a)$ (since $0 \times K$ is not a finitely generated ideal of $R$).

Example 4.4. Let $A$ be a non G-domain, $K = qf(A)$, $T = K \times K$ be the trivial ring extension of $K$ by $K$, and let $R := A \times K$ be the trivial ring extension of $A$ by $K$. Then:

1. $T$ is a G-ring since it is a total ring;
2. $R$ is not a G-ring by Corollary 3.2 since $A$ is not a G-domain;
3. $(0,1)T = 0 \times K \subseteq R$.

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References


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