Dumitru Popa

Copies of $l^n_p$'s uniformly in the spaces $\Pi_2(C[0, 1], X)$ and $\Pi_1(C[0, 1], X)$


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COPIES OF $l_p^n$'S UNIFORMLY IN THE SPACES
$\Pi_2(C[0,1], X)$ AND $\Pi_1(C[0,1], X)$

DUMITRU POPA, Constanța

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Abstract. We study the presence of copies of $l_p^n$'s uniformly in the spaces $\Pi_2(C[0,1], X)$ and $\Pi_1(C[0,1], X)$. By using Dvoretzky's theorem we deduce that if $X$ is an infinite-dimensional Banach space, then $\Pi_2(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of $l_\infty^n$'s and $\Pi_1(C[0,1], X)$ contains $\lambda$-uniformly copies of $l_2^n$'s for all $\lambda > 1$. As an application, we show that if $X$ is an infinite-dimensional Banach space then the spaces $\Pi_2(C[0,1], X)$ and $\Pi_1(C[0,1], X)$ are distinct, extending the well-known result that the spaces $\Pi_2(C[0,1], X)$ and $N(C[0,1], X)$ are distinct.

Keywords: $p$-summing linear operators; copies of $l_p^n$'s uniformly; local structure of a Banach space; multiplication operator; average

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1. Introduction and notation

The main purpose of this paper is to study the presence of copies of $l_p^n$'s uniformly in the spaces $\Pi_2(C[0,1], X)$ and $\Pi_1(C[0,1], X)$. Let us fix some notation and concepts used below. The scalar field $\mathbb{R}$ (or $\mathbb{C}$) is denoted by $\mathbb{K}$ and if $n \in \mathbb{N}$, $1 \leq p \leq \infty$, then $l_p^n = (\mathbb{K}^n, \| \cdot \|_p)$, where $\| (\alpha_1, \ldots, \alpha_n) \|_p = \left( \sum_{i=1}^{n} |\alpha_i|^p \right)^{1/p}$ if $p < \infty$ and $\| (\alpha_1, \ldots, \alpha_n) \|_\infty = \max_{1 \leq i \leq n} |\alpha_i|$. By $(e_i)_{1 \leq i \leq n}$ we denote the standard unit vectors in $\mathbb{K}^n$, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. For $1 \leq p \leq \infty$ we write, as usual, $p^*$ for the conjugate of $p$, i.e. $1/p + 1/p^* = 1$. If $\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n$, $1 \leq p, q \leq \infty$, $M_\alpha : l_p^n \to l_q^n$ is the multiplication operator, i.e. $M_\alpha((\xi_i)_{1 \leq i \leq n}) := (\alpha_i \xi_i)_{1 \leq i \leq n}$. By $r_n : [0,1] \to \mathbb{R}$, $r_n(t) = (-1)^{[2^nt]}$ we denote the Rademacher functions ($[\cdot]$ denotes the integer part) and $C[0,1]$ is the space of all scalar-valued continuous functions on $[0,1]$ under the uniform norm.

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Let $1 \leq p \leq \infty$ and $1 < \lambda < \infty$. We say that a Banach space $X$ contains $l_p^n$'s $\lambda$-uniformly or that $X$ contains $\lambda$-uniformly copies of $l_p^n$ if for every $n \in \mathbb{N}$ there exists a linear operator $J_n : l_p^n \to X$ such that

$$
\|\alpha\|_p \leq \|J_n(\alpha)\|_X \leq \lambda \|\alpha\|_p, \quad \alpha \in l_p^n
$$

(see [3], page 260). Let $X$, $Y$ be Banach spaces and $1 \leq p < \infty$. A linear operator $T : X \to Y$ is $p$-summing if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ the relation

$$
\left( \sum_{i=1}^n \|T(x_i)\|^p \right)^{1/p} \leq C \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^n |x^*(x_i)|^p \right)^{1/p}
$$

holds and the $p$-summing norm of $T$ is defined by $\pi_p(T) := \min \{C : C$ as above\}. We denote by $\Pi_p(X,Y)$ the class of all $p$-summing operators from $X$ into $Y$ (see [2], [3], [4], [6]). Let $X$ and $Y$ be Banach spaces. If $A$ is a set, the notation $(x_n)_{n \in \mathbb{N}} \subset A$ means that $x_n \in A$ for every $n \in \mathbb{N}$. A bounded linear operator $T : X \to Y$ is called nuclear if there exist $(x_n^*)_{n \in \mathbb{N}} \subset X^*$, $(y_n)_{n \in \mathbb{N}} \subset Y$ such that

$$
\sum_{n=1}^\infty \|x_n^*\| \|y_n\| < \infty
$$

and

$$
T(x) = \sum_{n=1}^\infty x_n^*(x)y_n
$$

for $x \in X$; such a representation is called a nuclear representation of $T$ and the nuclear norm of $T$ is defined by $\|T\|_{\text{nuc}} := \inf \{\sum_{n=1}^\infty \|x_n^*\| \|y_n\| : \text{such that } \sum_{n=1}^\infty \|x_n^*\| \|y_n\| < \infty \}$, where the infimum is taken over all the nuclear representations of $T$. We denote by $\mathcal{N}(X,Y)$ the space of all nuclear operators from $X$ into $Y$ (see [2], [3], [4], [6]).

In [10], Theorem 4.2, it was shown that, if $X$ is an infinite-dimensional Banach space, then $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$. As a natural consequence of our results, we recover the folklore result that if $X$ is an infinite dimensional Banach space, then $\Pi_1(C[0,1],X) \neq \Pi_2(C[0,1],X)$, and hence $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$, see Corollary 1.

All notation and terminology, not otherwise explained, are as in [2], [3], [4], [6].

**Preliminary results**

The next Lemma is essentially well-known (see [8], Lemma 10).

**Lemma 1.** Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$, $\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n$ and let $U^n_\alpha : C[0,1] \to l_p^n$ be the operator defined by $U^n_\alpha (f) = \left( \alpha_i \int_0^1 f(t) r_i(t) \, dt \right)_{1 \leq i \leq n}$. Then:

(i) $2^{-1/2} \|\alpha\|_r \leq \|U^n_\alpha\| \leq \pi_2(U^n_\alpha) \leq \|\alpha\|_r$ if $1 \leq p < 2$, where $1/p = 1/2 + 1/r$ and $2^{-1/2} \|\alpha\|_\infty \leq \|U^n_\alpha\| \leq \pi_2(U^n_\alpha) \leq \|\alpha\|_\infty$ if $2 \leq p < \infty$.

(ii) $\pi_1(U^n_\alpha) = \|\alpha\|_p$.

**Proof.** The representing measure of $U^n_\alpha$ is $G^n_\alpha : \Sigma \to l_p^n$ defined by $G^n_\alpha(E) := (\alpha_i \int_E r_i(t) \, dt)_{1 \leq i \leq n}$, where $\Sigma$ is the $\sigma$-algebra of all borelian subsets of $[0,1]$, see [4],
Theorem 1, page 152. Let \( h^n_\alpha \colon [0,1] \to l^p_\alpha \) be given by \( h^n_\alpha(t) = (\alpha_ir_i(t))_{1 \leq i \leq n} \) and observe that \( G^n_\alpha(E) = \int_E h^n_\alpha(t) \, dt \) for \( E \in \Sigma \) (the Bochner integral).

(i) From [4], Theorem 1, page 152, and Proposition 11, page 4, we have

\[
\|U^n_\alpha\| = \|G^n_\alpha([0,1])\| = \sup_{\|y^*\| \leq 1} |y^* \circ G^n_\alpha([0,1])| = \sup_{\|y^*\| \leq 1} \int_0^1 |\langle y^*, h^n_\alpha(t) \rangle| \, dt
\]

because \((y^* \circ G^n_\alpha)(E) = \int_E \langle y^*, h^n_\alpha(t) \rangle \, dt\) and \(|y^* \circ G^n_\alpha([0,1])| = \int_0^1 |\langle y^*, h^n_\alpha(t) \rangle| \, dt\).

However, for any \( y^* = (\xi_i)_{1 \leq i \leq n} \in (l^p_\alpha)^* = l^{p*}_\alpha \) we have \( \langle y^*, h^n_\alpha(t) \rangle = \sum_{i=1}^n \xi_i \alpha_ir_i(t) \) and by Khinchin’s inequality \( 2^{-1/2} \left( \sum_{i=1}^n |\xi_i \alpha_i|^2 \right)^{1/2} \leq \int_0^1 |\langle y^*, h^n_\alpha(t) \rangle| \, dt \), hence

\[
2^{-1/2} \|M_\alpha\| \leq \|G^n_\alpha([0,1])\|, \quad \text{where } M_\alpha \colon l^{p*}_\alpha \to l^2_\alpha \text{ is the multiplication operator.}
\]

Thus we have shown that \( 2^{-1/2} \|M_\alpha\| : l^{p*}_\alpha \to l^2_\alpha \| \leq \|U^n_\alpha\| \). Let us note that always \( \|U^n_\alpha\| \leq \pi_2(U^n_\alpha) \). Further, \( U^n_\alpha \colon C[0,1] \to L^2[0,1] \) is a factorization of \( U^n_\alpha \), where \( J \) is the canonical inclusion and \( R(f) = (\int_0^1 f(t)r_i(t) \, dt)_{1 \leq i \leq n} \). Since \( J \) is \( 2 \)-summing with \( \pi_2(J) = 1 \) and \( \|R\| = 1 \), we deduce that \( \pi_2(U^n_\alpha) \leq \|M_\alpha\| : l^p_\alpha \to l^2_\alpha \| \).

Now, as is well known, \( \|M_\alpha\| : l^{p*}_\alpha \to l^2_\alpha \| = \|M_\alpha\| : l^{p*}_\alpha \to l^{p*}_\alpha \| = \|\alpha\|_r \) if \( 1 \leq p < 2 \), where \( 1/p = 1/2 + 1/r \) and \( \|M_\alpha\| : l^{p*}_\alpha \to l^{2}_\alpha \| = \|M_\alpha\| : l^{2}_\alpha \to l^{p*}_\alpha \| = \max_{1 \leq i \leq n} |\alpha_i| = |\alpha|_\infty \) if \( 2 \leq p \leq \infty \), see [1], page 218, and the proof of (i) is finished.

(ii) From [4], Theorem 3, page 162, \( \pi_1(U^n_\alpha) = \|G^n_\alpha([0,1])\| = \int_0^1 \|h^n_\alpha(t)\|_p \, dt = \|\alpha\|_p. \)

□

In the sequel the technique named Average of a finite number of elements, introduced in [7], [9] is used to construct a useful kind of operators. Let us now fix some notation and recall this concept.

Let \( n \) be a natural number. For \((\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n\) we define the finite system denoted by \( \text{Average}(\lambda_i; 1 \leq i \leq n) \) as being the system with \( 2^n \) elements obtained by arranging in the lexicographical order of \( D_n := \{-1,1\}^n \) the elements \( \varepsilon_1 \lambda_1 + \ldots + \varepsilon_n \lambda_n \) for \((\varepsilon_1, \ldots, \varepsilon_n) \in D_n \). (On \([-1,1]\) we consider the natural order). Thus, as sets we have

\[
\text{Average}(\lambda_i; 1 \leq i \leq n) = \{\varepsilon_1 \lambda_1 + \ldots + \varepsilon_n \lambda_n : (\varepsilon_1, \ldots, \varepsilon_n) \in D_n\}.
\]

Let us note that if \((\lambda_i)_{1 \leq i \leq n} \in \mathbb{K}^n\) and \((e_{(\varepsilon_1, \ldots, \varepsilon_n)})_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n}\) are the standard unit vectors in \( \mathbb{K}^{2^n} \) ordered in the lexicographical order of \( D_n \), then the following equality in \( \mathbb{K}^{2^n} \) holds:

\[
\text{Average}(\lambda_i; 1 \leq i \leq n) = \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} (\varepsilon_1 \lambda_1 + \ldots + \varepsilon_n \lambda_n)e_{(\varepsilon_1, \ldots, \varepsilon_n)}.
\]

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If \(1 \leq p < \infty\), by Khinchin’s inequality we have
\[
A_p \| (\lambda_1, \ldots, \lambda_n) \|_2 \leq \left\| \text{Average} \left( \frac{1}{2^{n/p}} \lambda_i : 1 \leq i \leq n \right) \right\|_p \\
= \left( \frac{1}{2^n} \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} |\varepsilon_1 \lambda_1 + \ldots + \varepsilon_n \lambda_n|^p \right)^{1/p} \\
\leq B_p \| (\lambda_1, \ldots, \lambda_n) \|_2.
\]

Above and in the sequel \(A_p, B_p\) are Khinchin’s constants (see [3]).

**Lemma 2.** Let \(1 \leq p < \infty, n \in \mathbb{N}\), \(\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n\) and let \(\text{Av}_\alpha^n : C[0,1] \to l^2_p\) be the operator defined by
\[
\text{Av}_\alpha^n(f) = \text{Average} \left( \frac{\alpha_i}{2^{n/p}} \int_0^1 f(t) r_i(t) \, dt : 1 \leq i \leq n \right).
\]

Then:
(i) \(A_p 2^{-1/2} \| \alpha \|_\infty \leq \pi_2(\text{Av}_\alpha^n) \leq B_p \| \alpha \|_\infty\).
(ii) \(A_p \| \alpha \|_2 \leq \pi_1(\text{Av}_\alpha^n) \leq B_p \| \alpha \|_2\).

**Proof.** Let \(f \in C[0,1]\). From the relation (2) we have
\[
A_p \| U_\alpha^n(f) \|_2 \leq \| \text{Av}_\alpha^n(f) \| \leq B_p \| U_\alpha^n(f) \|_2
\]
where \(U_\alpha^n : C[0,1] \to l^2_p\) is defined by \(U_\alpha^n(f) = (\alpha_i \int_0^1 f(t) r_i(t) \, dt)_{1 \leq i \leq n}\). Thus
\[
A_p \pi_2(U_\alpha^n) \leq \pi_2(\text{Av}_\alpha^n) \leq B_p \pi_2(U_\alpha^n) \quad \text{and} \quad A_p \pi_1(U_\alpha^n) \leq \pi_1(\text{Av}_\alpha^n) \leq B_p \pi_1(U_\alpha^n).
\]
The conclusion follows, because in this case, by Lemma 1, \(2^{-1/2} \| \alpha \|_\infty \leq \pi_2(U_\alpha^n) \leq \| \alpha \|_\infty\) and \(\pi_1(U_\alpha^n) = \| \alpha \|_2\). \(\square\)

We need also the second average which we describe next. Let \(n\) be a natural number. Let us note that if \((\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n\) then
\[
\text{(3)} \quad c_\mathbb{K} \sum_{i=1}^n |\lambda_i| \leq \| \text{Average} (\lambda_i : 1 \leq i \leq n) \|_\infty \leq \sum_{i=1}^n |\lambda_i|
\]
where \(c_\mathbb{K} = 1\) if \(\mathbb{K} = \mathbb{R}\); \(c_\mathbb{K} = 1/2\) if \(\mathbb{K} = \mathbb{C}\) (in this case consider the real and the imaginary part).

For \((\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n\) let us denote the \(2^n\) elements of the set \(\text{Average}(\lambda_i : 1 \leq i \leq n)\) by \(\{\beta_1, \beta_2, \ldots, \beta_{2^n}\}\) and apply the same procedure; we define
\[
\text{Saverage}(\lambda_i : 1 \leq i \leq n) := \text{Average}(\beta_i : 1 \leq i \leq 2^n) \\
= \{\varepsilon_1 \beta_1 + \ldots + \varepsilon_{2^n} \beta_{2^n} : (\varepsilon_1, \ldots, \varepsilon_{2^n}) \in D_{2^n}\} \subset \mathbb{K}^{2^n}.
\]
From the relation (3) we have
\[ \frac{c_k}{2^n} \| (\beta_1, \ldots, \beta_{2^n}) \|_1 \leq \frac{1}{2^n} \| \text{Saverage}(\lambda_i: 1 \leq i \leq n) \|_\infty \leq \frac{1}{2^n} \| (\beta_1, \ldots, \beta_{2^n}) \|_1 \]
and since by Khinchin’s inequality
\[ \frac{1}{\sqrt{2}} \| (\lambda_1, \ldots, \lambda_n) \|_2 \leq \frac{1}{2^n} \sum_{i=1}^{2^n} |\beta_i| = \frac{1}{2^n} \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} |\varepsilon_1 \lambda_1 + \ldots + \varepsilon_n \lambda_n| \]
\[ \leq \| (\lambda_1, \ldots, \lambda_n) \|_2 \]
we get
\[ (4) \quad \frac{c_k}{\sqrt{2}} \| (\lambda_1, \ldots, \lambda_n) \|_2 \leq \frac{1}{2^n} \| \text{Saverage}(\lambda_i: 1 \leq i \leq n) \|_\infty \leq \| (\lambda_1, \ldots, \lambda_n) \|_2. \]

**Lemma 3.** (a) Let \( n \in \mathbb{N}, \alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n \) and let \( \text{Av}_n^\alpha: C[0, 1] \to l^2_{2^n} \) be the operator defined by
\[ \text{Av}_n^\alpha(f) = \text{Average} \left( \alpha_i \int_0^1 f(t) r_i(t) \, dt : 1 \leq i \leq n \right). \]
Then:
(i) \( c_k 2^{-1/2} \| \alpha \|_2 \leq \pi_2(\text{Av}_n^\alpha) \leq \| \alpha \|_2. \)
(ii) \( c_k \| \alpha \|_1 \leq \pi_1(\text{Av}_n^\alpha) \leq \| \alpha \|_1. \)

(b) Let \( n \in \mathbb{N}, \alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n \) and let \( \text{Sav}_n^\alpha: C[0, 1] \to l^2_{2^n} \) be the operator defined by
\[ \text{Sav}_n^\alpha(f) := \text{Saverage} \left( \frac{1}{2^n} \alpha_i \int_0^1 f(t) r_i(t) \, dt : 1 \leq i \leq n \right). \]
Then:
(i) \( c_k 2^{-1} \| \alpha \|_\infty \leq \pi_2(\text{Sav}_n^\alpha) \leq \| \alpha \|_\infty. \)
(ii) \( c_k 2^{-1/2} \| \alpha \|_2 \leq \pi_1(\text{Sav}_n^\alpha) \leq \| \alpha \|_2. \)

**Proof.** (a) Let \( f \in C[0, 1] \). From the relation (3) we have
\[ c_k \| U_n^\alpha(f) \|_1 \leq \| \text{Av}_n^\alpha(f) \|_\infty \leq \| U_n^\alpha(f) \|_1 \]
where \( U_n^\alpha: C[0, 1] \to l^1_1 \) is defined by \( U_n^\alpha(f) = (\alpha_i \int_0^1 f(t) r_i(t) \, dt)_{1 \leq i \leq n} \). Thus, easily,
\[ c_k \pi_2(U_n^\alpha) \leq \pi_2(\text{Av}_n^\alpha) \leq \pi_2(U_n^\alpha) \] and \( c_k \pi_1(U_n^\alpha) \leq \pi_2(\text{Av}_n^\alpha) \leq \pi_1(U_n^\alpha). \)
The conclusion follows, because in this case, by Lemma 1, \(2^{-1/2}\|\alpha\|_2 \leq \pi_2(U^n_\alpha) \leq \|\alpha\|_2\) and \(\pi_1(U^n_\alpha) = \|\alpha\|_1\).

(b) Let \(f \in C[0, 1]\). From the relation (4) we have

\[
\frac{c_K}{\sqrt{2}}\|U^n_\alpha(f)\|_2 \leq \|\text{Sav}^n_\alpha(f)\|_\infty \leq \|U^n_\alpha(f)\|_2
\]

where \(U^n_\alpha : C[0, 1] \rightarrow l^n_2\) is defined by \(U^n_\alpha(f) = (\alpha_i \int_0^1 f(t) \tau_i(t) \, dt)_{1 \leq i \leq n}\). Thus

\[
\frac{c_K}{\sqrt{2}} \pi_2(U^n_\alpha) \leq \pi_2(\text{Sav}^n_\alpha) \leq \pi_2(U^n_\alpha); \quad \frac{c_K}{\sqrt{2}} \pi_1(U^n_\alpha) \leq \pi_1(\text{Sav}^n_\alpha) \leq \pi_1(U^n_\alpha).
\]

The conclusion follows, because in this case, by Lemma 1, \(2^{-1/2}\|\alpha\|_\infty \leq \pi_2(U^n_\alpha) \leq \|\alpha\|_\infty\) and \(\pi_1(U^n_\alpha) = \|\alpha\|_2\).

\[\square\]

**The results**

In the next theorem, which is the main result of this paper, we show how the local structure of the spaces \(\Pi_2(C[0, 1], X)\) and \(\Pi_1(C[0, 1], X)\) depends on the local structure of \(X\).

**Theorem 4.** Let \(1 \leq p \leq \infty, 1 < \lambda < \infty\) and let \(X\) be a Banach space which contains \(l^n_p\)'s \(\lambda\)-uniformly. Then:

(i) For \(1 \leq p < 2\), \(\Pi_2(C[0, 1], X)\) contains \(\lambda\sqrt{2}\)-uniformly copies of \(l^n_1\)'s where \(1/p = 1/2 + 1/r\).

(ii) For \(2 \leq p \leq \infty\), \(\Pi_2(C[0, 1], X)\) contains \(\lambda\sqrt{2}\)-uniformly copies of \(l^n_\infty\)'s.

(iii) For \(1 \leq p < \infty\), \(\Pi_2(C[0, 1], X)\) contains \(\lambda B_p\sqrt{2}/A_p\)-uniformly copies of \(l^n_1\)'s.

(iv) \(\Pi_1(C[0, 1], X)\) contains \(\lambda\)-uniformly copies of \(l^n_p\)'s.

(v) For \(1 \leq p < \infty\), \(\Pi_1(C[0, 1], X)\) contains \(\lambda B_p/A_p\)-uniformly copies of \(l^n_2\)'s.

(vi) For \(1 \leq p < \infty\), the spaces \(\Pi_2(C[0, 1], X)\) and \(\Pi_1(C[0, 1], X)\) are distinct; in particular, \(\Pi_2(C[0, 1], X) \neq N(C[0, 1], X)\).

**Proof.** (i), (ii) and (iv). Let \(n \in \mathbb{N}\) be arbitrary. By hypothesis there exists a bounded linear operator \(J_n : l^n_p \rightarrow X\) such that

\[
\|\alpha\|_p \leq \|J_n(\alpha)\|_X \leq \lambda\|\alpha\|_p, \quad \alpha \in l^n_p.
\]

Let us define \(A_n : \mathbb{K}^n \rightarrow L(C[0, 1], X)\) by \(A_n(\alpha) = J_n \circ U^n_\alpha\), where \(U^n_\alpha : C[0, 1] \rightarrow l^n_p\) is the operator from Lemma 1. Though not needed in the sequel, let us note that if \(\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n\) and \(f \in C[0, 1]\) then

\[
A_n(\alpha)(f) = \sum_{i=1}^n \alpha_i \left( \int_0^1 f(t) \tau_i(t) \, dt \right) J_n(e_i).
\]
Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0, 1]$ by (5) we have

$$\|U^n_\alpha(f)\|_p \leq \|[A_n(\alpha)](f)\|_X = \|J_n(U^n_\alpha(f))\|_X \leq \lambda \|U^n_\alpha(f)\|_p$$

and by the definition of $p$-summing operators we deduce that

(6) $\pi_2(U^n_\alpha) \leq \pi_2(A_n(\alpha)) \leq \lambda \pi_2(U^n_\alpha)$ and $\pi_1(U^n_\alpha) \leq \pi_1(A_n(\alpha)) \leq \lambda \pi_1(U^n_\alpha)$.

From (6) and Lemma 1 we obtain

$$\|\alpha\|_r \leq \pi_2(\sqrt{2}A_n(\alpha)) \leq \lambda \sqrt{2}\|\alpha\|_r$$ if $1 \leq p < 2$, where $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$,

$$\|\alpha\|_\infty \leq \pi_2(\sqrt{2}A_n(\alpha)) \leq \lambda \sqrt{2}\|\alpha\|_\infty$$ if $2 \leq p < \infty$,

$$\|\alpha\|_p \leq \pi_1(A_n(\alpha)) \leq \lambda \|\alpha\|_p,$$

which ends the proof of (i), (ii) and (iv).

(iii) and (v). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n} : l_2^{2^n} \rightarrow X$ such that

(7) $\|\xi\|_p \leq \|J_{2^n}(\xi)\|_X \leq \lambda \|\xi\|_p$, $\xi \in l_2^{2^n}$.

We define $Av_n : \mathbb{K}^n \rightarrow L(C[0, 1], X)$ by $Av_n(\alpha) = J_{2^n} \circ Av_n$, where $Av_n : C[0, 1] \rightarrow l_2^{2^n}$ is the operator from Lemma 2. Again, though not needed in the sequel, let us note that if $\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n$ and $f \in C[0, 1]$ we have

$$[Av_n(\alpha)](f) = \frac{1}{2^{n/p}} \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} \left(\varepsilon_1 \alpha_1 \int_0^1 f(t)r_1(t) \, dt + \ldots \right. \left. + \varepsilon_n \alpha_n \int_0^1 f(t)r_n(t) \, dt\right)J_{2^n}(e(\varepsilon_1, \ldots, \varepsilon_n)).$$

Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0, 1]$ by (7) we have

$$\|Av^n_\alpha(f)\|_p \leq \|[Av_n(\alpha)](f)\|_X = \|J_{2^n}(Av^n_\alpha(f))\|_X \leq \lambda \|Av^n_\alpha(f)\|_p$$

and by the definition of $p$-summing operators we deduce that

(8) $\pi_2(Av^n_\alpha) \leq \pi_2(Av_n(\alpha)) \leq \lambda \pi_2(Av^n_\alpha)$ and $\pi_1(Av^n_\alpha) \leq \pi_1(Av_n(\alpha)) \leq \lambda \pi_1(Av^n_\alpha)$.

Since by Lemma 2

$$\frac{A_p}{\sqrt{2}} \|\alpha\|_\infty \leq \pi_2(Av_n(\alpha)) \leq B_p \|\alpha\|_\infty$$ and $A_p \|\alpha\|_2 \leq \pi_1(Av_n(\alpha)) \leq B_p \|\alpha\|_2,$

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from (8) we obtain
\[ \|\alpha\|_\infty \leq \pi_2\left(\frac{\sqrt{2}}{A_p} Av_n(\alpha)\right) \leq \frac{\lambda B_p \sqrt{2}}{A_p} \|\alpha\|_\infty; \quad \|\alpha\|_2 \leq \pi_1\left(\frac{Av_n(\alpha)}{A_p}\right) \leq \frac{\lambda B_p}{A_p} \|\alpha\|_2, \]

which ends the proof of (iii) and (v).

(vi) If \( \Pi_2(C[0,1],X) = \Pi_1(C[0,1],X) \), then by the open mapping theorem it follows that there exists \( C > 0 \) such that \( \pi_1(T) \leq C \pi_2(T) \) for all \( T \in \Pi_1(C[0,1],X) \). In particular, \( \pi_1(A_n(\alpha)) \leq C \pi_2(A_n(\alpha)) \) for all natural numbers \( n \) and all \( \alpha \in \mathbb{K}^n \).

By (i), (ii) and (iv) for all natural numbers \( n \) and all \( \alpha \in \mathbb{K}^n \) we have \( \|\alpha\|_p \leq C\|\alpha\|_r \) if \( 1 \leq p < 2 \), where \( 1/p = 1/2 + 1/r \), or \( \|\alpha\|_p \leq C\|\alpha\|_\infty \) if \( 2 \leq p < \infty \). Taking
\[ \alpha = (1, \ldots, 1) \]

we get that for all natural numbers \( n \) we have \( n \leq C^2 \) if \( 1 \leq p < 2 \), or \( n \leq C^p \) if \( 2 \leq p < \infty \), which is impossible. Let us note that a contradiction can be obtained if we use (iii) or (v). If \( \Pi_2(C[0,1],X) = \mathcal{N}(C[0,1],X) \) then, since \( \mathcal{N}(C[0,1],X) \subseteq \Pi_1(C[0,1],X) \subseteq \Pi_2(C[0,1],X) \), it follows that \( \Pi_1(C[0,1],X) = \Pi_2(C[0,1],X) \), which as we have shown above is impossible.

As a natural consequence of Theorem 4, we recover the folklore result that if \( X \) is an infinite-dimensional Banach space then the spaces \( \Pi_2(C[0,1],X) \) and \( \Pi_1(C[0,1],X) \) are distinct. This extends the well-known result that the spaces \( \Pi_2(C[0,1],X) \) and \( \mathcal{N}(C[0,1],X) \) are distinct, see [10], Theorem 4.2.

**Corollary 5.** Let \( X \) be an infinite dimensional Banach space. Then:

(i) \( \Pi_2(C[0,1],X) \) contains \( \lambda \sqrt{2} \)-uniformly copies of \( l^p_\infty \)'s for all \( \lambda > 1 \).

(ii) \( \Pi_1(C[0,1],X) \) contains \( \lambda \)-uniformly copies of \( l^p_\infty \)'s for all \( \lambda > 1 \).

(iii) The spaces \( \Pi_2(C[0,1],X) \) and \( \Pi_1(C[0,1],X) \) are distinct; in particular, \( \Pi_2(C[0,1],X) \neq \mathcal{N}(C[0,1],X) \).

**Proof.** Since \( X \) is infinite-dimensional, by the famous Dvoretzky theorem, see [3], Chapter 19, \( X \) contains \( l^p_\infty \)'s \( \lambda \)-uniformly for all \( 1 < \lambda < \infty \). The statement follows by taking \( p = 2 \) in Theorem 4.

Let us note that for \( p = \infty \) in Theorem 4 ((ii) and (iv)) it follows that if \( 1 < \lambda < \infty \) and \( X \) is a Banach space which contains \( l^p_\infty \)'s \( \lambda \)-uniformly, then \( \Pi_2(C[0,1],X) \) contains \( \lambda \sqrt{2} \)-uniformly copies of \( l^p_\infty \)'s and \( \Pi_1(C[0,1],X) \) contains \( \lambda \)-uniformly copies of \( l^p_\infty \)'s, so in this case, there is no distinction between these classes.

We prove now a natural completion of Theorem 4. It shows that for \( p = \infty \) in Theorem 4 we also have a distinct if we use the first and the second average.

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Theorem 6. Let $1 < \lambda < \infty$ and let $X$ be a Banach space which contains $l_\infty^n$’s $\lambda$-uniformly. Then:

(i) $\Pi_2(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of $l_2^n$’s in the real case ($2\lambda\sqrt{2}$-uniformly copies of $l_2^n$’s in the complex case).

(ii) $\Pi_1(C[0,1], X)$ contains $\lambda$-uniformly copies of $l_1^n$’s in the real case ($2\lambda$-uniformly copies of $l_1^n$’s in the complex case).

(iii) $\Pi_2(C[0,1], X)$ contains $2\lambda$-uniformly copies of $l_\infty^n$’s in the real case ($4\lambda$-uniformly copies of $l_\infty^n$’s in the complex case).

(iv) $\Pi_1(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of $l_2^n$’s in the real case ($2\lambda\sqrt{2}$-uniformly copies of $l_2^n$’s in the complex case).

Proof. (i) and (ii). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n}: l_\infty^{2^n} \to X$ such that

\begin{equation}
\|\xi\|_\infty \leq \|J_{2^n}(\xi)\|_X \leq \lambda \|\xi\|_\infty, \quad \xi \in l_\infty^{2^n}.
\end{equation}

We define $A_{2^n}: \mathbb{K}^n \to L(C[0,1], X)$ by $A_{2^n}(\alpha) = J_{2^n} \circ A_{2^n}$, where $A_{2^n}: C[0,1] \to l_\infty^{2^n}$ is the operator from Lemma 3. Let us note (not used in the sequel) the explicit expression,

\begin{equation}
[A_{2^n}(\alpha)](f) = \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in D_n} \left( \varepsilon_1 \alpha_1 \int_0^1 f(t) r_1(t) \, dt + \ldots + \varepsilon_n \alpha_n \int_0^1 f(t) r_n(t) \, dt \right) J_{2^n}(e_{(\varepsilon_1, \ldots, \varepsilon_n)})
\end{equation}

where $\alpha = (\alpha_i)_{1 \leq i \leq n} \in \mathbb{K}^n$ (see also the equality (1)). Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (9) we have

\begin{equation}
\|A_{2^n}(f)\|_\infty \leq \|[A_{2^n}(\alpha)](f)\|_X = \|J_{2^n}(A_{2^n}(f))\|_X \leq \lambda \|A_{2^n}(f)\|_\infty,
\end{equation}

and by the definition of $p$-summing operators we deduce that

\begin{equation}
\pi_2(A_{2^n}) \leq \pi_2(A_{2^n}(\alpha)) \leq \lambda \pi_2(A_{2^n})
\end{equation}

and

\begin{equation}
\pi_1(A_{2^n}) \leq \pi_1(A_{2^n}(\alpha)) \leq \lambda \pi_1(A_{2^n}).
\end{equation}

Since by Lemma 3

\[
\frac{c_\mathbb{K}}{\sqrt{2}} \|\alpha\|_2 \leq \pi_2(A_{2^n}(\alpha)) \leq \|\alpha\|_2 \quad \text{and} \quad c_\mathbb{K} \|\alpha\|_1 \leq \pi_1(A_{2^n}(\alpha)) \leq \|\alpha\|_1,
\]

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from (10) we obtain
\[ \|\alpha\|_2 \leq \pi_2 \left( \frac{\sqrt{2}}{c_K} Av_n(\alpha) \right) \leq \frac{\lambda \sqrt{2}}{c_K} \|\alpha\|_2 \quad \text{and} \quad \|\alpha\|_1 \leq \pi_1 \left( \frac{Av_n(\alpha)}{c_K} \right) \leq \frac{\lambda}{c_K} \|\alpha\|_1 \]
which ends the proof of (i) and (ii).

(iii) and (iv). Let \( n \in \mathbb{N} \) be arbitrary. By hypothesis there exists a bounded linear operator \( J_{2^n} : l^{2^n}_\infty \to X \) such that
\[
\|\xi\|_\infty \leq \|J_{2^n}(\xi)\|_X \leq \lambda \|\xi\|_\infty, \quad \xi \in l^{2^n}_\infty.
\]
We define \( S_{av}^n : K_n \to L(C[0,1], X) \) by
\[
S_{av}^n(\alpha) = J_{2^n} \circ S_{av}^n
\]
where \( S_{av}^n : C[0,1] \to l^{2^n}_\infty \) is the operator from Lemma 3. We leave for the interested reader to write the explicit expression for \([S_{av}^n(\alpha)](f)\), which again is not used in the sequel.

Let \( \alpha \in K_n \). For every \( f \in C[0,1] \) by (11) we have
\[
\|S_{av}^n(\alpha)(f)\|_\infty \leq \|[S_{av}^n(\alpha)](f)\|_X = \|J_{2^n} (S_{av}^n(\alpha)(f))\|_X \leq \lambda \|S_{av}^n(\alpha)(f)\|_\infty
\]
and by the definition of \( p \)-summing operators we deduce that
\[
\pi_2(S_{av}^n) \leq \pi_2(S_{av}(\alpha)) \leq \lambda \pi_2(S_{av}^n)
\]
and
\[
\pi_1(S_{av}^n) \leq \pi_1(S_{av}(\alpha)) \leq \lambda \pi_1(S_{av}^n).
\]
Since by Lemma 3
\[
\frac{c_K}{2} \|\alpha\|_\infty \leq \pi_2(S_{av}(\alpha)) \leq \|\alpha\|_\infty \quad \text{and} \quad \frac{c_K}{\sqrt{2}} \|\alpha\|_2 \leq \pi_1(S_{av}(\alpha)) \leq \|\alpha\|_2,
\]
from (12) we obtain
\[
\|\alpha\|_\infty \leq \pi_2 \left( \frac{2}{c_K} S_{av}(\alpha) \right) \leq \frac{2\lambda}{c_K} \|\alpha\|_\infty \quad \text{and} \quad \|\alpha\|_2 \leq \pi_1 \left( \frac{\sqrt{2}}{c_K} S_{av}(\alpha) \right) \leq \frac{\lambda \sqrt{2}}{c_K} \|\alpha\|_2,
\]
which ends the proof of (iii) and (iv). \( \square \)

In [5] was shown that the space \( \Pi_1(C[0,1], X) \) can be identified with the so called space \( l^{2\text{free}}_1(X) \); we refer the reader to the paper [5] for the definition of this space and more details. From Theorems 4, 6 and Corollary 5 we get
Corollary 7.  (a) Let $1 \leq p \leq \infty$, $1 < \lambda < \infty$ and let $X$ be a Banach space which contains $l^n_p$'s $\lambda$-uniformly. Then:
(i) $l^\text{tree}_1(X)$ contains $\lambda$-uniformly copies of $l^n_p$'s.
(ii) For $1 \leq p < \infty$, $l^\text{tree}_1(X)$ contains $\lambda B_p/A_p$-uniformly copies of $l^n_p$'s.
(b) Let $1 < \lambda < \infty$ and let $X$ be a Banach space which contains $l^n_\infty$'s $\lambda$-uniformly. Then:
(i) $l^\text{tree}_1(X)$ contains $\lambda \sqrt{2}$-uniformly copies of $l^n_1$'s in the real case ($2\lambda \sqrt{2}$-uniformly copies of $l^n_1$'s in the complex case).
(ii) $l^\text{tree}_1(X)$ contains $\lambda$-uniformly copies of $l^n_2$'s in the real case ($2\lambda$-uniformly copies of $l^n_2$'s in the complex case).
(c) Let $X$ be an infinite dimensional Banach space. Then $l^\text{tree}_1(X)$ contains $\lambda$-uniformly copies of $l^n_2$'s for all $\lambda > 1$.

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References


Author’s address: Dumitru Popa, Department of Mathematics, Ovidius University of Constanța, Bd. Mamaia 124, 900527 Constanța, Romania, e-mail: dpopa@univ-ovidius.ro.