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Generalized Higher Derivations on Lie Ideals of Triangular Algebras

Mohammad Ashraf, Nazia Parveen, Bilal Ahmad Wani

Abstract. Let $\mathfrak{A} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix}$ be the triangular algebra consisting of unital algebras $\mathcal{A}$ and $\mathcal{B}$ over a commutative ring R with identity 1 and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule. An additive subgroup $\mathfrak{L}$ of $\mathfrak{A}$ is said to be a Lie ideal of $\mathfrak{A}$ if $[\mathfrak{L}, \mathfrak{A}] \subseteq \mathfrak{L}$. A non-central square closed Lie ideal $\mathfrak{L}$ of $\mathfrak{A}$ is known as an *admissible Lie ideal*. The main result of the present paper states that under certain restrictions on $\mathfrak{A}$, every generalized Jordan triple higher derivation of $\mathfrak{L}$ into $\mathfrak{A}$ is a generalized higher derivation of $\mathfrak{L}$ into $\mathfrak{A}$.

1 Introduction

Let R be an associative ring with an identity element. Recall that a ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies $a = 0$). For any $x, y \in R, [x, y] = xy - yx$, denotes the well-known Lie product. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L of R is said to be a square closed Lie ideal of R if $x^2 \in L$ for all $x \in L$. A non-central square closed Lie ideal L of R is known as an admissible Lie ideal. An additive map $d\colon R \to R$ is said to be a derivation (resp. Jordan derivation) of R if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive map $d\colon R \to R$ is said to be a Jordan triple derivation of R if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$.

There has been a great deal of work concerning derivations, Jordan derivations and Jordan triple derivations in rings (see [3], [4], [14], [15]). Also, the derivation, Jordan derivation and Jordan triple derivation and their relationship with Lie ideals of prime rings have been extensively and systematically studied during the last four decades. One of the most remarkable result in this direction was given by Awtar [3]

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in the year 1984. In fact, he proved that if L is a Lie ideal of a prime ring R of characteristic different from 2 such that $x^2 \in L$ for all $x \in L$ and $d\colon R \to R$ is an additive mapping such that $d\,|_L$ is a Jordan derivation of L into R, then $d\,|_L$ is a derivation of L into R.

Further, the notion of derivation has also been generalized in various direction. In 1991, Brešar [5], introduced the concept of generalized derivations in rings as follows: an additive mapping $F : R \to R$ is said to be a generalized derivation (resp. generalized Jordan derivation) on R if there exists a derivation (resp. Jordan derivation) $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$), holds for all $x, y \in R$. An additive mapping $F\colon R \to R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation $d\colon R \to R$ such that $F(xyx) = F(x)yx + xd(y)x + xyd(x)$, holds for all $x, y \in R$.

The concept of derivation was extended to higher derivation by F. Hasse and F.K. Schmidt [13] (see [9] and [11] for a historical account and applications) as follows: let $\mathbb{N}$ be the set of non-negative integers. A family $D = \{d_n\}_{n\in\mathbb{N}}$ of additive mappings $d_n\colon R \to R$ such that $d_0 = id_R$, the identity map of R, is said to be a higher derivation (HD) if for every $n \in \mathbb{N}$, $d_n(xy) = \sum\limits_{i+j=n} d_i(x)d_j(y)$ holds for all $x, y \in R$. Higher derivations of a ring R have been studied by many authors in various directions. In 2002, M. Ferrero and C. Haetinger [10] generalized the above mentioned result obtained by R. Awtar [3] to higher derivations on Lie ideals. Further, motivated by the notion of generalized derivation in rings, the concept of generalized higher derivation was introduced by Cortes and Haetinger [7]. A family $F = \{f_n\}_{n\in\mathbb{N}}$ of additive mappings $f_n\colon R \to R$ such that $f_0 = \mathrm{id}_R$, is said to be a generalized higher derivation (GHD) if there exists a higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $f_n(xy) = \sum\limits_{i+j=n} f_i(x)d_j(y)$ holds for all $x, y \in R$.

We now turn our attention to triangular algebras. Let $\mathcal{A}, \mathcal{B}$ be unital algebras over a commutative ring R with identity 1 and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. The set

$$\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations forms an R-algebra. An algebra $\mathfrak{A}$ is called a triangular algebra if $\mathfrak{A}$ is (algebraicaly) isomorphic to $\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$. This algebra was first introduced by Chase [6]. Note that if $\mathfrak{L}$ is a Lie ideal of $\mathfrak{A}$, then $\mathfrak{L}$ can be written as $\mathfrak{L} = \mathrm{Tri}(\mathfrak{L}_A, M', \mathfrak{L}_B)$, where, $\mathfrak{L}_A$ and $\mathfrak{L}_B$ are Lie ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively and $\mathcal{M}'$ is an $(\mathfrak{L}_A, \mathfrak{L}_B)$-bimodule. For a non-central Lie ideal $\mathfrak{L}$ of a triangular algebra $\mathfrak{A}$, if $\mathfrak{L}_A$ and $\mathfrak{L}_B$ are admissible Lie ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively and $X^2 \in \mathfrak{L}$ for every $X \in \mathfrak{L}$ then we call $\mathfrak{L}$ an admissible Lie ideal of $\mathfrak{A}$.

Recently, in the year 2010 Xiao and Wei [18] introduced the notion of higher derivation and Jordan higher derivation on triangular algebras and proved that every JHD on triangular algebras is a HD. This result was extended to Lie ideals of triangular algebra by Han [12] who proved that every Jordan higher derivation (resp. Jordan triple higher derivation) of a Lie ideal into a triangular algebra is a higher derivation. We first recall the notions of higher derivation of Lie ideals into triangular algebras. Throughout the text, let $\mathfrak{A}$ be a triangular R-algebra consisting of unital R-algebras $\mathcal{A}, \mathcal{B}$ and an $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$. All the algebras and modules are 2-torsion free and $\mathfrak{L}$ will denote an admissible Lie ideal of $\mathfrak{A}$. In this paper, if $\mathfrak{L}$ is an admissible Lie ideal of $\mathfrak{A}$, then we always assume that either the action of each $a \neq 0 \in \mathfrak{L}_A$ on every $m \in \mathcal{M}$ is nontrivial or the action of each $b \neq 0 \in \mathfrak{L}_B$ on every $m \in \mathcal{M}$ is nontrivial.

Definition 1. Let $\mathbb{N}$ be the set of non-negative integers and $D = \{d_n\}_{n\in\mathbb{N}}$ be a family of linear mappings on $\mathfrak{A}$ such that $d_0 = \mathrm{id}_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$. Then for all $X, Y \in \mathfrak{L}$, D is called

(i) a higher derivation (HD) of $\mathfrak{L}$ into $\mathfrak{A}$ if for every $n \in \mathbb{N}$

$$d_n(XY) = \sum_{i+j=n} d_i(X)d_j(Y)\,;$$

(ii) a Jordan higher derivation (JHD) of $\mathfrak{L}$ into $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n(X^2) = \sum_{i+j=n} d_i(X)d_j(X)\,;$$

(iii) a Jordan triple higher derivation (JTHD) of $\mathfrak{L}$ into $\mathfrak{A}$ if for every $n \in \mathbb{N}$,

$$d_n(XYX) = \sum_{i+j+k=n} d_i(X)d_j(Y)d_k(X)\,.$$

Motivated by the notions of generalized derivations in rings and higher derivations on triangular algebras, we are able to define and introduce generalized higher derivation, generalized Jordan higher derivation and generalized Jordan triple higher derivation on triangular algebras as follows.

Definition 2. Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a family of linear mappings on $\mathfrak{A}$ such that $f_0 = \mathrm{id}_{\mathfrak{A}}$, the identity map on $\mathfrak{A}$. Then F is said to be

(i) a generalized higher derivation (GHD) of $\mathfrak{L}$ into $\mathfrak{A}$ if there exists a higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ into $\mathfrak{A}$ such that for every $n \in \mathbb{N}$,

$$f_n(XY) = \sum_{i+j=n} f_i(X)d_j(Y)$$

for all $X, Y \in \mathfrak{L}$;

(ii) a generalized Jordan higher derivation (GJHD) of $\mathfrak{L}$ into $\mathfrak{A}$ if there exists a Jordan higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ into $\mathfrak{A}$ such that for every $n \in \mathbb{N}$,

$$f_n(X^2) = \sum_{i+j=n} f_i(X)d_j(X)$$

for all $X \in \mathfrak{L}$;

(iii) a generalized Jordan triple higher derivation (GJTHD) of $\mathfrak{L}$ into $\mathfrak{A}$ if there exists a Jordan triple higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ into $\mathfrak{A}$ such that for every $n \in \mathbb{N}$,

$$f_n(XYX) = \sum_{i+j+k=n} f_i(X)d_j(Y)d_k(X)$$

for all $X, Y \in \mathfrak{L}$.

The main goal of the present paper is to consider the question of when a generalized Jordan higher derivation becomes a generalized higher derivation. Similarly, it is also an interesting question when a generalized Jordan triple higher derivation becomes a generalized higher derivation. This paper is organised as follows. In section 2, we obtain conditions under which every generalized Jordan higher derivation becomes a generalized higher derivation, whereas, in section 3, we study generalized Jordan triple higher derivations on admissible Lie ideals.

2 Generalized Jordan higher derivations on $\mathfrak{A}$

Assume that $\mathfrak{L}$ is a square closed Lie ideal of $\mathfrak{A}$. For any $X, Y \in \mathfrak{L}$,

$$(XY + YX) = (X + Y)^2 - (X^2 + Y^2)$$

and so $XY + YX \in \mathfrak{L}$. Also $[X, Y] = XY - YX \in \mathfrak{L}$ and it follows that $2XY \in \mathfrak{L}$. Hence $4XYZ = 2(2XY)Z \in \mathfrak{L}$ for all $X, Y, Z \in \mathfrak{L}$. These results will be used freely without any specific mention.

We begin with the following well-known results which will be used in the subsequent discussion.

Lemma 1 ([4], Lemma 4). *Let R be a prime ring of characteristic other than 2 and let L be a Lie ideal of R with $L \nsubseteq Z(R)$. If $a, b \in R$ and $aLb = \{0\}$, then either $a = 0$ or $b = 0$.*

Lemma 2 ([8], Lemma 2.2). *Let R be a 2-torsion free semiprime ring (resp. prime ring) and L be an* admissible Lie ideal *of R. If $a, b \in R$ (resp. $a \in L$ and $b \in R$) are such that $axb + bxa = 0$, for every $x \in R$ (resp. $x \in L$), then $axb = bxa = 0$ for every $x \in R$ (resp. $a = 0$ or $b = 0$).*

Lemma 3 ([8], Lemma 2.3). *Assume that R be a 2-torsion free semiprime ring (resp. prime ring) and L be an* admissible Lie ideal *of R. Let $G_1, G_2, ..., G_n$ be additive groups, $S\colon G_1 \times \cdots \times G_n \to R$ and $T\colon G_1 \times \cdots \times G_n \to R$ be mappings which are additive in each argument. If $S(a_1, \ldots, a_n)xT(a_1, \ldots, a_n) = 0$, for every $x \in R$ (resp. $x \in L$), $a_i \in G_i$, $i = 1, 2, \ldots, n$, then $S(a_1, \ldots, a_n)xT(b_1, \ldots, b_n) = 0$ for every $x \in R$, $a_i, b_i \in G_i, i = 1, 2, \ldots, n$ (resp. $S(a_1, \ldots, a_n) = 0$, for every $a_i \in G_i$, $i = 1, 2, \ldots, n$ or $T(b_1, \ldots, b_n) = 0$, for every $b_i \in G_i$, $i = 1, 2, \ldots, n$).*

Lemma 4 ([7], Lemma 7). *Let L_1 and L_2 be Lie ideals of R with $[L_1, L_2] \subseteq Z(R)$. Then $L_1 \subseteq Z(R)$ or $L_2 \subseteq Z(R)$.*

Lemma 5 ([12], Theorem 2.7). *Let $\mathfrak{A}$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are prime algebras and $\mathfrak{L}$ is an admissible Lie ideal of $\mathfrak{A}$. Then each JHD of $\mathfrak{L}$ into $\mathfrak{A}$ is a HD of $\mathfrak{L}$ into $\mathfrak{A}$.*

Given GJHD $F = \{f_n\}_{n\in\mathbb{N}}$ with associated JHD $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ in $\mathfrak{A}$, we now put

$$\xi_n(X,Y) = f_n(XY) - \sum_{i+j=n} f_i(X)d_j(Y)$$

for each fixed $n \in \mathbb{N}$ and for all $X, Y \in \mathfrak{L}$. Note that $\xi_n(Y,X) = -\xi_n(X,Y)$. It is straight forward to see that if $\xi_n(X,Y) = 0$ then $F = \{f_n\}_{n\in\mathbb{N}}$ is a GHD of $\mathfrak{L}$ into $\mathfrak{A}$.

Lemma 6. *Let $\mathfrak{A}$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, $\mathfrak{L}$ be a square closed Lie ideal of $\mathfrak{A}$ and $F = \{f_n\}_{n\in\mathbb{N}}$ a GJHD associated with Jordan higher derivation $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ into $\mathfrak{A}$. Then for each fixed $n \in \mathbb{N}$ and for every $X, Y, Z \in \mathfrak{L}$, the following statements hold:*

(i) $f_n(XY + YX) = \sum_{i+j=n} (f_i(X)d_j(Y) + f_i(Y)d_j(X))$,

(ii) $f_n(XYX) = \sum_{i+j+k=n} f_i(X)d_j(Y)d_k(X)$,

(iii) $f_n(XYZ + ZYX) = \sum_{i+j+k=n} (f_i(X)d_j(Y)d_k(Z) + f_i(Z)d_j(Y)d_k(X))$.

Proof. (i) We have

$$\begin{aligned} f_n(X+Y)^2 &= \sum_{i+j=n} f_i(X+Y)d_j(X+Y) \\ &= \sum_{i+j=n} \big(f_i(X)d_j(X) + f_i(X)d_j(Y) + f_i(Y)d_j(X) + f_i(Y)d_j(Y)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} f_n(X+Y)^2 &= f_n(X^2 + XY + YX + Y^2) \\ &= f_n(X^2) + f_n(Y^2) + f_n(XY + YX) \\ &= f_n(XY + YX) + \sum_{i+j=n} f_i(X)d_j(X) + \sum_{r+s=n} f_r(Y)d_s(Y). \end{aligned}$$

Comparing the above two equations, we obtain that

$$f_n(XY + YX) = \sum_{i+j=n} \big(f_i(X)d_j(Y) + f_i(Y)d_j(X)\big).$$

(ii) Replace Y by $XY+YX$ in (i)

$$\begin{aligned}
f_n(X(XY+YX)+(XY+YX)X) &= \sum_{i+j=n}(f_i(X)d_j(XY+YX)+f_i(XY+YX)d_j(X)) \\
&= \sum_{i+r+s=n} f_i(X)(d_r(X)d_s(Y)+d_r(Y)d_s(X)) \\
&\quad + \sum_{r+s+j=n}(f_r(X)d_s(Y)+f_r(Y)d_s(X))d_j(X) \\
&= \sum_{i+r+s=n} f_i(X)d_r(X)d_s(Y) \\
&\quad + 2\sum_{r+s+k=n} f_r(X)d_s(Y)d_k(X) \\
&\quad + \sum_{r+s+j=n} f_r(Y)d_s(X)d_j(X).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f_n(X(XY+YX)+(XY+YX)X) &= f_n(X^2Y+2XYX+YX^2) \\
&= \sum_{i+j=n}(f_i(X^2)d_j(Y)+f_i(Y)d_j(X^2))+2f_n(XYX) \\
&= \sum_{r+s+j=n} f_r(X)d_s(X)d_j(Y) \\
&\quad + \sum_{i+r+s=n} f_i(Y)d_r(X)d_s(X)+2f_n(XYX).
\end{aligned}$$

Comparing the above two equations and using the torsion restrictions, we obtain

$$f_n(XYX)=\sum_{i+j+k=n} f_i(X)d_j(Y)d_k(X).$$

(iii) Replace X by $X+Z$ in (ii)

$$\begin{aligned}
f_n\big((X+Z)Y(X+Z)\big) &= \sum_{i+j+k=n} f_i(X+Z)d_j(Y)d_k(X+Z) \\
&= \sum_{i+j+k=n}\big(f_i(X)d_j(Y)d_k(X)+f_i(Z)d_j(Y)d_k(X)\big) \\
&\quad + \sum_{i+j+k=n}\big(f_i(X)d_j(Y)d_k(Z)+f_i(Z)d_j(Y)d_k(Z)\big).
\end{aligned}$$

On the other hand using (ii)

$$\begin{aligned} f_n\big((X+Z)Y(X+Z)\big) &= f_n(XYZ+ZYX)+f_n(XYX)+f_n(ZYZ)\\ &= f_n(XYZ+ZYX)+\sum_{i+j+k=n} f_i(X)d_j(Y)d_k(X)\\ &\quad+\sum_{i+j+k=n} f_i(Z)d_j(Y)d_k(Z). \end{aligned}$$

These two equations imply that (iii) is true, completing the proof of this lemma. □

Lemma 7. *Let $\mathfrak{A}$ be a triangular algebra and $\mathfrak{L}$ be a Lie ideal of $\mathfrak{A}$. If $\xi_m(X,Y)=0$ for each $m<n$ and $X,Y\in\mathfrak{L}$, then $\xi_n(X,Y)Z[X,Y]=0$ for all $Z\in\mathfrak{L}$.*

Proof. Let $X,Y\in\mathfrak{L}$. By hypothesis $\xi_m(X,Y)=0$ for each $m<n$. Therefore $\xi_m(X,Y)Z[X,Y]=0$ for all $Z\in\mathfrak{L}$ and for all $m<n$. Suppose

$$\chi = 4(XYZYX+YXZXY)=(2XY)Z(2YX)+(2YX)Z(2XY),$$

where $X,Y,Z\in\mathfrak{L}$. Then by Lemma 6 (ii) we have

$$\begin{aligned} f_n(\chi) &= 4f_n(X(YZY)X+Y(XZX)Y)\\ &= 4\sum_{i+j+k=n}\big(f_i(X)d_j(YZY)d_k(X)+f_i(Y)d_j(XZX)d_k(Y)\big)\\ &= 4\sum_{i+j+k=n} f_i(X)\Big(\sum_{l+t+u=j} d_l(Y)d_t(Z)d_u(Y)\Big)d_k(X)\\ &\quad+4\sum_{i+j+k=n} f_i(Y)\Big(\sum_{l+t+u=j} d_l(X)d_t(Z)d_u(X)\Big)d_k(Y)\\ &= 4\sum_{i+l+t+u+k=n} f_i(X)d_l(Y)d_t(Z)d_u(Y)d_k(X)\\ &\quad+4\sum_{i+l+t+u+k=n} f_i(Y)d_l(X)d_t(Z)d_u(X)d_k(Y). \end{aligned}$$

However, by Lemma 6 (iii)

$$\begin{aligned} f_n(\chi) &= f_n\big((2XY)Z(2YX)+(2YX)Z(2XY)\big)\\ &= \sum_{p+q+r=n}\Big(f_p(2XY)d_q(Z)d_r(2YX)+f_p(2YX)d_q(Z)d_r(2XY)\Big). \end{aligned}$$

Equating the above two expressions, we obtain

$$\sum_{p+q+r=n} f_p(2XY)d_q(Z)d_r(2YX) - 4\sum_{i+l+t+u+k=n} f_i(X)d_l(Y)d_t(Z)d_u(Y)d_k(X) + \sum_{p+q+r=n} f_p(2YX)d_q(Z)d_r(2XY) - 4\sum_{i+l+t+u+k=n} f_i(Y)d_l(X)d_t(Z)d_u(X)d_k(Y) = 0. \tag{1}$$

Initially we calculate the first term of the above expression,

$$\begin{aligned}
&\sum_{p+q+r=n} f_p(2XY)d_q(Z)d_r(2YX)\\
&= \sum_{p+r=n} f_p(2XY)Zd_r(2YX) + \sum_{p+r=n-1} f_p(2XY)d_1(Z)d_r(2YX)\\
&\quad + \cdots + \sum_{p+r=1} f_p(2XY)d_{n-1}(Z)d_r(2YX) + 2XYd_n(Z)2YX\\
&= 2XYZd_n(2YX) + f_n(2XY)Z(2YX) + \sum_{\substack{p+r=n\\ 0<p,r\leq n-1}} f_p(2XY)Zd_r(2YX)\\
&\quad + \sum_{p+r=n-1} f_p(2XY)d_1(Z)d_r(2YX) + \cdots + 2XYd_{n-1}(Z)d_1(2YX)\\
&\quad + f_1(2XY)d_{n-1}(Z)2YX + 2XYd_n(Z)2YX.
\end{aligned}$$

Now by the hypothesis $\xi_m(X,Y) = 0$ for all $m < n$, we get

$$\begin{aligned}
&\sum_{p+q+r=n} f_p(2XY)d_q(Z)d_r(2YX)\\
&= 4\Big\{XYZd_n(YX) + f_n(XY)ZYX + \sum_{\substack{p+r=n\\ 0<p,r\leq n-1}} \Big(\sum_{i+j=p} f_i(X)d_j(Y)Z \sum_{u+k=r} d_u(Y)d_k(X)\Big)\\
&\quad + \cdots + XYd_{n-1}(Z)Yd_1(X) + XYd_{n-1}(Z)d_1(Y)X + f_1(X)Yd_{n-1}(Z)YX\\
&\quad + Xd_1(Y)d_{n-1}(Z)YX + XYd_n(Z)YX\Big\}\\
&= 4\Big\{XYZd_n(YX) + f_n(XY)ZYX + \sum_{\substack{i+j+u+k=n\\ 0<i+j,u+k\leq n-1}} f_i(X)d_j(Y)Zd_u(Y)d_k(X)\\
&\quad + \cdots + XYd_{n-1}(Z)Yd_1(X) + XYd_{n-1}(Z)d_1(Y)X + f_1(X)Yd_{n-1}(Z)YX\\
&\quad + Xd_1(Y)d_{n-1}(Z)YX + XYd_n(Z)YX\Big\}.
\end{aligned}$$

Also the second term of (1) becomes

$$
\begin{aligned}
4 \sum_{i+l+t+u+k=n} & f_i(X)d_l(Y)d_t(Z)d_u(Y)d_k(X) \\
&= 4\Big\{ \sum_{i+l+u+k=n} f_i(X)d_l(Y)Zd_u(Y)d_k(X) \\
&\quad + \cdots + \sum_{i+l+u+k=1} f_i(X)d_l(Y)d_{n-1}(Z)d_u(Y)d_k(X) + XYd_n(Z)YX \Big\} \\
&= 4\Big\{ \sum_{i+l=n} f_i(X)d_l(Y)ZYX + \sum_{u+k=n} XYZd_u(Y)d_k(X) \\
&\quad + \sum_{\substack{i+j+u+k=n \\ 0<i+j,u+k\leq n-1}} f_i(X)d_l(Y)Zd_u(Y)d_k(X) \\
&\quad + \cdots + \sum_{i+l=1} f_i(X)d_l(Y)d_{n-1}(Z)YX + \sum_{u+k=1} XYd_{n-1}(Z)d_u(Y)d_k(X) \\
&\quad + XYd_n(Z)YX \Big\} \\
&= 4\Big\{ \sum_{i+l=n} f_i(X)d_l(Y)ZYX + XYZd_n(YX) \\
&\quad + \sum_{\substack{i+j+u+k=n \\ 0<i+j,u+k\leq n-1}} f_i(X)d_l(Y)Zd_u(Y)d_k(X) \\
&\quad + \cdots + f_1(X)Yd_{n-1}(Z)YX + Xd_1(Y)d_{n-1}(Z)YX \\
&\quad + XYd_{n-1}(Z)d_1(Y)X + XYd_{n-1}(Z)Yd_1(X) + XYd_n(Z)YX \Big\}.
\end{aligned}
$$

Calculating the difference between the first two terms of (1) and using Lemma 5 we get;

$$
\begin{aligned}
\sum_{p+q+s=n} f_p(2XY)d_q(Z)d_r(2YX) - 4 \sum_{i+l+t+u+k=n} & f_i(X)d_l(Y)d_t(Z)d_u(Y)d_k(X) \\
&= \xi_n(X,Y)ZYX.
\end{aligned}
$$

With entirely analogous reasoning we can verify that

$$
\begin{aligned}
\sum_{p+q+s=n} f_p(2YX)d_q(Z)d_r(2XY) - 4 \sum_{i+l+t+u+k=n} & f_i(Y)d_l(X)d_t(Z)d_u(X)d_k(Y) \\
&= \xi_n(Y,X)ZXY.
\end{aligned}
$$

Therefore, from equation (1) it follows that

$$\xi_n(X,Y)ZYX + \xi_n(Y,X)ZXY = 0.$$

Using the property $\xi_n(Y,X) = -\xi_n(X,Y)$, we have

$$\begin{aligned} 0 &= \xi_n(X,Y)ZYX - \xi_n(X,Y)ZXY \\ &= \xi_n(X,Y)Z(XY - YX) \\ &= \xi_n(X,Y)Z[X,Y]. \end{aligned}$$

□

Theorem 1. *Let $\mathfrak{A}$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are prime algebras and $\mathfrak{L}$ is an admissible Lie ideal of $\mathfrak{A}$. Then each GJHD of $\mathfrak{L}$ into $\mathfrak{A}$ is a GHD of $\mathfrak{L}$ into $\mathfrak{A}$.*

Proof. Use induction on n. Since $\xi_0(X,Y) = 0$ for $n = 0$, we may assume that $\xi_k(X,Y) = 0$ for all $k < n$. By Lemma 7

$$\xi_n(X,Y)Z[X,Y] = 0 \tag{2}$$

for all $X, Y, Z \in \mathfrak{L}$.

Assume $X = \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}$, $Y = \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}$, $Z = \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix}$ and $X, Y, Z \in \mathfrak{L}$. Then

$$[X,Y] = \begin{pmatrix} [a_1, a_2] & m_0 \\ 0 & [b_1, b_2] \end{pmatrix}$$

for some $m_0 \in \mathcal{M}'$. Suppose that

$$\xi_n(X,Y) = \begin{pmatrix} \xi_{na}(X,Y) & \xi_{nm}(X,Y) \\ 0 & \xi_{nb}(X,Y) \end{pmatrix},$$

where $\xi_{na}(X,Y) \in \mathcal{A}, \xi_{nm}(X,Y) \in \mathcal{M}, \xi_{nb}(X,Y) \in \mathcal{B}$. Then (2) can be written as

$$\begin{pmatrix} \xi_{na}(X,Y) & \xi_{nm}(X,Y) \\ 0 & \xi_{nb}(X,Y) \end{pmatrix} \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix} \begin{pmatrix} [a_1, a_2] & m_0 \\ 0 & [b_1, b_2] \end{pmatrix} = 0\,.$$

This implies that

$$\begin{pmatrix} \xi_{na}(X,Y)a_3[a_1, a_2] & \xi_{na}(X,Y)a_3m_0 + \big(\xi_{na}(X,Y)m_3 + \xi_{nm}(X,Y)b_3\big)[b_1, b_2] \\ 0 & \xi_{nb}(X,Y)b_3[b_1, b_2] \end{pmatrix} = 0\,.$$

This gives

$$\xi_{na}(X,Y)a_3[a_1, a_2] = 0\,, \tag{3}$$

$$\xi_{nb}(X,Y)b_3[b_1, b_2] = 0 \tag{4}$$

and

$$\xi_{na}(X,Y)a_3m_0 + (\xi_{na}(X,Y)m_3 + \xi_{nm}(X,Y)b_3)[b_1, b_2] = 0 \tag{5}$$

for all $X, Y \in \mathfrak{L}$ and $m_0 \in \mathcal{M}'$. Using (3) and Lemma 3, either $\xi_{na}(X,Y) = 0$ for all $X, Y \in \mathfrak{L}$ or $[a_1', a_2'] = 0$ for all $a_1', a_2' \in \mathfrak{L}_A$. Thus $\xi_{na}(X,Y) = 0$ for all $X, Y \in \mathfrak{L}$ by Lemma 4. Similarly, $\xi_{nb}(X,Y) = 0$ for all $X, Y \in \mathfrak{L}$. Thus (5) becomes

$$\xi_{nm}(X,Y)b_3[b_1, b_2] = 0$$

for all $X, Y \in \mathfrak{L}$.
We can write above relation as

$$\xi_{nm}(a_1, a_2, b_1, b_2, m_1, m_2)b_3[b_1, b_2] = 0. \tag{6}$$

By the assumption, we assume that the action of each $b_3 \neq 0$ in $\mathfrak{L}_B$ on each $m \in \mathcal{M}$ is nontrivial. Also by Lemma 4, there exist $b_1', b_2' \in \mathfrak{L}_B$ such that $[b_1', b_2'] \neq 0$. Since $\mathfrak{L}_B$ is prime ideal, it follows from (6) that

$$\xi_{nm}(a_1, a_2, b_1', b_2', m_1, m_2) = 0 \tag{7}$$

for all $a_1, a_2 \in \mathfrak{L}_A$ and $m_1, m_2 \in \mathcal{M}'$.

Replacing b_1 by $b_1 + b_1'$ in (6) we get

$$\xi_{nm}(a_1, a_2, b_1', b_2, m_1, m_2)b_3[b_1, b_2] + \xi_{nm}(a_1, a_2, b_1, b_2, m_1, m_2)b_3[b_1', b_2] = 0 \tag{8}$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1, b_2, b_3 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. Replace b_2 by $b_2 + b_2'$ in the above equation. Then

$$\begin{aligned}\xi_{nm}(a_1, a_2, b_1, b_2, m_1, m_2)b_3[b_1', b_2'] + \xi_{nm}(a_1, a_2, b_1, b_2', m_1, m_2)b_3[b_1', b_2]\\ + \xi_{nm}(a_1, a_2, b_1', b_2, m_1, m_2)b_3[b_1, b_2'] + \xi_{nm}(a_1, a_2, b_1', b_2', m_1, m_2)b_3[b_1, b_2] = 0\end{aligned} \tag{9}$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1, b_2, b_3 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. Replace b_1 by b_1' (9) and using (7) we obtain $\xi_{nm}(a_1, a_2, b_1', b_2, m_1, m_2)b_3[b_1', b_2'] = 0$, which gives

$$\xi_{nm}(a_1, a_2, b_1', b_2, m_1, m_2) = 0 \tag{10}$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_2, b_3 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. Again replace b_2 by b_2' in (8) and use (7) to get

$$\xi_{nm}(a_1, a_2, b_1, b_2', m_1, m_2)b_3[b_1', b_2'] = 0$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1, b_3 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. Hence,

$$\xi_{nm}(a_1, a_2, b_1, b_2', m_1, m_2) = 0 \tag{11}$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. Combining (7), (10) and (11), (9) we get

$$\xi_{nm}(a_1, a_2, b_1, b_2, m_1, m_2)b_3[b_1', b_2'] = 0$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1, b_2, b_3 \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. So

$$\xi_{nm}(a_1, a_2, b_1, b_2, m_1, m_2) = 0$$

for all $a_1, a_2 \in \mathfrak{L}_A$, $b_1, b_2, \in \mathfrak{L}_B$ and $m_1, m_2 \in \mathcal{M}'$. That is $\xi_{nm}(X, Y) = 0$ for all $X, Y \in \mathfrak{L}$. Thus,

$$\xi_n(X, Y) = \begin{pmatrix} \xi_{na}(X, Y) & \xi_{nm}(X, Y) \\ 0 & \xi_{nb}(X, Y) \end{pmatrix} = 0$$

for all $X, Y \in \mathfrak{L}$. This completes the proof of our theorem. □

3 Generalized Jordan triple higher derivations on $\mathfrak{L}$

In a recent paper, Y.S. Jung [15] improved a result due to Jing and Lu [14, Theorem 3.5] for generalized Jordan triple derivations to generalized Jordan triple higher derivations and established that if R is a 2-torsion-free prime ring then every generalized Jordan triple higher derivation on R is a generalized higher derivation. In the previous section, we have seen that each GJHD on an admissible Lie ideal of a triangular algebra is a GHD. In this section, we extend the result mentioned above for triangular algebras, which is a generalization of Han's result (see for ref. [12]).

Let $F = \{f_n\}_{n\in\mathbb{N}}$ be a GJTHD with associated JTHD $D = \{d_n\}_{n\in\mathbb{N}}$ of $\mathfrak{L}$ into $\mathfrak{A}$. Define

$$\rho_n(X,Y,Z) = f_n(XYZ) - \sum_{i+j+k=n} f_i(X)d_j(Y)d_k(Z)$$

for each fixed $n \in \mathbb{N}$ and for all $X, Y, Z \in \mathfrak{L}$. Note that $\rho_n(X,Y,Z)$ is additive in all arguments. It is straightforward to see that $\rho_n(X,Y,X) = 0$. Given $X, Y, Z \in \mathfrak{L}$, put $[X,Y,Z] = XYZ - ZYX$.

Lemma 8. *Let $\mathfrak{A}$ be a triangular algebra. For all $X, Y, Z \in \mathfrak{L}$ and $n \in \mathbb{N}$, we have*

$$f_n(XYZ + ZYX) = \sum_{i+j+k=n} f_i(X)d_j(Y)d_k(Z) + \sum_{i+j+k=n} f_i(Z)d_j(Y)d_k(X).$$

Proof. This follows immediately from the relation $\rho_n(X+Z, Y, X+Z) = 0$. □

Lemma 9. *For all $X, Y, Z \in \mathfrak{L}$ and $n \in \mathbb{N}$, we have $\rho_n(X,Y,Z) = -\rho_n(Z,Y,X)$.*

Proof. Easily follows from Lemma 8. □

The proof of the following lemma can be seen in [8].

Lemma 10 ([8], Lemma 2.4). *Assume that R is a prime ring of characteristic other than 2 and $L \nsubseteq Z(R)$. Then there exist $a, b, c \in L$ such that $[a,b,c] \neq 0$.*

Now we prove the following lemma.

Lemma 11. *For $n \in \mathbb{N}$, if $\rho_n(X,Y,Z) = 0$ for all $k < n$ and $X, Y, Z \in \mathfrak{L}$. Then*

$$\rho_n(X,Y,Z)P[X,Y,Z] = 0$$

for all $X, Y, Z, P \in \mathfrak{L}$.

Proof. We assume that F is a GJTHD from $\mathfrak{L}$ into $\mathfrak{A}$. Take $X, Y, Z, P \in \mathfrak{L}$ and put

$$\eta = XYZPZYX + ZYXPXYZ = (XYZ)P(ZYX) + (ZYX)P(XYZ)\,.$$

Then

$$\begin{aligned} 2^4\eta = 2^4(XYZPZYX + ZYXPXYZ) &= X(4Y(4ZPZ)Y)X + Z(4Y(4XPX)Y)Z \\ &= (4XYZ)P(4ZYX) + (4ZYX)P(4XYZ)\,, \end{aligned}$$

where $4ZPZ$, $4Y(4ZPZ)Y$, $4XPX$, $4Y(4XPX)Y$, $4XYZ$, $4ZYX$ are in $\mathfrak{L}$. Since $\mathfrak{A}$ is a triangular algebra consisting of 2-torsion free algebras $\mathcal{A}$, $\mathcal{B}$ and a 2-torsion free module $\mathcal{M}$, the relations which will be obtained in the first part remain valid. By definition of GJTHD we have

$$\begin{aligned} f_n(\eta) &= f_n(X(YZPZY)X + Z(YXPXY)Z) \\ &= \sum_{i+j+k=n} \big(f_i(X)d_j(YZPZY)d_k(X) + f_i(Z)d_j(YXPXY)d_k(Z)\big) \\ &= \sum_{\upsilon=n} f_i(X)d_l(Y)d_s(Z)d_p(P)d_q(Z)d_u(Y)d_k(X) \\ &\quad + \sum_{\upsilon=n} f_i(Z)d_l(Y)d_s(X)d_p(P)d_q(X)d_u(Y)d_k(Z) \end{aligned}$$

where $\upsilon = i + l + s + p + q + u + k$.

On the other hand we have

$$\begin{aligned} f_n(\eta) &= f_n((XYZ)P(ZYX) + (ZYX)P(XYZ)) \\ &= \sum_{\alpha+\beta+\gamma=n} \Big(f_\alpha(XYZ)d_\beta(P)d_\gamma(ZYX) + f_\alpha(ZYX)d_\beta(P)d_\gamma(XYZ)\Big)\,. \end{aligned}$$

Equalizing the above two expressions, we obtain

$$\begin{aligned} &\sum_{\alpha+\beta+\gamma=n} f_\alpha(XYZ)d_\beta(P)d_\gamma(ZYX) - \sum_{\upsilon=n} f_i(X)d_l(Y)d_s(Z)d_p(P)d_q(Z)d_u(Y)d_k(X) \\ + &\sum_{\alpha+\beta+\gamma=n} f_\alpha(ZYX)d_\beta(P)d_\gamma(XYZ) - \sum_{\upsilon=n} f_i(Z)d_l(Y)d_s(X)d_p(P)d_q(X)d_u(Y)d_k(Z) = 0. \end{aligned}$$

Initially we calculate the first installment using the hypothesis $\rho_n(X, Y, Z) = 0$ for all $k < n$, we get

$$\begin{aligned} &\sum_{\alpha+\beta+\gamma=n} f_\alpha(XYZ)d_\beta(P)d_\gamma(ZYX) \\ &= XYZ \sum_{\beta+\gamma=n} d_\beta(P)d_\gamma(ZYX) + \sum_{i+j+k=1} f_i(X)d_j(Y)d_k(Z) \sum_{\beta+\gamma=n-1} d_\beta(P)d_\gamma(ZYX) \\ &\quad + \sum_{i+j+k=2} f_i(X)d_j(Y)d_k(Z) \sum_{\beta+\gamma=n-2} d_\beta(P)d_\gamma(ZYX) + \dots \\ &\quad + \sum_{i+j+k=n-1} f_i(X)d_j(Y)d_k(Z) \sum_{\beta+\gamma=1} d_\beta(P)d_\gamma(ZYX) \\ &\quad + f_n(XYZ)PZYX\,. \end{aligned}$$

Also we have

$$\sum_{\upsilon=n} f_i(X)d_l(Y)d_s(Z)d_p(P)d_q(Z)d_u(Y)d_k(X)$$
$$= XYZ \sum_{p+q+u+k=n} d_p(P)d_q(Z)d_u(Y)d_k(X)$$
$$+ \sum_{i+l+s=1} f_i(X)d_l(Y)d_s(Z) \sum_{p+q+u+k=n-1} d_p(P)d_q(Z)d_u(Y)d_k(X)$$
$$+ \cdots + \sum_{i+l+s=n} f_i(X)d_l(Y)d_s(Z)PZYX\,.$$

Calculating the difference between the first two installments, we get

$$\sum_{\alpha+\beta+\gamma=n} f_\alpha(XYZ)d_\beta(P)d_\gamma(ZYX) - \sum_{\upsilon=n} f_i(X)d_l(Y)d_s(Z)d_p(P)d_q(Z)d_u(Y)d_k(X)$$
$$= f_n(XYZ)PZYX - \sum_{i+l+s=n} f_i(X)d_l(Y)d_s(Z)PZYX$$
$$= \rho_n(X,Y,Z)PZYX\,.$$

With entirely analogous reasoning, we can verify that

$$\sum_{\alpha+\beta+\gamma=n} f_\alpha(ZYX)d_\beta(P)d_\gamma(XYZ) - \sum_{\upsilon=n} f_i(W)d_l(Y)d_s(X)d_p(P)d_q(X)d_u(Y)d_k(Z)$$
$$= \rho_n(Z,Y,X)PXYZ\,.$$

Therefore by combining, we have

$$\rho_n(X,Y,Z)PZYX + \rho_n(Z,Y,X)PXYZ = 0.$$

Using Lemma 9, we get

$$\rho_n(X,Y,Z)P[X,Y,Z] = 0.$$

Hence the proof is the same. □

Lemma 12. *Let $\mathfrak{A}$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are prime algebras and $\mathfrak{L}$ is an admissible Lie ideal of $\mathfrak{A}$. Then $\rho_n(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$.*

Proof. Use induction on n. Since $\rho_0(X,Y,Z) = 0$ for $n = 0$, we may assume that $\rho_k(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$ and $k < n$. By Lemma 11

$$\rho_n(X,Y,Z)P4[X,Y,Z] = 0 \tag{12}$$

for all $X,Y,Z,P \in \mathcal{U}$. Let

$$X = \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}, \quad Y = \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}, \quad Z = \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix}, \quad P = \begin{pmatrix} a_4 & m_4 \\ 0 & b_4 \end{pmatrix}.$$

Then

$$4[X,Y,Z] = \begin{pmatrix} 4[a_1,a_2,a_3] & 4m_0 \\ 0 & 4[b_1,b_2,b_3] \end{pmatrix};$$

for some $4m_0 \in \mathcal{M}'$. Assume that

$$\rho_n(X,Y,Z) = \begin{pmatrix} \rho_{na}(X,Y,Z) & \rho_{nm}(X,Y,Z) \\ 0 & \rho_{nb}(X,Y,Z) \end{pmatrix},$$

where $\rho_{na}(X,Y,Z) \in \mathcal{A}$, $\rho_{nm}(X,Y,Z) \in \mathcal{M}$, $\rho_{nb}(X,Y,Z) \in \mathcal{B}$. Then (12) can be written as

$$\begin{pmatrix} \rho_{na}(X,Y,Z) & \rho_{nm}(X,Y,Z) \\ 0 & \rho_{nb}(X,Y,Z) \end{pmatrix} \begin{pmatrix} a_4 & m_4 \\ 0 & b_4 \end{pmatrix} \begin{pmatrix} 4[a_1,a_2,a_3] & 4m_0 \\ 0 & 4[b_1,b_2,b_4] \end{pmatrix} = 0.$$

This gives,

$$\rho_{na}(X,Y,Z)a_4 4[a_1,a_2,a_3] = 0\,, \quad (13)$$

$$\rho_{nb}(X,Y,Z)b_4 4[b_1,b_2,b_3] = 0\,, \quad (14)$$

$$\rho_{na}(X,Y,Z)a_4 4m_0 + \big(\rho_{na}(X,Y,Z)m_4 + \rho_{nm}(X,Y,Z)b_4\big)4[b_1,b_2,b_3] = 0 \quad (15)$$

for all $X,Y,Z,P \in \mathfrak{L}$ and $m_0 \in \mathcal{M}'$. Using (13) and Lemma 3, either $\rho_{na}(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$ or $[a_1',a_2',a_3'] = 0$ for all $a_1',a_2',a_3' \in \mathfrak{L}_A$. Thus $\rho_{na}(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$ by Lemma 10. Similarly, $\rho_{nb}(X,Y,Z) = 0$ for all $X,Y,Z \in \mathfrak{L}$. Thus (15) becomes

$$\rho_{nm}(X,Y,Z)b_4[b_1,b_2,b_3] = 0$$

for all $X,Y,Z \in \mathfrak{L}$ and $b_4 \in \mathfrak{L}_B$, or we can write it as

$$\rho_{nm}(a_1,a_2,a_3,b_1,b_2,b_3,m_1,m_2,m_3)b_4[b_1,b_2,b_3] = 0 \quad (16)$$

for all $a_1,a_2,a_3 \in \mathfrak{L}_A$, $b_1,b_2,b_3,b_4 \in \mathfrak{L}_B$, $m_1,m_2,m_3 \in \mathcal{M}'$. By Lemma 10, there exists $b_1',b_2',b_3' \in \mathfrak{L}_B$ such that $[b_1',b_2',b_3'] \neq 0$. which gives

$$\rho_{nm}(a_1,a_2,a_3,b_1',b_2',b_3',m_1,m_2,m_3) = 0 \quad (17)$$

for all $a_1,a_2,a_3 \in \mathfrak{L}_A$ and $m_1,m_2,m_3 \in \mathcal{M}'$. Replacing b_1 by $b_1 + b_1'$ in (16) we get

$$\rho_{nm}(a_1,a_2,a_3,b_1',b_2,b_3,m_1,m_2,m_3)b_4[b_1,b_2,b_3] + \rho_{nm}(a_1,a_2,a_3,b_1,b_2,b_3,m_1,m_2,m_3)b_4[b_1',b_2,b_3] = 0 \quad (18)$$

for all $a_1,a_2,a_3 \in \mathfrak{L}_A$, $b_1,b_2,b_3,b_4 \in \mathfrak{L}_B$ and $m_1,m_2,m_3 \in \mathcal{M}'$. Replacing b_2 by $b_2 + b_2'$ in (18)

$$\begin{aligned}\rho_{nm}&(a_1,a_2,a_3,b_1,b_2,b_3,m_1,m_2,m_3)b_4[b_1',b_2',b_3'] \\ &+ \rho_{nm}(a_1,a_2,a_3,b_1,b_2',b_3,m_1,m_2,m_3)b_4[b_1',b_2,b_3] \\ &+ \rho_{nm}(a_1,a_2,a_3,b_1',b_2',b_3,m_1,m_2,m_3)b_4[b_1,b_2,b_3] \\ &+ \rho_{nm}(a_1,a_2,a_3,b_1',b_2,b_3,m_1,m_2,m_3)b_4[b_1,b_2',b_3] = 0 \quad (19)\end{aligned}$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2, b_3, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Replacing b_3 by $b_3 + b_3'$ in (19) we get

$$\begin{aligned}\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3', m_1, m_2, m_3)b_4[b_1', b_2', b_3]\\ + \rho_{nm}(a_1, a_2, a_3, b_1, b_2', b_3', m_1, m_2, m_3)b_4[b_1', b_2, b_3]\\ + \rho_{nm}(a_1, a_2, a_3, b_1', b_2', b_3', m_1, m_2, m_3)b_4[b_1, b_2, b_3]\\ + \rho_{nm}(a_1, a_2, a_3, b_1', b_2, b_3', m_1, m_2, m_3)b_4[b_1, b_2', b_3]\\ + \rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)b_4[b_1', b_2', b_3']\\ + \rho_{nm}(a_1, a_2, a_3, b_1, b_2', b_3, m_1, m_2, m_3)b_4[b_1', b_2, b_3']\\ + \rho_{nm}(a_1, a_2, a_3, b_1', b_2', b_3, m_1, m_2, m_3)b_4[b_1, b_2, b_3']\\ + \rho_{nm}(a_1, a_2, a_3, b_1', b_2, b_3, m_1, m_2, m_3)b_4[b_1, b_2', b_3'] = 0 \quad (20)\end{aligned}$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Replacing b_1 by b_1' in (20) and using (17) we obtain

$$2\rho_{nm}(a_1, a_2, a_3, b_1', b_2, b_3', m_1, m_2, m_3)b_4[b_1', b_2', b_3'] = 0 \quad (21)$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_2, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Again replacing b_2 by b_2' and b_3 by b_3' in (18) and using (17) we obtain

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2', b_3', m_1, m_2, m_3)b_4[b_1', b_2', b_3'] = 0$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Hence,

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2', b_3', m_1, m_2, m_3) = 0 \quad (22)$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Combining (20)–(22), we obtain

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3', m_1, m_2, m_3)b_4[b_1', b_2', b_3'] = 0.$$

which implies that

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3', m_1, m_2, m_3) = 0$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Similarly we get,

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2', b_3, m_1, m_2, m_3) = 0 \quad (23)$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. We can also have

$$\rho_{nm}(a_1, a_2, a_3, b_1', b_2, b_3, m_1, m_2, m_3) = 0 \quad (24)$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_2, b_3 \in \mathfrak{L}_A$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Take (23) and (24) into (19), we have

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)b_4[b_1', b_2', b_3] = 0 \quad (25)$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2, b_3, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. Replacing b_3 by $b_3 + b_3'$ in (25), we obtain

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)b_4[b_1', b_2', b_3'] = 0$$

for all $a_1, a_2, a_3 \in \mathfrak{L}_A$, $b_1, b_2, b_3, b_4 \in \mathfrak{L}_B$ and $m_1, m_2, m_3 \in \mathcal{M}'$. This implies

$$\rho_{nm}(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3) = 0$$

for all $X, Y, Z \in \mathfrak{L}$. Thus we have

$$\rho_n(X,Y,Z) = \begin{pmatrix} \rho_{na}(X,Y,Z) & \rho_{nm}(X,Y,Z) \\ 0 & \rho_{nb}(X,Y,Z) \end{pmatrix} = 0$$

for all $X, Y, Z \in \mathfrak{L}$. □

Theorem 2. *Let $\mathfrak{A}$ be a triangular algebra consisting of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are prime algebras and $\mathfrak{L}$ is an admissible Lie ideal of $\mathfrak{A}$. Then each GJTHD of $\mathfrak{L}$ into $\mathfrak{A}$ is a GHD of $\mathfrak{L}$ into $\mathfrak{A}$.*

Proof. We assume by induction $f_k(XY) = \sum_{i+j=k} f_i(X)d_j(Y)$ for all $X, Y \in \mathfrak{L}$ and $k < n$. Suppose $\chi = X(YPX)Y = (XY)P(XY)$ for all $X, Y, P \in \mathfrak{L}$. Using Lemma 12, we have

$$\begin{aligned} f_n(\chi) = f_n(X(YPX)Y) &= \sum_{i+j+k=n} f_i(X)d_j(YPX)d_k(Y) \\ &= \sum_{i+j+k=n} f_i(X) \sum_{l+m+s=j} d_l(Y)d_m(P)d_s(X)d_k(Y) \\ &= \sum_{i+l+m+s+k=n} f_i(X)d_l(Y)d_m(P)d_s(X)d_k(Y) \\ &= XY \sum_{m+s+k=n} d_m(P)d_s(X)d_k(Y) \\ &\quad + \sum_{i+l=1} f_i(X)d_l(Y) \sum_{m+s+k=n-1} d_m(P)d_s(X)d_k(Y) \\ &\quad + \sum_{i+l=2} f_i(X)d_l(Y) \sum_{m+s+k=n-2} d_m(P)d_s(X)d_k(Y) + \cdots \\ &\quad + \sum_{i+l=n-1} f_i(X)d_l(Y) \sum_{m+s+k=1} d_m(P)d_s(X)d_k(Y) + \sum_{i+l=n} f_i(X)d_l(Y)PXY. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} f_n(\chi) = f_n\big((XY)P(XY)\big) &= \sum_{i+j+k=n} f_i(XY)d_j(P)d_k(XY) \\ &= f_n(XY)PXY + f_{n-1}(XY) \sum_{j+k=1} d_j(P)d_k(XY) \\ &\quad + f_{n-2}(XY) \sum_{j+k=2} d_j(P)d_k(XY) + \cdots + f_2(XY) \sum_{j+k=n-2} d_j(P)d_k(XY) \\ &\quad + f_1(XY) \sum_{j+k=n-1} d_j(P)d_k(XY) + XY \sum_{j+k=n} d_j(P)d_k(XY). \end{aligned}$$

Comparing the above two equations, we get

$$f_n(XY)PXY - \sum_{i+l=n} f_i(X)d_l(Y)PXY = 0.$$

This can also be written as

$$\Big(f_n(XY) - \sum_{i+l=n} f_i(X)d_l(Y)\Big)PXY = 0 \tag{26}$$

for all $X, Y, P \in \mathfrak{L}$.

We define $\Upsilon_n(X,Y) = f_n(XY) - \sum\limits_{i+l=n} f_i(X)d_l(Y)$ and assume

$$\Upsilon_n(X,Y) = \begin{pmatrix} \Upsilon_{na}(X,Y) & \Upsilon_{nm}(X,Y) \\ 0 & \Upsilon_{nb}(X,Y) \end{pmatrix},$$

where $\Upsilon_{na}(X,Y) \in \mathcal{A}$, $\Upsilon_{nm}(X,Y) \in \mathcal{M}$ and $\Upsilon_{nb}(X,Y) \in \mathcal{B}$. So (26) can be written as

$$\Upsilon_n(X,Y)PXY = 0$$

for all $X, Y, P \in \mathfrak{L}$. Then by similar induction as used in Theorem 1 we claim that $\Upsilon_n(X,Y) = 0$. The proof is complete in this case. □

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