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REGULARLY WEAKLY BASED MODULES OVER RIGHT PERFECT RINGS AND DEDEKIND DOMAINS

Michal Hrbek, Pavel Růžička, Praha

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Abstract. A weak basis of a module is a generating set of the module minimal with respect to inclusion. A module is said to be regularly weakly based provided that each of its generating sets contains a weak basis. We study
(1) rings over which all modules are regularly weakly based, refining results of Nashier and Nichols, and
(2) regularly weakly based modules over Dedekind domains.

Keywords: weak basis; regularly weakly based ring; Dedekind domain; perfect ring

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1. Introduction

By a module we always mean a right unitary module over a ring R with identity element. Let M be a module and let X, Y be subsets of M. We say that the set X is *weakly independent over* Y if $x \notin \operatorname{Span}((X \setminus \{x\}) \cup Y)$ for all $x \in X$. We say shortly that X is *weakly independent* in the case of $Y = \emptyset$. A generating weakly independent subset of a module M is called a *weak basis* of M. A module M is *weakly based* if it contains a weak basis. Finally, a module M is called *regularly weakly based* if any generating set of M contains a weak basis.

Nashier and Nichols characterized right perfect rings as rings over which every quasi-cyclic right R-module (i.e. every finitely generated submodule is contained in a cyclic submodule) is cyclic (i.e. every submodule is contained in a cyclic submodule). As a consequence of this they have proved that rings over which all right

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modules are regularly weakly based are necessarily right perfect ([9], Theorem 2, together with [10], page 110). They raised a question whether, conversely, all modules over right perfect rings are regularly weakly based. We refine their result, proving that infinitely generated free modules over non-right perfect rings are not regularly weakly based and we observe that their question regarding right perfect rings easily reduces to semisimple rings.

The other topic of the paper is a study of regularly weakly based modules over Dedekind domains. This is motivated by the characterization of weakly based modules over abelian groups ([6]) and later Dedekind domains ([5]) by the authors. For regularly weakly based modules we will not obtain the full characterization, however, we reduce the problem to a question of characterizing regularly weakly based modules over commutative semisimple rings, which indeed is a special case of the more general open question regarding right perfect rings introduced above. We remark that our results from Section 4 generalize those obtained in [3], from torsion abelian groups to general modules over Dedekind domains. However, even in the case of abelian groups (see [3], Remark 3), the last remaining case of torsion groups which are bounded, but not primary, remains open.

There are a few simple facts regarding regularly weakly based modules which we will freely use within the paper. Namely, it is clear that a finitely generated module is regularly weakly based. Also observe that unlike in the case of weakly based modules, a direct summand of a regularly weakly based module is regularly weakly based. Also the next elementary lemma, in different variations, will be repeatedly used with no reference. Its proof is left to the reader.

Lemma 1.1. *Let R be a ring and M a right R-module. Let X, Y, Z be subsets of M. Suppose that X is weakly independent over $Y \cup Z$ and Y is weakly independent over $X \cup Z$. Then $X \cup Y$ is weakly independent over Z.*

2. Modules over right perfect rings

We start with the natural task of characterizing rings R such that all right R-modules are regularly weakly based. We refine the result of [9], Theorem 2, that all such rings must be right perfect. In particular, in Lemma 2.2, we prove that an infinitely generated free module over a non-perfect ring is not regularly weakly based. Nashier and Nichols suggested, conversely, that all modules over right perfect rings are regularly weakly based. We discuss this question in the final part of this section, adding an observation that we can factor out the Jacobson radical, and so reduce the question to semisimple rings.

Lemma 2.1 ([9], Proposition 1 and Theorem 2). *A ring R is right perfect if and only if for each sequence $(r_n\colon\ n \in \omega)$ of elements of R there is $n_0 \in \omega$ such that for all $n \geqslant n_0$ there is $j \geqslant 1$ such that $r_{n+j} \dots r_{n+1}R = r_{n+j} \dots r_{n+1}r_nR$.*

Lemma 2.2. *Let R be a ring that is not right perfect. Then a free right R-module is regularly weakly based if and only if it is finitely generated.*

Proof. A finitely generated module is regularly weakly based. Thus, it suffices to show that an infinitely generated free right module is not regularly weakly based. Since a direct summand of a regularly weakly based module is regularly weakly based, we can restrict ourselves to a countably generated free right R module $F = R^{(\aleph_0)}$. Now fix a free basis $B = \{b_n\colon\ n \in \omega\}$ of F. Since R is not right perfect, there is by Lemma 2.1 a sequence $(r_n\colon\ n \in \omega)$ of elements of R such that for any $n \in \omega$ and all $j \geqslant 1$ we have that $r_{n+j} \dots r_{n+1}R \supsetneq r_n \dots r_{n+1}r_nR$. In particular, this implies that all r_n are not right invertible in R. For each $n \in \omega$, we define the following elements from F:

$$x_n = b_{n+1}r_n \quad \text{and} \quad y_n = b_n - x_n = b_n - b_{n+1}r_n. \tag{2.1}$$

Put $Z = \{x_n, y_n\colon\ n \in \omega\}$ and $Y = \{y_n\colon\ n \in \omega\}$. Clearly $B \subseteq \operatorname{Span}(Z)$, hence Z generates F. We claim that Z does not contain any weak basis of F. Suppose otherwise and pick a weak basis $W \subseteq Z$ of F. As each r_n is not right invertible, $Y \subseteq W$.

Let $n \in \omega$ and suppose $x_n \in W$. Observe that then b_k and thus also $x_k = b_k - y_k$ belong to $\operatorname{Span}(W)$ for all $k \leqslant n$. Since W is weakly independent and $Y \subseteq W$, it contains at most one x_n, that is, $W \subseteq Y \cup \{x_n\}$ for some $n \in \omega$. We claim that $b_{n+1} \notin \operatorname{Span}(W)$. Indeed, otherwise

$$b_{n+1} = x_n s + \sum_{i \in \omega} y_i s_i \tag{2.2}$$

for some $s, s_0, s_1, \dots$ from R such that all but finitely many s_i are 0. Using the substitution (2.1) we get that

$$b_{n+1} = b_{n+1}r_n s + \sum_{i \in \omega} (b_i s_i - b_{i+1} r_i s_i). \tag{2.3}$$

From this we get that $s_0 = \dots = s_n = 0$, $s_{n+1} = (1 - r_n s)$ and $s_{n+1+j} = r_{n+j} \dots r_{n+1}(1 - r_n s)$ for all $j > 0$. Since all but finitely many s_i equal 0, there is $j > 0$ such that $s_{n+1+j} = 0$. Then we get that $r_{n+j} \dots r_{n+1} = r_{n+j} \dots r_{n+1} r_n s$, which gives $r_{n+j} \dots r_{n+1}R = r_{n+j} \dots r_{n+1}r_nR$. This contradicts our choice of the sequence $(r_n\colon\ n \in \omega)$. $\square$

Recall that a subset I of a ring R is right T-nilpotent provided that for every sequence $a_1, a_2, \ldots$ there is a positive integer n such that $a_n \ldots a_1 = 0$. A right ideal J of a ring R is T-nilpotent if and only if $MJ \neq M$ for every nonzero right R-module by [1], Lemma 28.3. By the Theorem of Bass [1], Theorem 28.4, a ring R is right perfect if and only if its Jacobson radical J is T-nilpotent and the ring R/J is semisimple.

Lemma 2.3. *Let J be a right T-nilpotent right ideal of a ring R, let M be a right R-module. Then every $X \subseteq M$ lifting a weak basis of M/MJ over MJ is a weak basis of M.*

P r o o f. Since X lifts a weak basis of M/MJ over MJ we have that X is weakly independent and $M = \operatorname{Span}(X) + MJ$. From the second equality we infer that $(M/\operatorname{Span}(X))J = M/\operatorname{Span}(X)$. Since the ideal J is right T-nilpotent, we conclude that $M/\operatorname{Span}(X) = 0$, that is, $M = \operatorname{Span}(X)$. □

Proposition 2.4. *Let R be a ring.*

(1) ([9], page 311) *If all right R-modules are regularly weakly based, then R is right perfect.*
(2) *Let J denote a Jacobson radical of R. If R is right perfect, then all right R-modules are regularly weakly based if and only if all right modules over the semisimple ring R/J are regularly weakly based.*

P r o o f. (1) follows readily from Lemma 2.2, while (2) follows from Lemma 2.3. □

Proposition 2.4 reduces the characterization of rings over which all modules are regularly weakly based to a question whether all modules over a semisimple ring are regularly weakly based. The answer to this seems surprisingly nontrivial (see [4]).

We conclude the section with a straight consequence of Proposition 2.4.

Corollary 2.5. *Every module over a local perfect ring is regularly weakly based.*

3. Factoring out a finitely generating submodule

It can be easily seen that a module M is weakly based if and only if the factor M/K is weakly based for a finitely generated submodule K of M. The situation becomes less apparent when weakly based is replaced with a regularly weakly based. We will apply this fact in the subsequent section. Before we proceed to its proof, we introduce the following notions (taken from [5]).

Let M, N be modules, let $\varphi\colon M \to N$ be a homomorphism, and let X be a subset of M. We say that X *lifts a subset* Y *of* N *via* φ provided that $\varphi_{\restriction X}$ is a bijection onto Y. If N is a quotient module of M, we say shortly that X lifts Y, meaning that X lifts Y via the canonical projection.

Let M be a module and let X, Y be subsets of M. Let

$$X^Y = \{x + \operatorname{Span}(Y)\colon\ x \in X\}$$

denote the image of the set X in the canonical projection $M \to M/\operatorname{Span}(Y)$.

Lemma 3.1. *Let R be a ring, let M be a right R-module and let K be a finitely generated submodule of M. Then M is regularly weakly based if and only if the factor module M/K is regularly weakly based.*

Proof. First suppose that the module M is regularly weakly based. Let $\overline{X}$ be a generating set of M/K, and let X be a subset of M which lifts $\overline{X}$, i.e. $X^K = \overline{X}$. Then $X \cup K$ generates M, and since M is regularly weakly based, $X \cup K$ contains a weak basis of M, say Y. Since K is finitely generated, there is a finite subset F of Y such that $K \subseteq \operatorname{Span}(F)$. Put $Y_0 = Y \setminus F$. As Y is a weak basis of M, Y_0^K is weakly independent in M/K.

Since Y generates M, the factor-module $M/(K+\operatorname{Span}(Y_0))$ is generated by $F^{K\cup Y_0}$. As finitely generated modules are regularly weakly based, there is $F_0 \subseteq F$ that lifts a weak basis of $M/(K + \operatorname{Span}(Y_0))$. Since Y_0 is weakly independent over F, $K \subseteq \operatorname{Span}(F)$, and F_0 lifts a weak basis of $M/(K + \operatorname{Span}(Y_0))$, we conclude that $Y_0^K \cup F_0^K$ is a weak basis of M/K. Using $Y_0 \cup F_0 \subseteq X \cup K$ we infer that $Y_0 \cup F_0 \subseteq X$, whence $Y_0^K \cup F_0^K \subseteq \overline{X}$. We have proved that the module M/K is regularly weakly based.

Now suppose that the factor-module M/K is regularly weakly based. Let X be a generating subset of M. Since K is finitely generated, there is a finite subset F of X such that $K \subseteq \operatorname{Span}(F)$. The already proved implication gives that $M/\operatorname{Span}(F)$ is regularly weakly based. Thus, we can pick a subset X_0 of X lifting a weak basis of $M/\operatorname{Span}(F)$. Observe that $F^{\operatorname{Span}(X_0)}$ generates the factor-module $M/\operatorname{Span}(X_0)$ and since a finitely generated module is regularly weakly based, there is $F_0 \subseteq F$ lifting a weak basis of $M/\operatorname{Span}(X_0)$. We conclude that $X_0 \cup F_0$ is a weak basis of M contained in X. $\square$

4. Regularly weakly based modules over Dedekind domains

From now on we restrict ourselves to the case of Dedekind domains. Let R be a Dedekind domain. We denote by m-Spec(R) the set of all nonzero prime (and thus maximal) ideals of R. An R-module T is *torsion* if $\operatorname{Ann}(m) \neq 0$ for any $m \in T$. Recall that any torsion R-module T has a primary decomposition, that is, $T = \bigoplus_{\mathfrak{p} \in \text{m-Spec}(R)} T_{\mathfrak{p}}$, where $T_{\mathfrak{p}} = \{m \in T\colon \operatorname{Ann}(m) = \mathfrak{p}^k \text{ for some } k\}$. We say that T is $\mathfrak{p}$-primary if $T = T_{\mathfrak{p}}$. Alternatively, the $\mathfrak{p}$-primary part $T_{\mathfrak{p}}$ corresponds naturally to the localization $T \otimes_R R_{\mathfrak{p}}$. In particular, we can view a $\mathfrak{p}$-primary R-module naturally as a module over the localization $R_{\mathfrak{p}}$.

Let us recall a notion from abelian group theory which will prove useful in what follows. We say that a submodule B of a $\mathfrak{p}$-primary module T is *basic* if B is a pure submodule of T, B is isomorphic to a direct sum of cyclic modules, and T/B is divisible. As all these notions hold the same meaning independent of whether we view T as an R-module or as an $R_{\mathfrak{p}}$-module, we can use [8], Theorem 9.4, to infer that any $\mathfrak{p}$-primary module has a basic submodule (determined uniquely up to isomorphism).

Module M is said to be *bounded* if $IM = 0$ for some nonzero ideal I. The following two lemmas generalize [3], Corollary, from abelian groups to modules over Dedekind domains.

Lemma 4.1. *Let R be a Dedekind domain and let T be a torsion R-module. If T is regularly weakly based, then T is bounded.*

Proof. Let T be an unbounded torsion R-module. First suppose that T is $\mathfrak{p}$-primary for some $\mathfrak{p} \in$ m-Spec(R). We claim that there is a projection from T onto a nonzero divisible module. In order to prove this, choose a basic submodule B of T (existence of which is discussed above). If $B \subsetneq T$, then T/B is nonzero divisible and $T \to T/B$ is the desired projection. If $B = T$, then T is a direct sum of $\mathfrak{p}$-primary cyclic modules of unbounded annihilators, and hence T contains a submodule S isomorphic to $\bigoplus_{n\in\mathbb{N}} R/\mathfrak{p}^n$. It is well known that the indecomposable $\mathfrak{p}$-primary divisible R-module can be constructed as a direct limit of the system of inclusions $R/\mathfrak{p} \to R/\mathfrak{p}^2 \to R/\mathfrak{p}^3 \to \ldots$, and thus it is a quotient of S. As divisible R-modules are injective, this projection can be extended to the entire T.

We showed that there is a projection $\pi\colon T \to D$, where D is nonzero divisible module. Denote by K the kernel of π and choose a generating set X' of D. Since D is divisible, there is a subset X of $\mathfrak{p}T$ lifting X' via π. Put $Z = X \cup K$ and note that Z generates T. Suppose that $W \subseteq Z$ is a weak basis of T. By [5], Corollary 3.3

and Lemma 5.2, any weak basis of T lifts a basis of $T/\mathfrak{p}T$ over $\mathfrak{p}T$. Hence $W \subseteq K$, which is a contradiction with W being a generating set.

Let now T be an unbounded (not necessarily $\mathfrak{p}$-primary) torsion R-module. Since regularly weakly based modules are closed under direct summands, the first part of this proof implies that $T_{\mathfrak{p}}$ is bounded for each $\mathfrak{p} \in \text{m-Spec}(R)$. As T is unbounded, there must be an infinite subset $\mathcal{P}$ of m-Spec(R) such that $T_{\mathfrak{p}} \neq 0$ for each $\mathfrak{p} \in \mathcal{P}$. If there is a nonzero divisible submodule of T, it is a non-weakly based direct summand of T (see [5], Corollary 3.6). Thus, T is not regularly weakly based. We can thus assume that T is reduced and apply [7], Theorem 9, to infer that there is a nonzero cyclic direct summand $C_{\mathfrak{p}}$ of $T_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathcal{P}$. Since $\mathcal{P}$ is infinite, we can pick a countable infinite sequence $\mathfrak{p}_n$, $n \in \omega$, of pairwise distinct primes from $\mathcal{P}$. It will suffice to show that $\bigoplus\limits_{n\in\omega} C_{\mathfrak{p}_n}$ is not regularly weakly based. Fix a generator x_n of $C_{\mathfrak{p}_n}$ and put $y_n = x_0 + x_1 + \ldots + x_n$ for each $n \in \omega$. It follows easily that $\text{Span}(y_m) \subsetneq \text{Span}(y_n)$ whenever $m < n$, and so the generating set $\{y_n \colon n \in \omega\}$ of $\bigoplus\limits_{n\in\omega} C_{\mathfrak{p}_n}$ does not contain a weak basis. □

Lemma 4.2. *Let R be a Dedekind domain and let $\mathfrak{p} \in$ m-Spec(R). Every bounded $\mathfrak{p}$-primary R-module is regularly weakly based.*

P r o o f. Let B be a bounded $\mathfrak{p}$-primary R-module. Then $B\mathfrak{p}^n = 0$ for some positive integer n and B can be naturally viewed as an $R/\mathfrak{p}^n$ module. Since R is a Dedekind domain, the factor ring $R/\mathfrak{p}^n$ is local perfect, hence B is regularly weakly based by Corollary 2.5. □

Before proving the main lemma of the paper, we need the following auxiliary lemma:

Lemma 4.3. *Let R be a Dedekind domain and let N be a torsion-free R-module. If N is an extension of a free module by a torsion bounded module, then N is projective.*

P r o o f. Let F be a free submodule of N such that the factor-module $B = N/F$ is bounded torsion. Enumerate a free basis $X = \{x_\alpha \colon \alpha < \lambda\}$ of F by an ordinal λ and put $F_\beta = \text{Span}(\{x_\alpha \colon \alpha < \beta\})$ for all $\beta < \lambda$. For each $\alpha < \lambda$ let N_α denote the smallest pure submodule of N containing F_α. It follows that $N = \bigcup\limits_{\alpha<\lambda} N_\alpha$ is a filtration of N with $N_{\alpha+1}/N_\alpha$ torsion-free for each $\alpha < \lambda$. Finitely generated torsion free modules over Dedekind domains are projective, see [2], Theorem 6.3.23, and therefore it will suffice to prove that all $N_{\alpha+1}/N_\alpha$ are finitely generated (and thus, projective). Indeed, then $N \simeq \bigoplus\limits_{\alpha<\lambda} N_{\alpha+1}/N_\alpha$ and so N is projective.

Put $B_\alpha = N_\alpha/F_\alpha$ for each $\alpha < \lambda$. As $F_\alpha = N_\alpha \cap F$ by the linear independence of set X, we have the isomorphism $B_\alpha = N_\alpha/(N_\alpha \cap F) \simeq (N_\alpha + F)/F$ and so we can view naturally B_α as a submodule of B. Denote by Q the field of quotients of R. For each $\alpha < \lambda$ we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N_\alpha + \mathrm{Span}(x_\alpha))/N_\alpha & \longrightarrow & N_{\alpha+1}/N_\alpha & \longrightarrow & B_{\alpha+1}/B_\alpha & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \subseteq & & \downarrow \subseteq & & \\ 0 & \longrightarrow & R & \longrightarrow & Q & \longrightarrow & Q/R & \longrightarrow & 0. \end{array}$$

Both exact sequences in the rows are given by the obvious quotient maps. For the maps in columns, the left most isomorphism follows from the fact, that $\mathrm{Span}(x_\alpha) \cap N_\alpha = 0$, as N_α is the purification of X_α, and $x_\alpha \notin X_\alpha$. The middle inclusion is given by $N_{\alpha+1}/N_\alpha$ being torsion-free module of rank 1, and the right most column map is induced by the two other ones. It is well known that $(Q/R)_\mathfrak{p}$ is uniserial for each $\mathfrak{p} \in \text{m-Spec}(R)$. As $B_{\alpha+1}/B_\alpha$ is bounded, it has only finitely many nonzero $\mathfrak{p}$-primary parts, and as each of them is a bounded submodule of a uniserial module, they are all finitely generated. Therefore $B_{\alpha+1}/B_\alpha$ is finitely generated. We conclude that $N_{\alpha+1}/N_\alpha$ is an extension of a cyclic module by a finitely generated module, hence it is finitely generated. This finishes the proof. □

Lemma 4.4. *A regularly weakly based module over a Dedekind domain splits into a direct sum of a projective module and a bounded torsion module.*

Proof. Let M be a regularly weakly based module over a Dedekind domain R. Let T denote the torsion submodule of M and let $F = M/T$ be the torsion-free quotient of M. If F is projective, then M decomposes as $T \oplus F$, and both the direct summands are regularly weakly based, in particular, T is bounded torsion by Lemma 4.1.

Suppose now that F is not projective. Then we start with the following claim:

Claim. *There is an ideal $\mathfrak{p} \in \text{m-Spec}(R)$ and a subset X of M which lifts a basis of $M/M\mathfrak{p}$ over $M\mathfrak{p}$, such that $M/\mathrm{Span}(X)$ is not regularly weakly based.*

Proof of Claim. We choose an arbitrary $\mathfrak{p} \in \text{m-Spec}(R)$ and a subset X' of T lifting a basis of $T/T\mathfrak{p}$. As T is a pure submodule of M, we can extend X' to a subset X of M containing X' such that X lifts a basis of $M/M\mathfrak{p}$. Put $Y = X \setminus X'$ and note that Y^T lifts a basis of $F/F\mathfrak{p}$ over $F\mathfrak{p}$. By [5], Lemma 7.1, Y^T is a linearly independent subset of $F = M/T$, hence, $\mathrm{Span}(Y^T)$ is free. Set $D = M/\mathrm{Span}(X)$. We claim that D is not regularly weakly based.

If D is torsion, then $M/(T + \mathrm{Span}(X)) \simeq F/\mathrm{Span}(Y^T)$ is torsion too. As F is an extension of $\mathrm{Span}(Y^T)$ by $M/(T + \mathrm{Span}(X))$, the latter module is not bounded by Lemma 4.3, otherwise F would be projective. Hence, D is also an unbounded torsion module, and by Lemma 4.1 D is not regularly weakly based as desired.

Finally, suppose that D is not torsion. In this case, choose any element $d \in D$ with $\mathrm{Ann}(d) = 0$, and put $D' = D/d\mathfrak{p}$. Because $dR \simeq R$, we have that $d\mathfrak{p} \subsetneq dR \subseteq D$, and thus there is a submodule of D' isomorphic to $R/\mathfrak{p}$, showing that the $\mathfrak{p}$-primary component of D' is nonzero. Since $D = D\mathfrak{p}$, also $D' = D'\mathfrak{p}$. As the $\mathfrak{p}$-primary component of D' is a pure submodule of D', it is divisible by $\mathfrak{p}$, and therefore divisible. Hence, D' contains a nonzero divisible direct summand, and thus D' is not regularly weakly based by [5], Corollary 3.6. As $d\mathfrak{p}$ is a finitely generated submodule of D, Lemma 3.1 shows that D is not regularly weakly based as desired. This concludes the proof of the Claim. □

We pick a generating set Y' of $M/\mathrm{Span}(X)$ which does not contain any weak basis. As $M/\mathrm{Span}(X)$ is divisible by $\mathfrak{p}$, we can find a subset Y of $\mathfrak{p}M$ lifting Y' over $\mathrm{Span}(X)$. Then $X \cup Y$ is a generating set, which does not contain any weak basis of M. Indeed, any subset of $X \cup Y$ generating M must contain the entire X, and Y' does not contain any weak basis of $M/\mathrm{Span}(X)$. □

Theorem 4.5. *Let R be a Dedekind domain that is not a division ring. Then a regularly weakly based R-module splits into a direct sum of a finitely generated projective module and a bounded torsion module.*

Proof. Let M be regularly weakly based module over a Dedekind domain R. Then $M = P \oplus B$, where P is projective and B a bounded torsion R-module, by Lemma 4.4. Since R is not a division ring, it is not perfect, indeed the only perfect domains are division rings. Applying Lemma 2.2 and the fact that regularly weakly based modules are closed under direct summands, we conclude that P is finitely generated (recall that infinitely generated projective modules over a Dedekind domain are free by [11], Theorem 7.7). □

Lemma 4.6. *Let R be a Dedekind domain, F a finitely generated module and B a bounded $\mathfrak{p}$-primary module. Then $F \oplus B$ is regularly weakly based.*

Proof. Apply Lemma 3.1 and Lemma 4.2. □

Corollary 4.7. *Let R be a discrete valuation ring and M an R-module. Then M is regularly weakly based if and only if $M \simeq F \oplus B$, where F is finitely generated free module and B is bounded torsion module.*

Corollary 4.8. *Let A be an abelian group. If A is regularly weakly based, then $A \simeq F \oplus B$, where F is finitely generated free and $nB = 0$ for some positive integer n.*

5. Closing remarks

The remaining question is whether any bounded torsion module over a Dedekind domain is regularly weakly based. In other words, we ask whether all R/I-modules are regularly weakly based for any nonzero ideal I of a Dedekind domain R. Since nonzero ideals over Dedekind domains are products of prime ideals, $I = P_1^{n_1} \dots P_k^{n_k}$, where $P_1, \dots, P_k$ are distinct prime ideals and $n_1, \dots, n_k$ are positive integers. The Jacobson radical of the ring R/I corresponds to the ideal $(P_1 \dots P_k)/I$ and it is clearly nilpotent. Applying Lemma 2.3 we can reduce the question to the case when I is a product of distinct primes. In this case $R/I = R/(P_1 \dots P_k) \simeq (R/P_1) \times \dots \times (R/P_k)$ is a product of fields, i.e, it is a commutative semisimple ring. Thus, we arrived to a particular case of the question discussed at the end of Section 2. Let us formulate it as an open problem:

Problem 5.1. Is every module over a semisimple ring regularly weakly based? In particular, is every module over a product of division rings (fields) regularly weakly based?

The class of regularly weakly based modules is not closed under submodules in general. A counterexample can be obtained as follows. Let R be a commutative Von Neumann regular ring with infinitely generated socle S (e.g. an infinite product of fields). The regular module R, being finitely generated, is regularly weakly based. We show that the R-module S is not. There is a submodule (and thus, a direct summand) S' of S of length $\aleph_0$, say $S' \simeq \bigoplus_{n\in\omega} S_n$, with S_n simple for each $n \in \omega$. As R is regular, S_n has a direct complement M_n in R for each n. Choose a generator x_n of S_n for each $n \in \omega$ and put $y_n = x_0+x_1+\dots+x_n$. We claim that $Y = \{y_n : n \in \omega\}$ is a generating set of S' which does not contain any weak basis. As $M_0 \cap M_1 \cap \dots \cap M_{n-1} \nsubseteq M_n$, we conclude that $\operatorname{Span}(y_n) \subseteq \operatorname{Span}(y_m)$ for each $n \leqslant m$, and that $\operatorname{Span}(x_n) \subseteq \operatorname{Span}(y_n)$ for each $n \in \omega$. Hence, Y generates S', and as S' is not finitely generated, Y contains no weak bases of S'.

Problem 5.2. Is the class of regularly weakly based modules always closed under quotients?

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Authors' address: Michal Hrbek, Pavel Růžička, Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha, Czech Republic, e-mail: hrbmich@gmail.com, ruzicka@karlin.mff.cuni.cz.