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Spaces with property \((DC(\omega_1))\)

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Abstract. We prove that if \(X\) is a first countable space with property \((DC(\omega_1))\) and with a \(G_\delta\)-diagonal then the cardinality of \(X\) is at most \(c\). We also show that if \(X\) is a first countable, DCCC, normal space then the extent of \(X\) is at most \(c\).

Keywords: \(G_\delta\)-diagonal; property \((DC(\omega_1))\); cardinal; DCCC

Classification: Primary 54D20, 54E35

1. Introduction

Diagonal property is useful in estimating the cardinality of a space. For example, Ginsburg and Woods in [6] proved that the cardinality of a space with countable extent and a \(G_\delta\)-diagonal is at most \(c\). Therefore, if \(X\) is Lindelöf and has a \(G_\delta\)-diagonal then \(|X| \leq c\). However, the cardinality of a regular space with the countable Souslin number and a \(G_\delta\)-diagonal need not have an upper bound [11], [12]. Buzyakova in [4] proved that if a space \(X\) with the countable Souslin number has a regular \(G_\delta\)-diagonal then the cardinality of \(X\) does not exceed \(c\). Recently, Xuan and Shi in [13] show that if \(X\) is a DCCC space with a rank 3-diagonal then the cardinality of \(X\) is at most \(c\).

In this paper, we prove that if \(X\) is a first countable space with property \((DC(\omega_1))\) (defined below) and with a \(G_\delta\)-diagonal then the cardinality of \(X\) is at most \(c\). We also show that if \(X\) is a first countable, DCCC, normal space then the extent of \(X\) is at most \(c\).

2. Notation and terminology

All the spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set \(X\) is denoted by \(|X|\), and \([X]^2\) will denote the set of two-element subsets of \(X\). We write \(\omega\) for the first infinite cardinal and \(c\) for the cardinality of the continuum.

Definition 2.1. We say that a topological space \(X\) has a \(G_\delta\)-diagonal if there exists a sequence \(\{G_n : n \in \omega\}\) of open sets in \(X^2\) such that \(\Delta_X = \bigcap \{G_n : n < \omega\}\), where \(\Delta_X = \{(x, x) : x \in X\}\).
Definition 2.2. A space \( X \) has a strong rank 1-diagonal \([2]\) if there exists a sequence \( \{U_n : n < \omega\} \) of open covers of \( X \) such that for each \( x \in X \), \( \{x\} = \bigcap \{\text{St}(x, U_n) : n < \omega\} \).

Note that a space having a strong rank 1-diagonal always has a \( G_\delta \)-diagonal.

Definition 2.3. A topological space \( X \) is pracompact if it has a dense subspace every infinite subset of which has a limit point in \( X \).

Clearly, every countably compact space is pracompact and it is easy to see that every countably pracompact space is DFCC, i.e., every infinite family \( \xi \) of open sets of \( X \) has an accumulation point in \( X \). It should be pointed out that for Tychonoff spaces DFCC is equivalent to pseudocompactness, i.e., every continuous real-valued function on \( X \) is bounded.

\( (DC(\omega_1)) \) is a property which is the analog of countable pracompactness.

Definition 2.4. A topological space \( X \) has property \( (DC(\omega_1)) \) if it has a dense subspace every uncountable subset of which has a limit point in \( X \).

This notion was first introduced and studied in \([7]\) by Ikenaga. It is clear that every countably pracompact space and every space with a dense subspace of countable extent is \( (DC(\omega_1)) \).

Definition 2.5. We say that a space \( X \) satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of non-empty open subsets of \( X \) is countable.

All notation and terminology not explained here is given in \([5]\).

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó.

Lemma 3.1 ([8, p.8]). Let \( X \) be a set with \( |X| > \mathfrak{c} \) and suppose \( |X|^2 = \bigcup \{P_n : n \in \omega\} \). Then there exist \( n_0 < \omega \) and a subset \( S \) of \( X \) with \( |S| > \omega \) such that \( |S|^2 \subset P_{n_0} \).

Theorem 3.2. Let \( X \) be a first countable space with property \( (DC(\omega_1)) \) and with a \( G_\delta \)-diagonal. Then the cardinality of \( X \) is at most \( \mathfrak{c} \).

Proof: Since \( X \) has a \( G_\delta \)-diagonal, there exists a sequence \( \{G_k : k < \omega\} \) of open sets of \( X^2 \) such that \( \Delta_X = \bigcap \{G_k : k < \omega\} \). For each \( k \in \omega \) and \( x \in X \), there exists an open subset \( V_k(x) \) of \( X \) such that \( (x, x) \in V_k(x) \times V_k(x) \subset G_k \). Thus without loss of generality, we assume that \( G_k = \bigcup \{V_k(x) \times V_k(x) : x \in X\} \) and \( G_{k+1} \subset G_k \).

Assume that \( Y \) is a dense subspace of \( X \) which witnesses that \( X \) has property \( (DC(\omega_1)) \). We shall show that \( |Y| \leq \mathfrak{c} \). Suppose not. For \( n < \omega \), let

\[
P_n = \left\{ \{x, y\} \in [Y]^2 : (x, y) \notin G_n \right\}.
\]
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Clearly, for any \(\{x, y\} \in [Y]^2\), there exists \(n < \omega\) such that \(\{x, y\} \in P_n\). Thus, 
\[ [Y]^2 = \bigcup \{P_n : n < \omega\}. \]
Then by Lemma 3.1 there exists a subset \(S\) of \(Y\) with 
\(|S| > \omega\) and \([S]^2 \subseteq P_{n_0}\) for some \(n_0 < \omega\). Since \(X\) has property \((DC(\omega_1))\), it follows that \(S\) has a limit point \(x \in X\). Since \(X\) is \(T_1\) each neighborhood of \(x\) meets infinitely many members of \(S\). In particular, there exist distinct points \(y\) and \(z\) in \(S \cap V_{n_0}(x)\). Thus \((y, z) \notin G_{n_0}\), which is a contradiction. This shows 
\[ |Y| \leq c. \]

From Theorem 4.4 of [8, p.55] that every first countable Hausdorff space with 

a dense subset of cardinality \(\leq c\) has cardinality \(\leq c\), we conclude that 
\(|X| \leq c\). This completes the proof. \(\Box\)

The authors do not know whether the condition “first countable” is necessary 
in Theorem 3.2. However, we know that if we drop the condition “property 

\((DC(\omega_1))\)” or “\(G_\delta\)-diagonal”, the conclusion will be no longer true, as can be 

seen in the following examples.

**Example 3.3.** Let \(D\) be the discrete space with \(|D| = 2^c\). It is evident that 

\(D\) is first countable and has a \(G_\delta\)-diagonal. However it does not have property 

\((DC(\omega_1))\).

**Example 3.4.** Let \(X\) be the subspace of \([0, 2^c]\), consisting of all ordinals of countable cofinality, equipped with the ordered topology. Then \(X\) has cardinality \(2^c\). Moreover \(X\) is first countable and countably compact, and hence \(X\) is \((DC(\omega_1))\). However, it does not have a \(G_\delta\)-diagonal.

Clearly, every point of any space with a \(G_\delta\)-diagonal is a \(G_\delta\)-point. By applying 

Lemma 2 of [3] that if every point of a regular DFCC space \(X\) is a \(G_\delta\)-point then 

\(X\) is first countable, we can conclude the following conclusion.

**Corollary 3.5.** Let \(X\) be a regular countably pracompact space with a \(G_\delta\)-diagonal. Then the cardinality of \(X\) is at most \(c\).

However, a Tychonoff pseudocompact space with a \(G_\delta\)-diagonal can have arbitrarily big cardinality.

**Example 3.6.** For every cardinal \(\tau\), there exists a pseudocompact space of cardinality \(> \tau\) but having a \(G_\delta\)-diagonal [9, p.34].

**Theorem 3.7.** Let \(X\) be a regular DFCC space with a strong rank \(1\)-diagonal. Then the cardinality of \(X\) is at most \(c\).

**Proof:** By Theorem 3.7 of [2], it is easy to deduce that \(X\) is a Moore space, and hence \(X\) is perfect. Moreover, every DFCC, perfect space has countable chain condition (short for CCC) [10, Proposition 2.3]. Since the cardinality of a first countable, CCC space is at most \(c\), it follows that 
\(|X| \leq c\). \(\Box\)

Recall that the extent \(e(X)\) of \(X\) is the supremum of the cardinalities of closed discrete subsets of \(X\). A space \(X\) is star countable if whenever \(U\) is an open cover of \(X\), there is a countable subset \(A \subseteq X\) such that \(St(A, U) = X\). It is clear that a space with countable extent is star countable.
Theorem 3.8. Let $X$ be a first countable, DCCC, normal space. Then the extent $e(X)$ of $X$ is at most $\mathfrak{c}$.

**Proof:** Suppose that $e(X) > \mathfrak{c}$. Then there exists a closed and discrete subset $Y$ of $X$ such that $|Y| > \mathfrak{c}$. Let $\mathcal{B}(x) = \{B_n(x) : n < \omega\}$ be a local base for $x$. Assume $B_{n+1}(x) \subset B_n(x)$ for any $n < \omega$. For each $n < \omega$ let

$$P_n = \left\{ \{x, y\} \in [Y]^2 : B_n(x) \cap B_n(y) = \emptyset \right\}.$$ 

Thus, $[Y]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 3.1 there exists a subset $S$ of $Y$ with $|S| > \omega$ and $[S]^2 \subset P_n$ for some $n_0 < \omega$. Since $S \subset Y$, one easily sees that $S$ is closed and discrete. Besides, it is evident that for any two distinct points $x, y \in S$, $B_{n_0}(x) \cap B_{n_0}(y) = \emptyset$.

Since $X$ is normal, there exists an open subset $U$ of $X$ such that $S \subset U \subset \overline{U} \subset \bigcup\{B_{n_0}(x) : x \in S\}$. Let $\mathcal{U} = \{B_{n_0}(x) \cup U : x \in S\}$. It must have a cluster point $y \in X$, since $X$ is DCCC. Since $B_{n_0}(x) \cap \overline{U} \subset \bigcup\{B_{n_0}(x) : x \in S\}$, we can conclude that $y \in \bigcup\{B_{n_0}(x) : x \in S\}$. Now we assume that $y \in B_{n_0}(x_0)$ for some $x_0 \in S$. It is clear to see that $B_{n_0}(x_0) \cap B_{n_0}(x) = \emptyset$ for any $x \in S \setminus \{x_0\}$. This shows that $y$ is not a cluster point of $\{B_{n_0}(x) \cap U : x \in S\}$. A contradiction! This proves that $e(X) \leq \mathfrak{c}$. \hfill \Box

**Remark 3.9.** Theorem 3.8 would be compared to a recent result of [1]: if $X$ is first countable and $e(X) > \mathfrak{c}$, then $X$ is not star countable. It can be proved by using our method in the proof of Theorem 3.8 that, clearly, $X$ has an uncountable closed discrete subset $S$ whose points can be separated by pairwise disjoint open sets. For each $x \in S$, let $U_x \subset X$ be an open set containing $x$ such that for each $y \in S \setminus \{x\}$, $U_x \cap U_y = \emptyset$. Then $\mathcal{U} = \{U_x : x \in S\} \cup \{X \setminus S\}$ is an open cover for which there is no countable $A$ of $X$ such that $\text{St}(A, \mathcal{U}) = X$. This shows that $X$ is not star countable.

We finish this paper by the following question.

**Question 3.10.** Must a first countable, DCCC, normal space be star countable?

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**References**


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