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Conformal Ricci Soliton in Lorentzian $\alpha$-Sasakian Manifolds

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Abstract

In this paper we have studied conformal curvature tensor, conharmonic curvature tensor, projective curvature tensor in Lorentzian $\alpha$-Sasakian manifolds admitting conformal Ricci soliton. We have found that a Weyl conformally semi symmetric Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton is $\eta$-Einstein manifold. We have also studied conharmonically Ricci symmetric Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton. Similarly we have proved that a Lorentzian $\alpha$-Sasakian manifold $M$ with projective curvature tensor admitting conformal Ricci soliton is $\eta$-Einstein manifold. We have also established an example of 3-dimensional Lorentzian $\alpha$-Sasakian manifold.

Key words: Conformal Ricci soliton, conformal curvature tensor, conharmonic curvature tensor, Lorentzian $\alpha$-Sasakian manifolds, projective curvature tensor.

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1 Introduction

In 1982 Hamilton [11] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston’s geometric conjecture

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which says that each closed three manifold admits a geometric decomposition. Hamilton also [12] classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature.

The Ricci flow equation is given by
\[
\frac{\partial g}{\partial t} = -2S
\]
(1.1)
on a compact Riemannian manifold \(M\) with Riemannian metric \(g\). Ricci soliton emerges as the limit of the solutions of Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. Ramesh Sharma [28] started the study of Ricci soliton in contact manifolds and after him M. M. Tripathi [31], Bejan, Crasmareanu [4] studied Ricci soliton in contact metric manifolds. The Ricci soliton equation is given by
\[
\mathcal{L}_X g + 2S + 2\lambda g = 0,
\]
(1.2)
where \(\mathcal{L}_X\) is the Lie derivative, \(S\) is Ricci tensor, \(g\) is Riemannian metric, \(X\) is a vector field and \(\lambda\) is a scalar. The \(\varphi\)-vector fields are special type Ricci soliton studied in [14, 15].

In 2005, A.E. Fischer [9] introduced a new concept called conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by
\[
\frac{\partial g}{\partial t} + 2 \left( S + \frac{g}{n} \right) = -pg
\]
(1.3)
and \(R(g) = -1\), where \(p\) is a scalar non-dynamical field(time dependent scalar field), \(R(g)\) is the scalar curvature of the manifold and \(n\) is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows
\[
\mathcal{L}_X g + 2S = \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right] g.
\]
(1.4)
The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

A Riemannian manifold is said to be locally symmetric if its curvature tensor \(R\) satisfies \(\nabla R = 0\), where \(\nabla\) is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor \(R\) satisfies
\( R(X, Y).R = 0 \) for all \( X, Y \in TM \), where \( R(X, Y) \) acts on \( R \) as a derivation. N. S. Sinyukov, J. Mikeš, I. Hinterleitner and others studied geodesic mappings of symmetric and semisymmetric spaces [29, 10, 18, 13, 19, 17, 22, 23, 24, 25, 16]. K. Sekigawa [27], Z. I. Szabo [30] studied Riemannian manifolds or hypersurfaces of such manifold satisfying the condition \( R(X, Y).R = 0 \) or condition similar to it. It is easy to see that \( R(X, Y).R = 0 \) implies \( R(X, Y).C = 0 \). So it is meaningful to undertake the study of manifolds satisfying such type of conditions.

### 1.1 Definition of Einstein manifold

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold with Ricci tensor is proportional to the metric. If \( M \) is the underlying \( n \)-dimensional manifold and \( g \) is its metric tensor then the Einstein condition means that

\[
S(X, Y) = \lambda g(X, Y),
\]

for some constant \( \lambda \), where \( S \) denotes the Ricci tensor of \( g \). Einstein manifolds with \( \lambda = 0 \) are called Ricci-flat manifolds.

### 1.2 Definition of \( \eta \)-Einstein manifold

A trans-Sasakian manifold \( M^n \) is said to be \( \eta \)-Einstein manifold if its Ricci tensor \( S \) is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

where \( a, b \) are smooth functions.

### 2 Basic concepts of Lorentzian \( \alpha \)-Sasakian manifolds

A differentiable manifold of dimension \((2n+1)\) is called Lorentzian \( \alpha \)--Sasakian manifold [1] if it admits a \((1, 1)\) tensor field \( \varphi \), a vector field \( \xi \) and 1-form \( \eta \) and Lorentzian metric \( g \) which satisfy on \( M \) respectively such that

\[
\varphi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0, \quad (2.1)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (2.2)
\]

\[
\nabla_X \xi = \alpha \varphi X, \quad (\nabla_X \eta) Y = \alpha g(\varphi X, Y), \quad (2.3)
\]

where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \) on \( M \). Geometry of Sasakian spaces was studied in [21, 20, 26, 19].
On an Lorentzian $\alpha$-Sasakian manifold $M$ the following relations hold [1]:

\begin{align*}
R(X,Y)\xi &= \alpha^2[\eta(Y)X - \eta(X)Y], \\
R(\xi,X)Y &= \alpha^2[g(X,Y)\xi - \eta(Y)X], \\
S(X,\xi) &= 2n\alpha^2\eta(X), \\
Q\xi &= 2n\alpha^2\xi, \\
S(\xi,\xi) &= -2n\alpha^2,
\end{align*}

where $\alpha$ is some constant, $R$ is the Riemannian curvature, $S$ is the Ricci tensor and $Q$ is the Ricci operator given by $S(X,Y) = g(QX,Y)$ for all $X,Y \in \chi(M)$.

Now from definition of Lie derivative we have

\begin{equation}
(\mathcal{L}_\xi g)(X,Y) = (\nabla_\xi g)(X,Y) + g(\alpha\varphi X,Y) + g(X,\alpha\varphi Y) \\
= 2\alpha g(\varphi X,Y), \quad [\cdot : g(X,\varphi Y) = g(\varphi X,Y)].
\end{equation}

Applying (2.9) in (1.4) we get

\begin{equation}
S(X,Y) = \frac{1}{2} \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right] g(X,Y) - \alpha g(\varphi X,Y) \\
= Ag(X,Y) - \alpha g(\varphi X,Y),
\end{equation}

where

\begin{equation}
A = \frac{1}{2} \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right].
\end{equation}

Since $S(X,Y) = g(QX,Y)$ for the Ricci operator $Q$, we have

\begin{equation}
g(QX,Y) = Ag(X,Y) - \alpha g(\varphi X,Y)
\end{equation}
i.e.

\begin{equation}
QX = AX - \alpha\varphi X, \quad \forall Y.
\end{equation}

Also

\begin{equation}
S(Y,\xi) = A\eta(Y), \quad S(\xi,\xi) = -A, \quad Q\xi = A\xi.
\end{equation}

If we put $X = Y = e_i$ in (2.10), where $\{e_i\}$ is orthonormal basis of the tangent space $TM$ where $TM$ is a tangent bundle of $M$ and summing over $i$, we get

\begin{equation}
R(g) = An - \alpha g(\varphi e_i,e_i)
\end{equation}

As $R = -1$, we have

\begin{equation}
-1 = An - \alpha.(\text{tr } \varphi) \quad \text{i.e.} \quad A = \frac{1}{n}(\alpha.(\text{tr } \varphi) - 1).
\end{equation}
2.1 Example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold

In this section we construct an example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold. To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

\[ e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^{-z} \frac{\partial}{\partial y}, \quad e_3 = -e^{-z} \frac{\partial}{\partial z} \]

are linearly independent at each point of $M$.

Let $g$ be the Lorentzian metric defined by

\[ g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \]
\[ g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \]

Let $\eta$ be the 1-form which satisfies the relation $\eta(e_3) = -1$. Let $\varphi$ be the $(1, 1)$ tensor field defined by $\varphi(e_1) = -e_1$, $\varphi(e_2) = -e_2$, $\varphi(e_3) = 0$. Then we have

\[ \varphi^2(Z) = Z + \eta(Z)e_3, \]
\[ g(\varphi Z, \varphi W) = g(Z, W) + \eta(Z)\eta(W), \]

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi$, $(\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. Now, after calculating we have

\[ [e_1, e_3] = -e^{-z} e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = -e^{-z} e_2. \]

The Riemannian connection $\nabla$ of the metric is given by the Koszul’s formula which is

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \]
\[ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (2.13) \]

By Koszul’s formula we get

\[ \nabla_{e_1} e_1 = -e^{-z} e_3, \quad \nabla_{e_2} e_2 = -e^{-z} e_3, \quad \nabla_{e_3} e_3 = 0. \]

From the above we have found that $\alpha = e^{-z}$ and it can be easily shown that $M^3(\varphi, \xi, \eta, g)$ is a Lorentzian $\alpha$-Sasakian manifold.

3 Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton and $\tilde{R}(\xi, X).\tilde{C} = 0$

Let $M$ be an $(2n + 1)$ dimensional Lorentzian $\alpha$-Sasakian manifold admitting a conformal Ricci soliton $(g, V, \lambda)$. The conformal curvature tensor $\tilde{C}$ on $M$ is
defined by [2]

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{R}{2n(n-1)}[g(Y, Z)X - g(X, Z)Y],
\]

(3.1)

where \( R \) is scalar curvature.

Now we prove the following theorem:

**Theorem 3.1.** If a Lorentzian \( \alpha \)-Sasakian manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. \( R(\xi, X)\tilde{C} = 0 \), then the manifold is \( \eta \)-Einstein manifold where \( \tilde{C} \) is Conformal curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

**Proof.** Let \( M \) be an \((2n + 1)\) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \((g, V, \lambda)\). So we have \( R = -1 \) [9].

After putting \( R = -1 \) and \( Z = \xi \) in (3.1) we have

\[
\tilde{C}(X, Y)\xi = R(X, Y)\xi - \frac{1}{2n-1}[S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY] - \frac{1}{2n(n-1)}[g(Y, \xi)X - g(X, \xi)Y].
\]

(3.2)

Using (2.2), (2.4), (2.11) and (2.12) in (3.2) we get

\[
\tilde{C}(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y] - \frac{1}{2n-1}[A\eta(Y)X - A\eta(X)Y + \eta(Y)(AX - \alpha\varphi X) - \eta(X)(AY - \alpha\varphi Y)] - \frac{1}{2n(n-1)}[\eta(Y)X - \eta(X)Y].
\]

(3.3)

Using (3.1) and after a brief simplification we obtain

\[
\tilde{C}(X, Y)\xi = [\alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)}](\eta(Y)X - \eta(X)Y).
\]

(3.4)

Considering

\[
B = \alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)},
\]

(3.4) becomes

\[
\tilde{C}(X, Y)\xi = B[\eta(Y)X - \eta(X)Y]
\]

(3.5)

and

\[
g(\tilde{C}(X, Y)\xi, Z) = B[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)],
\]
which implies

\[-\eta(\tilde{C}(X, Y)Z) = B[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]. \quad (3.6)\]

Now we consider that the Lorentzian \(\alpha\)-Sasakian manifold \(M\) admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. \(R(\xi, X).\tilde{C} = 0\) holds in \(M\) (the manifold is locally isometric to the hyperbolic space \(H^{n+1}(-\alpha^2)\) [32]), which implies

\[
R(\xi, X)(\tilde{C}(Y, Z)W) - \tilde{C}(R(\xi, X)Y, Z)W - \tilde{C}(Y, R(\xi, X)Z)W \\
- \tilde{C}(Y, Z)R(\xi, X)W = 0, \quad (3.7)
\]

for all vector fields \(X, Y, Z, W\) on \(M\).

Using (2.5) in (3.7) and putting \(W = \xi\) we get

\[
g(X, \tilde{C}(Y, Z)\xi)\xi - \eta(\tilde{C}(Y, Z)\xi)X - g(X, Y)\tilde{C}(\xi, Z)\xi \\
+ \eta(Y)\tilde{C}(X, Z)\xi - g(X, Z)\tilde{C}(Y, \xi)\xi + \eta(Z)\tilde{C}(Y, X)\xi \\
- g(X, \xi)\tilde{C}(Y, Z)\xi + \eta(\xi)\tilde{C}(Y, Z)X = 0. \quad (3.8)
\]

Taking inner product with \(\xi\) in (3.8) and using (2.1) we obtain

\[
- g(X, \tilde{C}(Y, Z)\xi) - g(X, Y)\eta(\tilde{C}(\xi, Z)\xi) \\
+ \eta(Y)\eta(\tilde{C}(X, Z)\xi) - g(X, Z)\eta(\tilde{C}(Y, \xi)\xi) + \eta(Z)\eta(\tilde{C}(Y, X)\xi) \\
- \eta(X)\eta(\tilde{C}(Y, Z)\xi) - \eta(\tilde{C}(Y, Z)X) = 0. \quad (3.9)
\]

Using (3.5) in (3.9) we have

\[
-B\eta(Z)g(X, Y) + B\eta(Y)g(X, Z) - \eta(\tilde{C}(Y, Z)X) = 0. \quad (3.10)
\]

Putting \(Z = \xi\) in (3.10) and using (2.1) we get

\[
Bg(X, Y) + B\eta(Y)\eta(X) - \eta(\tilde{C}(Y, \xi)X) = 0. \quad (3.11)
\]

Now from (3.1) we can write

\[
\tilde{C}(Y, \xi)X = R(\xi, Y)X - \frac{1}{2n-1}[S(\xi, X)Y - S(Y, X)\xi + g(\xi, X)QY - g(Y, X)Q\xi] \\
- \frac{1}{2n(n-1)}[g(\xi, X)Y - g(Y, X)\xi]. \quad (3.12)
\]

Taking inner product with \(\xi\) and using (2.1), (2.5), (2.12) in (3.12) we get

\[
\eta(\tilde{C}(Y, \xi)X) = \alpha^2\eta(X)\eta(Y) + \alpha^2g(X, Y) \\
- \frac{A}{2n-1}\eta(X)\eta(Y) - \frac{1}{2n-1}S(X, Y) - \frac{A}{2n-1}\eta(X)\eta(Y) \\
- \frac{A}{2n-1}g(X, Y) - \frac{1}{2n(n-1)}\eta(X)\eta(Y) - \frac{1}{2n(n-1)}g(X, Y). \quad (3.13)
\]
After putting (3.13) in (3.11) the equation reduces to
\[ Bg(X,Y) + B\eta(Y)\eta(X) - \alpha^2 \eta(X)\eta(Y) - \alpha^2 g(X,Y) \]
\[ + \frac{A}{2n-1} \eta(X)\eta(Y) + \frac{1}{2n-1} S(X,Y) + \frac{A}{2n-1} \eta(X)\eta(Y) + \frac{A}{2n-1} g(X,Y) \]
\[ + \frac{1}{2n(n-1)} \eta(X)\eta(Y) + \frac{1}{2n(n-1)} g(X,Y) = 0. \] (3.14)

Simplifying (3.14) we have
\[ g(X,Y) \left[ B - \alpha^2 + \frac{A}{2n-1} + \frac{1}{2n(n-1)} \right] \]
\[ + \eta(X)\eta(Y) \left[ B - \alpha^2 + \frac{2A}{2n-1} + \frac{1}{2n(n-1)} \right] + \frac{1}{2n-1} S(X,Y) = 0, \] (3.15)

which can be written in the form
\[ S(X,Y) = \rho g(X,Y) + \sigma \eta(X)\eta(Y), \] (3.16)

where
\[ \rho = (2n-1) \left( \alpha^2 - B - \frac{A}{2n-1} - \frac{1}{2n(n-1)} \right) \]
and
\[ \sigma = (2n-1) \left( \alpha^2 - B - \frac{2A}{2n-1} - \frac{1}{2n(n-1)} \right). \]

So from (3.16) we conclude that the manifold becomes \( \eta \)-Einstein manifold.

\section{Lorentzian \( \alpha \)-Sasakian manifold admitting conformal Ricci soliton and \( K(\xi,X).S = 0 \)}

Let \( M \) be an \( (2n+1) \) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \((g,V,\lambda)\). The conharmonic curvature tensor \( K \) on \( M \) is defined by [8]
\[ K(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1} [S(Y,Z)X - S(X,Z)Y \]
\[ + g(Y,Z)QX - g(X,Z)QY]. \] (4.1)

for all \( X, Y, Z \in \chi(M) \), \( R \) is the curvature tensor and \( Q \) is the Ricci operator.

Now we prove the following theorem:

**Theorem 4.1.** If a Lorentzian \( \alpha \)-Sasakian manifold admits conformal Ricci soliton and the manifold is conharmonically Ricci symmetric i.e. \( K(\xi,X).S = 0 \) then the Ricci operator \( Q \) satisfies the quadratic equation \( FQ^2 + Q - D = 0 \) for all \( X \in \chi(M) \) where \( F,D \) are constants, \( K \) is conharmonic curvature tensor and \( S \) is a Ricci tensor.
Proof. Let $M$ be an $(2n + 1)$ dimensional Lorentzian $\alpha$-Sasakian manifold admitting a conformal Ricci soliton $(g, V, \lambda)$. From (4.1) we can write

$$K(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n - 1} [S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX].$$

(4.2)

Using (2.5), (2.12) in (4.2) we have

$$K(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n - 1} [S(X, Y)\xi - A\eta(Y)X + Ag(X, Y)\xi - \eta(Y)QX].$$

(4.3)

Similarly from (4.2) we get

$$K(\xi, X)Z = R(\xi, X)Z - \frac{1}{2n - 1} [S(X, Z)\xi - S(\xi, Z)X + g(X, Z)Q\xi - g(\xi, Z)QX] = \alpha^2 [g(X, Z)\xi - \eta(Z)X] - \frac{1}{2n - 1} [S(X, Z)\xi - A\eta(Z)X + Ag(X, Z)\xi - \eta(Z)QX].$$

(4.4)

Now we consider that the tensor derivative of $S$ by $K(\xi, X)$ is zero i.e. $K(\xi, X).S = 0$. Then the Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton is conharmonically Ricci symmetric (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]). It gives

$$S(K(\xi, X)Y, Z) + S(Y, K(\xi, X)Z) = 0.$$  

(4.5)

Using (4.3) and (4.4) in (4.5) we get

$$S(\alpha^2 g(X, Y)\xi - \alpha^2 \eta(Y)X$$

$$- \frac{1}{2n - 1} S(X, Y)\xi + \frac{A}{2n - 1} \eta(Y)X - \frac{A}{2n - 1} g(X, Y)\xi + \frac{\eta(Y)}{2n - 1} QX, Z)$$

$$+ S(\alpha^2 g(X, Z)\xi - \alpha^2 \eta(Z)X - \frac{1}{2n - 1} S(X, Z)\xi + \frac{A}{2n - 1} \eta(Z)X$$

$$- \frac{A}{2n - 1} g(X, Z)\xi + \frac{\eta(Z)}{2n - 1} QX, Y) = 0.$$  

(4.6)

Putting $Z = \xi$ and using (2.1), (2.12) in (4.6) we get

$$\left(\frac{A^2}{2n - 1} - A\alpha^2\right) g(X, Y) + \alpha^2 [S(X, Y) - \frac{1}{2n - 1} S(QX, Y) = 0$$
which implies

\[ E g(X,Y) + \frac{1}{2n-1} S(QX,Y) = -\alpha^2 S(X,Y), \]  

(4.7)

where \( E = \frac{A^2}{2n-1} - A \alpha^2 \).

From (4.7) we can write

\[ S(X,Y) = D g(X,Y) - \frac{1}{\alpha^2(2n-1)} S(QX,Y), \]  

(4.8)

where \( D = -\frac{1}{\alpha^2} E \), which implies

\[ QX = DX - FQ^2 X \quad \forall Y \in \chi(M), \]  

(4.9)

where \( F = \frac{1}{\alpha^2(2n-1)} \), i.e.

\[ FQ^2 + Q - D = 0 \quad \forall X. \]  

(4.10)

\[ \square \]

5 Lorentzian \( \alpha \)-Sasakian manifold admitting conformal Ricci soliton and \( P(\xi,X)\tilde{C} = 0 \)

Let \( M \) be an \((2n + 1)\) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \((g,V,\lambda)\). The Weyl projective curvature tensor \( P \) on \( M \) is given by [2]

\[ P(X,Y)Z = R(X,Y)Z - \frac{1}{2n} [S(Y,Z)X - S(X,Z)Y]. \]

Now we prove the following theorem:

**Theorem 5.1.** If a Lorentzian \( \alpha \)-Sasakian manifold \( M \) admits conformal Ricci soliton and \( P(\xi,X).\tilde{C} = 0 \) holds, then the manifold becomes \( \eta \)-Einstein manifold, where \( P \) is projective curvature tensor and \( \tilde{C} \) is conformal curvature tensor.

**Proof.** We know from (3.1) that

\[ \tilde{C}(\xi,X)Y = R(\xi,X)Y \]
\[ - \frac{1}{2n-1}[S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX] \]
\[ - \frac{1}{2n(n-1)}[g(X,Y)\xi - g(\xi,Y)X], \]

(5.1)

since for conformal Ricci soliton the scalar curvature \( R = -1 \) [9].
From (2.5), (2.12) and taking inner product with $\xi$ on (5.1) we have
\[
\eta(\tilde{C}(\xi, X)Y) = \alpha^2 g(X, Y)\eta(\xi) - \alpha^2 \eta(Y)\eta(X)
- \frac{1}{2n-1} S(X, Y)\eta(\xi) + \frac{A}{2n-1} \eta(Y)\eta(X) - \frac{A}{2n-1} \eta(\xi)g(X, Y)
+ \frac{1}{2n-1} \eta(Y)\eta(QX) - \frac{1}{2n(n-1)} [g(X, Y)\eta(\xi) - \eta(Y)\eta(X)]
= g(X, Y) \left[ \frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)} \right]
+ \eta(Y)\eta(X) \left[ \frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)} \right]
+ \frac{1}{2n-1} S(X, Y) = F g(X, Y) + G \eta(Y)\eta(X) + TS(X, Y),
\]
where
\[
F = \frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)},
\]
\[
G = \frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)}
\]
and
\[
T = \frac{1}{2n-1}.
\]
Also
\[
\eta(\tilde{C}(X, Y)\xi) = B[\eta(Y)\eta(X) - \eta(X)\eta(Y)] = 0
\]
and
\[
\eta(\tilde{C}(Y, \xi)\xi) = B[\eta(Y)\eta(\xi) - \eta(\xi)\eta(Y)] = 0.
\]
Now
\[
P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n} [S(X, Y)\xi - S(\xi, Y)X]. \tag{5.2}
\]
Using (2.5), (2.12) in (5.2) we get
\[
P(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n} [S(X, Y)\xi - A\eta(Y)X]. \tag{5.3}
\]
Here we consider that the tensor derivative of $\tilde{C}$ by $P(\xi, X)$ is zero i.e. conformally symmetric with respect to projective curvature tensor i.e. $P(\xi, X)\tilde{C} = 0$ holds (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]).

So
\[
P(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(P(\xi, X)Y, Z)W - \tilde{C}(Y, P(\xi, X)Z)W
- \tilde{C}(Y, Z)P(\xi, X)W = 0, \tag{5.4}
\]
for all vector fields $X, Y, Z, W$ on $M$. 

Using (5.3) in (5.4) and putting $W = \xi$ we have

\[
\alpha^2 g(X, \tilde{C}(Y, Z)\xi) - \alpha^2 \eta(\tilde{C}(Y, Z)\xi)X \\
- \frac{1}{2n} S(X, \tilde{C}(Y, Z)\xi)\xi + \frac{A}{2n} \eta(\tilde{C}(Y, Z)\xi)X - \alpha^2 g(X, Y)\tilde{C}(\xi, Z)\xi \\
+ \alpha^2 \eta(Y)\tilde{C}(X, Z)\xi + \frac{1}{2n} S(X, Y)\tilde{C}(\xi, Z)\xi - \frac{A}{2n} \eta(Y)\tilde{C}(X, Z)\xi \\
- \alpha^2 g(X, Z)\tilde{C}(Y, \xi)\xi + \alpha^2 \eta(Z)\tilde{C}(Y, X)\xi + \frac{1}{2n} S(X, Z)\tilde{C}(Y, \xi)\xi \\
- \frac{A}{2n} \eta(Z)\tilde{C}(Y, X)\xi - \alpha^2 g(X, \xi)\tilde{C}(Y, Z)\xi + \alpha^2 \eta(\xi)\tilde{C}(Y, Z)X \\
+ \frac{1}{2n} S(X, \xi)\tilde{C}(Y, Z)\xi - \frac{A}{2n} \eta(\xi)\tilde{C}(Y, Z)X = 0. \tag{5.5}
\]

Taking inner product with $\xi$ on (5.5) we get

\[
-\alpha^2 g(X, \tilde{C}(Y, Z)\xi) + \frac{1}{2n} S(X, \tilde{C}(Y, Z)\xi) = 0. \tag{5.6}
\]

From (3.2) and (5.6) we have

\[
-\alpha^2 B\eta(Z)g(X, Y) + \alpha^2 \eta(Y)Bg(X, Z) + \frac{B}{2n} \eta(Z)S(X, Y) - \frac{B}{2n} \eta(Y)S(X, Z) = 0. \tag{5.7}
\]

Putting $z = \xi$ in (5.7) and using (2.1), (2.12) we obtain

\[
\alpha^2 Bg(X, Y) + B\alpha^2 \eta(Y)\eta(X) - \frac{B}{2n} S(X, Y) - \frac{AB}{2n} \eta(Y)\eta(X) = 0,
\]

which implies

\[
S(X, Y) = 2n\alpha^2 g(X, Y) + 2n(\alpha^2 - \frac{A}{2n})\eta(Y)\eta(X). \tag{5.8}
\]

So the manifold becomes $\eta$-Einstein manifold. \hfill \Box

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**References**


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