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Algebraic Connections and Curvature in Fibrations Bundles of Associative Algebras

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Abstract

In this article fibrations of associative algebras on smooth manifolds are investigated. Sections of these fibrations are spinor, co spinor and vector fields with respect to a gauge group. Invariant differentiations are constructed and curvature and torsion of invariant differentiations are calculated.

Key words: Algebraic fibration, spinor, co spinor, vector field, field of connection, invariant differentiation, curvature, torsion.

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Fibrations of linear algebras are a specification of vector fibrations on smooth manifolds where the standard fiber is a linear algebra. Such specification allows for a smooth manifold to introduce some connection which is compatible with the algebraic structure of a standard fiber.

Let us consider an arbitrary associative unitary algebra $A$, $\dim A = n$, with basis space $T_m$, $\dim T_m = m$. Let $M$ be a differentiable manifold, $\dim M = m$. Denote by $T_m(x)$ the tangent space in a point $x \in M$ and by $A(x)$ the algebra with basis space $T_m(x)$. By this we for the manifold $M$ obtain a vector fiber bundle $AM$, the standard fiber of which is a linear space of algebra $A$ (see [1]).

However, $AM$ is not only a vector space, because in every fiber $A(x)$ we may define not only linear operations but also a product of vectors. Therefore it is useful to introduce for fiber bundle $AM$ a special denomination algebraic fibration (see [2]). Herewith the module $A(M)$ of smooth sections of algebraic fibration is an infinite algebra, the restriction of which to a point $x \in M$ coincides with algebra $A(x)$. This algebra will be called a gauge algebra of fibration $AM$ (analogously to modules of gauge field in time-space manifolds, see [3]). Elements of this algebra, i.e. sections of fibration, will be called algebraic (gauge) fields on manifold $M$. 
Herewith the algebra $A(M)$ is unitary because an algebra $A$ is unitary. Therefore the module $F(M)$ of smooth functions on a manifold $M$ is a subalgebra of the algebra $A(M)$.

Now, let us denote by $\mathcal{R}(A(M))$ a multiplicative group of all algebraic fields and call it by a regular group of the algebra $A(M)$.

Let $\Phi \in F(M)$ be an arbitrary multiplicative function. This function defines a subgroup $G_{\Phi}(M) \subset \mathcal{R}(A(M))$, elements of which $\alpha = \alpha(x)$ fulfills the identity $\Phi(\alpha) = 1$. By this way, in a fibration $AM$ we obtain a geometric structure, gauge motions of which are given by linear algebraic functions. Fields $\xi = \xi(x) \in A(M)$ which for an action of a gauge group $G_{\Phi}(M)$ satisfy

$$\psi_L(\xi(x)) = \alpha(x) \cdot \xi(x),$$

for any $\alpha(x) \in G_{\Phi}(M)$, are called $G$-spinor fields (analogously to spinor field in time-space manifolds, see [4]).

Fields $\eta = \eta(x) \in A(M)$ which for an action of a gauge group $G_{\Phi}(M)$ satisfy

$$\psi_R(\eta(x)) = \eta(x) \cdot \alpha^{-1}(x),$$

are called $G$-co spinor fields (by the same physical analogy).

Finally, fields $\zeta = \zeta(x) \in A(M)$ which for an action of a gauge group $G_{\Phi}(M)$ satisfy

$$\psi(\zeta(x)) = \alpha(x) \cdot \zeta(x) \cdot \alpha^{-1}(x),$$

are called $G$-vector fields.

Let us consider some differentiation of fields $\xi \in A(M)$, i.e. a linear operator $\partial: A(M) \to A(M)$ which satisfies the Leibnitz identity:

$$\partial(\xi \cdot \eta) = \partial(\xi) \cdot \eta + \xi \cdot \partial(\eta),$$

for any fields $\xi = \xi(x), \eta = \eta(x) \in A(M)$. According to the Leibnitz identity the operator of differentiation is not invariant with respect to gauge action, generally. It means that the identities $\partial(\psi_L(\xi)) = \psi_L(\partial(\xi)), \partial(\psi_R(\xi)) = \psi_R(\partial(\xi)), \partial(\psi(\zeta)) = \psi(\partial(\zeta))$ are not satisfied for it. However, for any differentiation we may construct some new operators which will be invariant with respect to the action of the group $G_{\Phi}(M)$ on $G$-spinor, $G$-co spinor and $G$-vector fields.

For this purpose, we will in every point $x \in M$ consider an arbitrary set of differentiations $\partial_V = \partial_V(x)$. Denote by $D(x)$ the linear space which is generated by such set and construct a fibration $DAM$, fibers of which are Cartesian products $D(x) \times A(x)$. Sections of this fibration $\Gamma\{\partial\}$, where $\partial(x) \in D(x)$, which for an action of a gauge group $G_{\Phi}(M)$ satisfy

$$\psi_C(\Gamma\{\partial\}) = \alpha \cdot \Gamma\{\partial\} \cdot \alpha^{-1} - \partial(\alpha) \cdot \alpha^{-1},$$

are called $G$-connection and the set of them will be denoted by $\delta A(M)$.

It is clear to see that if $a, b \in R$, $a + b = 1$, then for any connections $\Gamma_1(\partial), \Gamma_2(\partial) \in \delta A(M)$ a field $a\Gamma_1(\partial) + b\Gamma_2(\partial)$ is also a connection.

If fields of connection are given we may construct a differential operator invariant with respect to the (gauge) motion. Especially, for any $\delta(x) \in D(x)$ we have the following theorem.
Theorem 1 (invariance theorem). Let \( \xi = \xi(x) \), \( \eta = \eta(x) \) and \( \zeta = \zeta(x) \) be arbitrary G-spinor, G-co spinor, and G-vector fields. The operators defined by the following formulas

\[
\nabla_L \{ \partial \} \xi = \partial \xi + \Gamma \{ \partial \} \cdot \xi, \quad (6)
\]

\[
\nabla_R \{ \partial \} \eta = \partial \eta - \eta \cdot \Gamma \{ \partial \}, \quad (7)
\]

\[
\nabla \{ \partial \} \zeta = \partial \zeta + \Gamma_1 \{ \partial \} \cdot \zeta - \zeta \cdot \Gamma_2 \{ \partial \}. \quad (8)
\]

are invariant with respect to motions of the group \( G \). 

In fact, if \( \psi_L(\xi) = \alpha \cdot \xi \) then we may write:

\[
\nabla_L \{ \partial \}(\psi_L(\xi)) = \partial(\psi_L(\xi)) + \psi_C(\Gamma \{ \partial \}) \cdot \psi_L(\xi)
\]

\[
= (\partial \alpha) \cdot \xi + \alpha \cdot (\partial \xi) + (\alpha \cdot \Gamma \{ \partial \} \cdot \alpha^{-1}) \cdot (\alpha \cdot \xi) - (\partial \alpha) \cdot \alpha^{-1} \cdot (\alpha \cdot \xi)
\]

\[
= \alpha \cdot (\partial \xi) + \alpha \cdot \Gamma \{ \partial \} \cdot \xi = \alpha \cdot (\nabla_L \{ \partial \} \xi) = \psi_L(\nabla_L \{ \partial \} \xi).
\]

The invariance of operators \( \nabla_R \{ \partial \} \) and \( \nabla \{ \partial \} \) may be proved analogously.

Operators are called operators of invariant G-spinor, G-co spinor, and G-vector differentiation, respectively.

In this case, if connections \( \Gamma_1 \{ \partial \} \) and \( \Gamma_2 \{ \partial \} \) of operator \( \nabla \{ \partial \} \) are identical, the operator of invariant G-vector differentiation is called symmetric and we denote it by

\[
\nabla \{ \partial \} \xi = \partial \zeta + \Gamma \{ \partial \} \cdot \zeta - \zeta \cdot \Gamma \{ \partial \} = \partial \xi + [\Gamma \{ \partial \}, \xi].
\]

Let us remark that an action of an arbitrary operator of G-vector invariant differentiation \( \nabla \{ \partial \} \zeta \) may be represented as an action of symmetric G-vector operator with a sum of anti-commutator of a given G-vector field \( \xi \) and another G-vector field. For this purpose we for the operator (8) introduce a G-connection \( \Gamma \{ \partial \} = (\Gamma_1 \{ \partial \} + \Gamma_2 \{ \partial \})/2 \) and we remark, that a difference \( S \{ \partial \} = (\Gamma_1 \{ \partial \} - \Gamma_2 \{ \partial \})/2 \) is a G-vector field (it will be called G-torsion of a couple of G-connection \( \Gamma_1 \{ \partial \} \) and \( \Gamma_2 \{ \partial \} \)). Now we may write

\[
\nabla \{ \partial \} \zeta = \partial \zeta + \Gamma_1 \{ \partial \} \cdot \zeta - \zeta \cdot \Gamma_2 \{ \partial \}
\]

\[
= \partial \zeta + \Gamma_1 \{ \partial \} \cdot \zeta - \zeta \cdot \Gamma_2 \{ \partial \} + S \{ \partial \} \cdot \zeta + S \{ \partial \} \cdot \zeta
\]

\[
= \partial \zeta + [\Gamma \{ \partial \}, \zeta] + S \{ \partial \} \cdot \zeta = \nabla \{ \partial \} \zeta + \langle S \{ \partial \}, \zeta \rangle.
\]

For operators \( \nabla_L \{ \partial \} \), \( \nabla_R \{ \partial \} \) and \( \nabla \{ \partial \} \) the following theorems holds.

Theorem 2 (on curvature). Let differentiations \( \partial_1(x), \partial_2(x) \in D(x) \) be given. Then commutators of invariant G-differentiations \( \nabla_L \{ \partial \}, \nabla_R \{ \partial \}, \nabla \{ \partial \} \) are reduced to linear functions coefficients of which are some G-vector fields \( K \{ \partial_1, \partial_2 \} \) depending on G-connections \( \Gamma \{ \partial_1 \}, \Gamma \{ \partial_2 \}, \Gamma \{ [\partial_2, \partial_1] \} \).
In fact, if $\nabla_L(\partial)\xi = \partial\xi + \Gamma(\partial) \cdot \xi$ then we obtain

$$
\nabla_L(\partial_2)\nabla_L(\partial_1)\xi = \partial_2\partial_1\xi + \partial_2\Gamma(\partial_1) \cdot \xi + \Gamma(\partial_2) \cdot \partial_1\xi
+ \Gamma(\partial_2) \cdot \partial_1\xi + \Gamma(\partial_2) \cdot \Gamma(\partial_1) \cdot \xi,
$$

$$
\nabla_L(\partial_1)\nabla_L(\partial_2)\xi = \partial_1\partial_2\xi + \partial_1\Gamma(\partial_2) \cdot \xi + \Gamma(\partial_2) \cdot \partial_1\xi
+ \Gamma(\partial_2) \cdot \partial_1\xi + \Gamma(\partial_2) \cdot \Gamma(\partial_1) \cdot \xi,
$$

$$
\nabla_L([\partial_2, \partial_1])\xi = \partial_2\partial_1\xi - \partial_1\partial_2\xi + \Gamma([\partial_2, \partial_1]) \cdot \xi.
$$

Therefore

$$(\nabla_L(\partial_2)\nabla_L(\partial_1) - \nabla_L(\partial_1)\nabla_L(\partial_2) - \nabla_L([\partial_2, \partial_1]))\xi = K(\partial_1, \partial_2) \cdot \xi,$$

where

$$
K(\partial_1, \partial_2) = \partial_2\Gamma(\partial_1) - \partial_1\Gamma(\partial_2) + \Gamma(\partial_2) \cdot \Gamma(\partial_1) - \Gamma(\partial_1) \cdot \Gamma(\partial_2) - \Gamma([\partial_2, \partial_1]).
$$

By na analogous way, we may prove

$$(\nabla_L(\partial_2)\nabla_L(\partial_1) - \nabla_L(\partial_1)\nabla_L(\partial_2) - \nabla_L([\partial_2, \partial_1]))\eta = -\eta \cdot K(\partial_1, \partial_2),$$

and

$$(\nabla_2\nabla_1 - \nabla_1\nabla_2 - \nabla([\partial_2, \partial_1]))\zeta = K(\partial_1, \partial_2) \cdot \zeta - \zeta \cdot K(\partial_1, \partial_2).$$

It remains to prove, that $K(\partial_1, \partial_2)$ is a $G$-vector field:

$$
\partial_2\psi_C(\Gamma(\partial_1)) - \partial_1\psi_C(\Gamma(\partial_2)) + \psi_C(\Gamma(\partial_2)) \cdot \psi_C(\Gamma(\partial_1))
= (\partial_2\alpha) \cdot \Gamma(\partial_1) \cdot \alpha^{-1} + \alpha \cdot (\partial_1\Gamma(\partial_1)) \cdot \alpha^{-1} - \partial_2\Gamma(\partial_1) \cdot \alpha^{-1} - \partial_1\alpha \cdot \alpha^{-1} - \partial_1\Gamma(\partial_1) \cdot \alpha^{-1} - \partial_2\alpha \cdot \alpha^{-1} - \partial_1\alpha \cdot \alpha^{-1} - \partial_2\Gamma(\partial_1) \cdot \alpha^{-1} + \partial_1\alpha \cdot \alpha^{-1} + \partial_2\alpha \cdot \alpha^{-1} - \partial_1\Gamma(\partial_1) \cdot \alpha^{-1} + \partial_2\alpha \cdot \alpha^{-1} - \partial_1\Gamma(\partial_1) \cdot \alpha^{-1} + \partial_2\alpha \cdot \alpha^{-1} \cdot \alpha^{-1}.
$$

In conclusion, if on a manifold $\mathbf{M}$ Riemannian metric is defined and if as an algebraic fibration over such manifold the fibration of Clifford algebras is given, then Spin($\mathbf{M}$) is such gauge group actions of which on vector and spinor fields preserve Riemannian metric. In this case $G$-connection for differential operators $\partial = \xi^k \partial x^k$ will be a Riemannian connection and $G$-vector field $K(\partial_1, \partial_2)$ will be a tensor field of Riemannian curvature (see [5]).
References


