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The Regularization of the Second Order Lagrangians in Example

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Abstract

This paper is devoted to geometric formulation of the regular (resp. strongly regular) Hamiltonian system. The notion of the regularization of the second order Lagrangians is presented. The regularization procedure is applied to concrete example.

Key words: Hamilton extremals, Dedecker–Hamilton extremals, Hamilton equations, Lagrangian, Lepagean equivalents, Poincaré–Cartan form, regular and strongly regular systems.

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1 Introduction

In general, a second order Lagrangian gives rise to an Euler–Lagrange form on 4th order jet prolongation, i.e. the Euler–Lagrange equations are of the 4th order. In this paper we are interested in second order Lagrangians which give rise to Euler–Lagrange equations of the 3rd order.

We consider 3rd order Hamiltonian systems for a given second order Lagrangian. The Lagrangian is quadratic or affine in second derivatives. All these Lagrangians are singular in the standard Hamilton–De Donder theory [2]. However, in the generalized setting, the question on existence of regular Hamilton equations has sense. We apply to this case regularity conditions found in and find their explicit expression for the above mentioned type of Lagrangians.

The results (the regularity resp. strong regularity) can be directly applied to concrete Lagrangian. A regularization procedure is illustrated on concrete example of Lagrangian which is quadratic in second derivatives. The Lagrangian does not satisfy the regularity condition in the classical De Donder–Hamilton sence but the Hamiltonian system is strongly regular in sense [6]. This geometrical meaning of regular Lagrangians is possible to apply to physical theories.

A regularization procedure of first order Lagrangians has been studied in [1, 7, 9] and some second order Lagrangians have been studied in [10, 11].

Throughout the paper all manifolds and mappings are smooth and summation convention is used. We consider a fibered manifold (i.e., surjective submersion) $\pi\colon Y \to X$, $\dim X = n$, $\dim Y = n+m$, its r-jet prolongation $\pi_r\colon J^rY \to X$, $r \geq 1$ and canonical jet projections $\pi_{r,k}\colon J^rY \to J^kY$, $0 \leq k < r$ (with an obvious notations $J^0Y = Y$). A fibered char on Y (resp. associated fibered chart on J^rY) is denoted by (V,ψ), $\psi = (x^i, y^\sigma)$ (resp. (V_r,ψ_r), $\psi_r = (x^i, y^\sigma, y^\sigma_i, \ldots, y^\sigma_{i_1\ldots i_r})$).

A vector field ξ on J^rY is called *π_r-vertical* (resp. *$\pi_{r,k}$-vertical*) if it projects onto the zero vector field on X (resp. on J^kY).

Recall that every q-form η on J^rY admits a unique (canonical) decomposition into a sum of q-forms on $J^{r+1}Y$ as follows:

$$\pi^*_{r+1,r}\eta = h\eta + \sum_{k=1}^{q} p_k\eta,$$

where $h\eta$ is a horizontal form, called the *horizontal part of* η, and $p_k\eta$, $1 \leq k \leq q$, is a *k-contact part of* η (see [3]).

We use the following notations:

$$\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \omega_i = i_{\frac{\partial}{\partial x^i}}\omega_0, \quad \omega_{ij} = i_{\frac{\partial}{\partial x^j}}\omega_i,$$

and

$$\omega^\sigma = dy^\sigma - y^\sigma_j dx^j, \ldots, \omega^\sigma_{i_1 i_2 \ldots i_k} = dy^\sigma_{i_1 i_2 \ldots i_k} - y^\sigma_{i_1 i_2 \ldots i_k j} dx^j$$

For more details on fibered manifolds and the corresponding geometric structures we refer e.g. to [8].

2 Hamiltonian systems and regularity

In this section we briefly recall basic concepts on Lepagean equivalents of Lagrangians, due to Krupka [3], [4], and on Lepagean equivalents of Euler–Lagrange forms and generalized Hamiltonian field theory, due to Krupková [5, 6].

By an *r-th order Lagrangian* we shal mean a horizontal n-form λ on J^rY.

A n-form ρ is called a *Lepagean equivalent of a Lagrangian* λ if (up to a projection) $h\rho = \lambda$, and $p_1 d\rho$ is a $\pi_{r+1,0}$-horizontal form.

For an r-th order Lagrangian we have all its Lepagean equivalents of order $(2r-1)$ characterized by the following formula

$$\rho = \Theta + \mu, \tag{2.1}$$

where Θ is a (global) Poincaré–Cartan form associated to λ and μ is an arbitrary n-form of order of contactness ≥ 2, i.e., such that $h\mu = p_1\mu = 0$. Recall that for a Lagrangian of order 1, $\Theta = \theta_\lambda$ where θ_λ is the classical Poincaré–Cartan form of λ. If $r \geq 2$, Θ is no more unique, however, there is an *non-invariant* decomposition

$$\Theta = \theta_\lambda + p_1 d\nu, \tag{2.2}$$

where

$$\theta_\lambda = L\omega_0 + \sum_{k=0}^{r-1} \left(\sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1 \dots j_k p_1 \dots p_l i}} \right) \omega^\sigma_{j_1 \dots j_k} \wedge \omega_i, \tag{2.3}$$

and ν is an arbitrary at least 1-contact $(n-1)$-form.

A *closed* $(n+1)$-form α is called a *Lepagean equivalent of an Euler–Lagrange form* $E = E_\sigma \omega^\sigma \wedge \omega_0$ if $p_1 \alpha = E$.

Recall that the Euler–Lagrange form corresponding to an r-th order $\lambda = L\omega_0$ is the following $(n+1)$-form of order $\leq 2r$

$$E = \left(\frac{\partial L}{\partial y^\sigma} - \sum_{l=1}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{p_1 \dots p_l}} \right) \omega^\sigma \wedge \omega_0.$$

By definition of a Lepagean equivalent of E, one can find using Poincaré lemma local forms ρ, such that $\alpha = d\rho$, and ρ is an Lepagean equivalent of a Lagrangian for E. The family of Lepagean equivalents of E is also called a *Lagrangian system*, and denoted by $[\alpha]$. The corresponding Euler–Lagrange equations now take the form

$$J^s\gamma^* i_{J^s\xi}\alpha = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y, \tag{2.4}$$

where α is any representative of order s of the class $[\alpha]$. A (single) Lepagean equivalent α of E on J^sY is also called a *Hamiltonian system of order* s and the equations

$$\delta^* i_\xi \alpha = 0 \quad \text{for every } \pi_s\text{-vertical vector field } \xi \text{ on } J^sY \tag{2.5}$$

are called *Hamilton equations*. They represent equations for integral sections δ (called *Hamilton extremals*) of the *Hamiltonian ideal*, generated by the system $\mathcal{D}^s_\alpha$ of n-forms $i_\xi\alpha$, where ξ runs over π_s-vertical vector fields on J^sY. Also, considering π_{s+1}-vertical vector fields on $J^{s+1}Y$, one has the ideal $\mathcal{D}^{s+1}_{\hat{\alpha}}$ of n-forms $i_\xi\hat{\alpha}$ on $J^{s+1}Y$, where $\hat{\alpha}$ (called *principal part* of α) denotes the at most 2-contact part of α. Its integral sections which moreover annihilate all at least 2-contact forms, are called *Dedecker–Hamilton extremals*. It holds that if γ is an extremal then its s-prolongation (resp. $(s+1)$-prolongation) is a Hamilton (resp. Dedecker–Hamilton) extremal, and (up to projection) every Dedecker–Hamilton extremal is a Hamilton extremal.

Denote by r_0 the minimal order of Lagrangians corresponding to E. A Hamiltonian system α on J^sY, $s \geq 1$, associated with E is called *regular* if the system of local generators of $\mathcal{D}^{s+1}_{\hat{\alpha}}$ contains all the n-forms

$$\omega^\sigma \wedge \omega_i, \; \omega^\sigma_{(j_1} \wedge \omega_{i)}, \; \dots, \; \omega^\sigma_{(j_1 \dots j_{r_0 - 1}} \wedge \omega_{i)}, \tag{2.6}$$

where $(\dots)$ denotes symmetrization in the indicated indices. If α is regular then every Dedecker–Hamilton extremal is holonomic up to the order r_0, and its projection is an extremal. (In case of first order Hamiltonian systems there is an bijection between extremals and Dedecker–Hamilton extremals). α is called *strongly regular* if the above correspondence holds between extremals and Hamilton extremals. It can be proved that every strongly regular Hamiltonian system is regular, and it is clear that if α is regular and such that $\alpha = \hat{\alpha}$ then it is strongly regular. A Lagrangian system is called *regular* (resp. *strongly regular*) if it has a regular (resp. strongly regular) associated Hamiltonian system.

3 Hamiltonian systems for second order Lagrangians

In general, a second order Lagrangian gives rise to an Euler–Lagrange form on J^4Y. We shall consider second order Lagrangians λ which satisfy one of the following conditions

1) The corresponding Euler–Lagrange form is of order 3, i.e. the Lagrangians satisfy the conditions

$$\left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_{kl}}\right)_{Sym(ijkl)} = 0, \tag{3.1}$$

where $Sym(ijkl)$ means symmetrization in the indicated indices,

2) The Euler–Lagrange expressions λ of are of the second order, "non-affine" in the second derivatives

$$\frac{\partial^2 E_\sigma}{\partial y^\nu_{kl} \partial y^\kappa_{ij}} \neq 0. \tag{3.2}$$

An interesting case of condition 1) is the Lagrangian affine in second derivatives, i.e. its Lagrangial function takes the form $L = L_0 + L^{ij}_{\sigma\nu} y^{\sigma\nu}_{ij}$, where the functions L_0, $L^{ij}_{\sigma\nu}$ do not depepend on $y^{\alpha\beta}_{kl}$.

In what follows, we shall study Hamiltonian systems corresponding to a special choice of a Lepagean equivalent of such Lagrangians, namely, α of order 3, $\alpha = d\rho$, where

$$\begin{aligned} \rho = L\omega_0 &+ \left(\frac{\partial L}{\partial y^\sigma_j} - d_k \frac{\partial L}{\partial y^\sigma_{jk}}\right) \omega^\sigma \wedge \omega_j + \frac{\partial L}{\partial y^\sigma_{ij}} \omega^\sigma_i \wedge \omega_j + \bar{\mu} \\ &+ a^{ij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu_k \wedge \omega_{ij} + c^{klij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu_{kl} \wedge \omega_{ij}, \end{aligned} \tag{3.3}$$

with an arbitrary at least 3-contact n-form $\bar{\mu}$ and functions $a^{ij}_{\sigma\nu}$, $b^{kij}_{\sigma\nu}$, $c^{klij}_{\sigma\nu}$ dependent on variables x^k, y^κ, y^κ_k, y^κ_{kl} and satisfying the conditions

$$a^{ij}_{\sigma\nu} = -a^{ji}_{\sigma\nu},\ a^{ij}_{\sigma\nu} = -a^{ij}_{\nu\sigma};\ b^{kij}_{\sigma\nu} = -b^{kji}_{\sigma\nu};\ c^{klij}_{\sigma\nu} = c^{lkij}_{\sigma\nu},\ c^{klij}_{\sigma\nu} = -c^{klji}_{\sigma\nu}. \tag{3.4}$$

Theorem. [11] *Let* $\dim X \geq 2$. *Let* $\lambda = L\omega_0$ *be a second order Lagrangian with the Euler–Lagrange form (3.1) or (3.2), and* $\alpha = d\rho$ *with* ρ *of the form (3.3),* *(3.4), be its Lepagean equivalent. Assume that the matrix*

$$P^{ijkl}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^{\nu}_{ij} \partial y^{\sigma}_{kl}} + 2\; c^{klij}_{\nu\sigma} \right)_{Sym(jkl)} , \tag{3.5}$$

with mn^3 rows (resp. mn columns) labelled by σjkl (resp. νi) has maximal rank equal to mn and matrix

$$Q^{ijkl}_{\sigma\nu} = \left(\frac{\partial^2 L}{\partial y^{\sigma}_{ij} \partial y^{\nu}_{kl}} - 2 c^{klij}_{\sigma\nu} \right) , \tag{3.6}$$

with mn^2 rows (resp. mn^2 columns) labelled by σij (resp. νkl) has maximal rank equal to $mn\,(n+1)\,/2$. Then the Hamiltonian system $\alpha = d\rho$ is regular (i.e. every Dedecker–Hamilton extremal is of the form $\pi_{3,2} \circ \delta_D = J^2\gamma$, where γ is an extremal of λ).

If moreover $\bar{\mu}$ is closed then the Hamiltonian system $\alpha = d\rho$ is strongly regular (i.e. every Hamilton extremal is of the form $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is an extremal of λ).

Proof of the above theorem follows from explicit computation. The following proposition is straightforward application of the theorem to the special case of the second order Lagrangians affine in second derivatives.

Proposition. *Let $\dim X \geq 2$. Let $\lambda = L\omega_0$ be a second order Lagrangian of the form $L = L_0 + L^{ij}_{\sigma\nu} y^{\sigma\nu}_{ij}$, where the functions L_0, $L^{ij}_{\sigma\nu}$ do not depepend on $y^{\alpha\beta}_{kl}$ and $\alpha = d\rho$ with ρ of the form (3.3), (3.4), be its Lepagean equivalent. Assume that the matrix*

$$\left(c^{klij}_{\nu\sigma} \right)_{Sym(jkl)} , \tag{3.7}$$

with mn^3 rows (resp. mn columns) labelled by σjkl (resp. νi) has maximal rank equal to mn and matrix

$$\left(c^{klij}_{\sigma\nu} \right) , \tag{3.8}$$

with mn^2 rows (resp. mn^2 columns) labelled by σij (resp. νkl) has maximal rank equal to $mn(n+1)/2$. Then the Hamiltonian system $\alpha = d\rho$ is regular (i.e. every Dedecker–Hamilton extremal is of the form $\pi_{3,2} \circ \delta_D = J^2\gamma$, where γ is an extremal of λ).

If moreover $\bar{\mu}$ is closed then the Hamiltonian system $\alpha = d\rho$ is strongly regular (i.e. every Hamilton extremal is of the form $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is an extremal of λ).

Proof. Explicit computation $\alpha = d\rho$ gives:

$$\pi^*_{4,3}\alpha = E_\sigma \omega^\sigma \wedge \omega_0 + \left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 d_k a^{ij}_{\sigma\nu} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4 a^{ik}_{\nu\sigma} - 2 d_j b^{kij}_{\sigma\nu} \right) \omega^\nu_k \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 (b^{kil}_{\sigma\nu})_{Sym(kl)} - 2 d_j c^{klij}_{\sigma\nu} \right) \omega^\nu_{kl} \wedge \omega^\sigma \wedge \omega_i$$
$$- \left(2 c^{klij}_{\sigma\nu}\right)_{Sym(jkl)} \omega^\nu_{jkl} \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4 (b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))} \right) \omega^\nu_k \wedge \omega^\sigma_j \wedge \omega_i$$
$$+ \left(2 c^{klij}_{\sigma\nu}\right) \omega^\nu_{kl} \wedge \omega^\sigma_j \wedge \omega_i + \left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa} \right)_{Alt(\kappa\sigma\nu)} \omega^\kappa \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$$
$$+ \left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa_p} + \frac{\partial b^{pij}_{\nu\kappa}}{\partial y^\sigma} \right)_{Alt(\sigma\nu)} \omega^\kappa_p \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$$
$$+ \left(\left(\frac{\partial a^{ij}_{\sigma\nu}}{\partial y^\kappa_{pq}} \right)_{Sym(pq)} + \left(\frac{\partial c^{pqij}_{\nu\kappa}}{\partial y^\sigma_{pq}} \right)_{Alt(\sigma\nu)} \right) \omega^\kappa_{pq} \wedge \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$$
$$+ \left(\frac{\partial b^{qij}_{\sigma\nu}}{\partial y^\kappa_p} \right)_{Alt((\kappa p)(\nu q))} \omega^\sigma \wedge \omega^\nu_q \wedge \omega^\kappa_p \wedge \omega_{ij} + \left(\frac{\partial b^{kij}_{\sigma\nu}}{\partial y^\kappa_{pq}} - \frac{\partial c^{pqij}_{\sigma\kappa}}{\partial y^\nu_k} \right)_{Sym(pq)}$$
$$\omega^\sigma \wedge \omega^\nu_k \wedge \omega^\kappa_{pq} \wedge \omega_{ij} - \left(\frac{\partial c^{klij}_{\sigma\nu}}{\partial y^\kappa_{pq}} \right)_{Alt((\kappa pq)(\nu kl))} \omega^\sigma \wedge \omega^\kappa_{pq} \wedge \omega^\nu_{kl} \wedge \omega_{ij} + d\bar{\mu}, \qquad (3.9)$$

where $Alt((\dots)\dots(\dots))$ means alternation in the indicated multiindices and $Sym(\dots)$ means symmetrization in the indicated indices.

In the notation (3.7), (3.8) the principal part of α (3.9) takes form

$$\hat{\alpha} = E_\sigma \omega^\sigma \wedge \omega_0 + \left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 d_k a^{ij}_{\sigma\nu} \right) \omega^\nu \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4 a^{ik}_{\nu\sigma} - 2 d_j b^{kij}_{\sigma\nu} \right) \omega^\nu_k \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2 (b^{kil}_{\sigma\nu})_{Sym(kl)} - 2 d_j c^{klij}_{\sigma\nu} \right) \omega^\nu_{kl} \wedge \omega^\sigma \wedge \omega_i$$
$$+ \left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4 (b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))} \right) \omega^\nu_k \wedge \omega^\sigma_j \wedge \omega_i$$
$$- \left(2 c^{klij}_{\sigma\nu}\right)_{Sym(jkl)} \omega^\nu_{jkl} \wedge \omega^\sigma \wedge \omega_i - 2 c^{klij}_{\sigma\nu} \omega^\nu_{kl} \wedge \omega^\sigma_j \wedge \omega_i, \qquad (3.10)$$

Expressing the generators of the ideal $\mathcal{D}^4_{\hat{\alpha}}$ we get

$$\begin{aligned}
i_{\frac{\partial}{\partial y^\nu}}\hat{\alpha} &= E_\nu\omega_0 + 2\left(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu} - \frac{\partial}{\partial y^\nu} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2d_k a^{ij}_{\sigma\nu}\right)\omega^\sigma \wedge \omega_i \\
&- \left(\frac{\partial^2 L}{\partial y^\nu_i \partial y^\sigma_k} - \frac{\partial^2 L}{\partial y^\nu \partial y^\sigma_{ik}} - \frac{\partial}{\partial y^\sigma_k} d_j \frac{\partial L}{\partial y^\nu_{ij}} + 4a^{ik}_{\sigma\nu} - 2d_j b^{kij}_{\nu\sigma}\right)\omega^\sigma_k \wedge \omega_i \\
&- \left(\frac{\partial^2 L}{\partial y^\nu_i \partial y^\sigma_{kl}} - \frac{\partial}{\partial y^\sigma_{kl}} d_j \frac{\partial L}{\partial y^\nu_{ij}} - 2(b^{kil}_{\nu\sigma})_{Sym(kl)} - 2d_j c^{klij}_{\nu\sigma}\right)\omega^\sigma_{kl} \wedge \omega_i \\
&+ \left(2c^{klij}_{\sigma\nu}\right)_{Sym(jkl)}\omega^\sigma_{jkl} \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_k}}\hat{\alpha} &= \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_k} - \frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_{ik}} - \frac{\partial}{\partial y^\nu_k} d_j \frac{\partial L}{\partial y^\sigma_{ij}} + 4a^{ik}_{\nu\sigma} - 2d_j b^{kij}_{\sigma\nu}\right)\omega^\sigma \wedge \omega_i \\
&+ 2\left(\frac{\partial^2 L}{\partial y^\sigma_{ij} \partial y^\nu_k} - 4(b^{kij}_{\sigma\nu})_{Alt((\sigma j)(\nu k))}\right)\omega^\sigma_j \wedge \omega_i - 2c^{klij}_{\sigma\nu}\omega^\sigma_{jl} \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_{kl}}}\hat{\alpha} &= \left(\frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_{kl}} - \frac{\partial}{\partial y^\nu_{kl}} d_j \frac{\partial L}{\partial y^\sigma_{ij}} - 2(b^{kil}_{\sigma\nu})_{Sym(kl)} - 2d_j c^{klij}_{\sigma\nu}\right)\omega^\sigma \wedge \omega_i \\
&- 2c^{klij}_{\sigma\nu}\omega^\sigma_j \wedge \omega_i, \\
i_{\frac{\partial}{\partial y^\nu_{jkl}}}\hat{\alpha} &= -\left(2c^{klij}_{\sigma\nu}\right)_{Sym(jkl)}\omega^\nu \wedge \omega_i
\end{aligned} \tag{3.11}$$

Since the ranks of the matrices $\left(c^{klij}_{\nu\sigma}\right)_{Sym(jkl)}$, $\left(c^{klij}_{\nu\sigma}\right)$ are maximal then the $\omega^\sigma \wedge \omega_i$ and $\omega^\sigma_{(j} \wedge \omega_{i)}$ are generators of ideal $\mathcal{D}^4_{\hat{\alpha}}$. We obtain for Dedecker–Hamilton extremals δ_D $\pi_{3,2} \circ \delta_D = J^2\gamma$, where γ is a section of π. Substituting this into (2.5) we get

$$\delta_D{}^* i_{\frac{\partial}{\partial y^\sigma}}\hat{\alpha} = E_\sigma \circ J^3\gamma$$

for 3rd order Euler–Lagrange form (3.1) and

$$\delta_D{}^* i_{\frac{\partial}{\partial y^\sigma}}\hat{\alpha} = E_\sigma \circ J^2\gamma$$

for 2nd order Euler–Lagrange form (3.2) and γ is an extremal of λ.

Let us prove strong regularity: We have to show that under our assumptions, for every section δ satisfying Hamilton equations, one has $\pi_{3,2} \circ \delta = J^2\gamma$, where γ is a solution of the Euler–Lagrange equations of the Lagrangian λ. Assuming $d\bar{\mu} = 0$, we obtain: $\delta^*(i_{\partial/\partial y^\sigma_{jkl}}\alpha) = \delta^*((2c^{klij}_{\nu\sigma})_{Sym(jkl)}\omega^\nu \wedge \omega_i) = 0$, i.e. $\delta^*\omega^\nu = 0$ by the rank condition on

$$\left(c^{klij}_{\nu\sigma}\right)_{Sym(jkl)},$$

i.e. $\partial y^\sigma / \partial x^i = y^\sigma_i$. Hence,

$$\delta^*(i_{\partial/\partial y^\nu_{kl}}\alpha) = \delta^*(\left(-2c^{klij}_{\nu\sigma}\right)\omega^\sigma_j \wedge \omega_i) = 0.$$

Note that matrix $(c^{klij}_{\nu\sigma})$ is symmetric in indices kl and its maximal rank is $mn(n+1)/2$. Due to the rank condition on $(c^{klij}_{\nu\sigma})$, $\delta^*\omega^\sigma_j = 0$, i.e.

$$\left(\partial y^\sigma_j / \partial x^i\right)_{Sym(ij)} = y^\sigma_{ij}.$$

The above obtained conditions on δ mean that every solution of Hamilton equations is holonomic up to the second order, i.e., we can write $\pi_{3,2}\circ\delta = J^2\gamma$, where γ is a section of π. Now, the equations $J^3(\pi_{3,0}\circ\delta)^*(i_{\partial/\partial y^\sigma_k}\alpha) = 0$ are satisfied identically, and the last set of Hamilton equations, i.e., $J^3(\pi_{3,0}\circ\delta)^*(i_{\partial/\partial y^\sigma}\alpha) = 0$ take the form $E_\sigma \circ J^3\gamma = 0$ (3.1), resp. $E_\sigma \circ J^2\gamma = 0$ (3.2) proving that γ is an extremal of λ. This completes the proof. □

4 Example

The above results (the regularity conditions) can be directly applied to concrete Lagrangians. Let us consider the following example as an illustration. We find to a given Lagrangian 3 different Hamiltonian systems satisfying:

Let $X = R^2$, $Y = R^2 \times R^2$ (i.e., $n = 2$, $m = 2$). Denote (V, ψ), $\psi = (x^i, y^\sigma)$ a fibered chart on $R^2 \times R^2$. Let us consider the following Lagrangian

$$\lambda = L\omega_0, \quad L = y^1_{11}y^2_{22} - y^1_{22}y^2_{11} \tag{4.1}$$

which satisfies (3.1).

In view of the above considerations we take a Lepagean equivalent ρ (of the Euler–Lagrange form E of Lagrangian (4.1)) in the form $\alpha = d\rho$, where ρ is (3.3), (3.4).

We consider functions $a^{ij}_{\sigma\nu}$, $b^{kij}_{\sigma\nu}$, $c^{ijkl}_{\sigma\nu}$ (3.4) on an open set $U \subset J^3R^2$ where the conditions $y^1_1 \neq 0$, $y^1_2 \neq 0$, $y^1_{12} \neq 0$ and $y^2_{12} \neq 0$.

The functions $a^{ij}_{\sigma\nu}$ and $b^{ijp}_{\kappa\sigma}$ are arbitrary. We assume that $c^{ijkl}_{\sigma\nu}$ are constant functions. We have again only 8 non-zero constants, we choose $c^{1212}_{11} = c^{2112}_{11} = -c^{2121}_{11} = -c^{1221}_{11} = 1$ and $c^{1212}_{22} = c^{2112}_{22} = -c^{2121}_{22} = -c^{1221}_{22} = 1$.

Then the Lepagean equivalent takes the form

$$\begin{aligned}\rho = {} & \theta_\lambda + a^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij} + b^{kij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu_k \wedge \omega_{ij} \\ & + 4\,\omega^1 \wedge \omega^1_{12} \wedge \omega_{12} + 4\,\omega^2 \wedge \omega^2_{12} \wedge \omega_{12} + \bar{\mu},\end{aligned}$$

where $\bar{\mu}$ is an arbitrary n-form.

The matrices (3.5) and (3.6) take the following form

$$(P^{ijkl}_{\sigma\nu})^T = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & -4 & -4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$Q^{ijkl}_{\sigma\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can easily see that $\operatorname{rank}(P^{ijkl}_{\sigma\nu}) = 4$ and $\operatorname{rank}(Q^{ijkl}_{\sigma\nu}) = 6$. The form $\alpha = d\rho + d\bar{\mu}$ is regular.

If moreover $\bar{\mu}$ is closed then $\alpha = d\rho$ is strongly regular.

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