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A note on the solutions of a second-order evolution inclusion in non separable Banach spaces

Aurelian Cernea

Abstract. We consider a Cauchy problem associated to a second-order evolution inclusion in non separable Banach spaces under Filippov type assumptions and we prove the existence of mild solutions.

Keywords: Lusin measurable multifunctions; differential inclusion; selection

Classification: 34A60

1. Introduction

In this note we study second-order evolution inclusions of the form

\[ x''(t) \in A(t)x(t) + F(t, x(t)), \quad x(0) = x_0, \quad x'(0) = y_0, \]

where \( F : [0, T] \times X \to P(X) \) is a set-valued map, \( X \) is a separable Banach space, \( x_0, y_0 \in X \) and \( \{A(t)\}_{t \geq 0} \) is a family of linear closed operators from \( X \) into \( X \) that generates an evolution system of operators \( \{U(t, s)\}_{t, s \in [0, T]} \). The general framework of evolution operators \( \{A(t)\}_{t \geq 0} \) that define problem (1.1) has been developed by Kozak ([9]) and improved by Henriquez ([7]).

The present paper is motivated by several recent papers ([1]–[3], [8], [9]) where existence results and qualitative properties of mild solutions for problem (1.1) have been obtained by using fixed point techniques. All these approaches are obtained provided that the Banach space \( X \) is separable.

De Blasi and Pianigiani ([5]) established the existence of mild solutions for semilinear differential inclusions on an arbitrary, not necessarily separable, Banach space \( X \). Even the results in [5] are based on Filippov’s ideas ([6]), the approach in [6] has a fundamental difference which consists in the construction of the measurable selections of the multifunction. This construction does not use classical selection theorems as Kuratowsky and Ryll-Nardzewski ([10]) or Bressan and Colombo ([4]).

In the present paper we obtain an existence result for problem (1.1) similar to the one in [5]. We will prove the existence of solutions for problem (1.1) in an arbitrary space \( X \) under assumptions on \( F \) of Filippov type.
The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. Preliminaries

Let us denote by $I$ the interval $[0, T]$, $T > 0$ and let $X$ be a real Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot, \cdot)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(\cdot) : I \to X$ endowed with the norm $|x(\cdot)|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(\cdot) : I \to X$ endowed with the norm $|x(\cdot)|_1 = \int_0^T |x(t)| dt$. By $B(X)$ we denote the Banach space of linear bounded operators on $X$.

Let $\mathcal{P}(X)$ be the space of all bounded nonempty subsets of $X$ endowed with the Hausdorff pseudometric

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup_{a \in A} d(a, B),$$

where $d(a, X) = \inf_{a \in A} |x - a|, A \subset X, x \in X$.

Let $\mathcal{L}$ be the $\sigma$-algebra of the (Lebesgue) measurable subsets of $R$ and, for $A \in \mathcal{L}$, let $\mu(A)$ be the Lebesgue measure of $A$.

Let $X$ be a Banach space and $Y$ be a metric space. An open (resp. closed) ball in $Y$ with center $y$ and radius $r$ is denoted by $B_Y(y, r)$ (resp. $\overline{B}_Y(y, r)$). In what follows $B = B_X(0, 1)$.

A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be $d_H$-continuous at $y_0 \in Y$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_Y(y_0, \delta)$ we have $d_H(F(y), F(y_0)) \leq \varepsilon$. $F$ is called $d_H$-continuous if it is so at each point $y_0 \in Y$.

Let $A \in \mathcal{L}$ with $\mu(A) < \infty$. A multifunction $F : Y \to \mathcal{P}(X)$ with closed bounded nonempty values is said to be \textit{Lusin measurable} if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$ with $\mu(A \setminus K_\varepsilon) < \varepsilon$ such that $F$ restricted to $K_\varepsilon$ is $d_H$-continuous.

It is clear that if $F, G : A \to \mathcal{P}(X)$ and $f : A \to X$ are Lusin measurable then so are $F$ restricted to $B$ ($B \subset A$ measurable), $F + G$ and $t \to d(f(t), F(t))$. Moreover, the uniform limit of a sequence of Lusin measurable multifunctions is also Lusin measurable.

In what follows $\{A(t)\}_{t \geq 0}$ is a family of linear closed operators from $X$ into $X$ that generates an evolution system of operators $\{U(t, s)\}_{t, s \in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in $X$ and is independent of $t$.

**Definition 2.1** ([7], [9]). A family of bounded linear operators $U(t, s) : X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t)$$

if the following conditions hold:

(i) for any $x \in X$, the map $(t, s) \to U(t, s)x$ is continuously differentiable and
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(2.2) \( U(t) = 0, t \in I \);

(2.3) \( U(t) \) is continuous;

(i) if \( t, s \in \Delta \), then \( \frac{\partial^2}{\partial s^2} U(t, s) x \in D(A(t)), \) the map \( (t, s) \rightarrow U(t, s) x \) is of class \( C^2 \) and

(2.4) \( \frac{\partial^3}{\partial t^3} U(t, s) x = A(t) U(t, s) x; \)

(2.5) \( \frac{\partial^3}{\partial s^3} U(t, s) x = A(t) \frac{\partial^3}{\partial s^2} U(t, s) x; \)

(2.6) \( \frac{\partial^3}{\partial t^3} U(t, s) x = 0. \)

(ii) If \( (t, s) \in \Delta \), then there exist \( \frac{\partial^3}{\partial t^3} U(t, s) x \), \( \frac{\partial^3}{\partial s^3} U(t, s) x \) and

(a) \( \frac{\partial^3}{\partial t^3} U(t, s) x = A(t) \frac{\partial^3}{\partial s^2} U(t, s) x \) and the map \( (t, s) \rightarrow A(t) \frac{\partial^3}{\partial s^2} U(t, s) x \) is continuous;

(b) \( \frac{\partial^3}{\partial s^3} U(t, s) x = \frac{\partial^3}{\partial t^3} U(t, s) A(s) x. \)

As an example for equation (2.1) one may consider the problem (e.g. [7])

\[
\frac{\partial^2}{\partial t^2} z(t, \tau) = \frac{\partial^2}{\partial \tau^2} z(t, \tau) + a(t) \frac{\partial}{\partial \tau} z(t, \tau), \quad t \in [0, T], \tau \in [0, 2\pi],
\]

\[
z(t, 0) = z(t, \Pi) = 0, \quad \frac{\partial z}{\partial \tau}(t, 0) = \frac{\partial z}{\partial \tau}(t, 2\pi), \quad t \in [0, T],
\]

where \( a(\cdot) : I \rightarrow \mathbb{R} \) is a continuous function. This problem is modeled in the space \( X = L^2(\mathbb{R}, \mathbb{C}) \) of \( 2\pi \)-periodic 2-integrable functions from \( \mathbb{R} \) to \( \mathbb{C} \), \( A_1 z = \frac{d^2 z(\tau)}{d\tau^2} \) with domain \( H^2(\mathbb{R}, \mathbb{C}) \), the Sobolev space of \( 2\pi \)-periodic functions whose derivatives belong to \( L^2(\mathbb{R}, \mathbb{C}) \). It is well known that \( A_1 \) is the infinitesimal generator of strongly continuous cosine functions \( C(t) \) on \( X \). Moreover, \( A_1 \) has discrete spectrum; namely the spectrum of \( A_1 \) consists of eigenvalues \(-n^2\), \( n \in \mathbb{Z} \) with associated eigenvectors \( z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}, n \in \mathbb{N} \). The set \( z_n, n \in \mathbb{N} \) is a normal basis of \( X \). In particular, \( A_1 z = \sum_{n \in \mathbb{Z}} -n^2 \langle z, z_n \rangle z_n, z \in D(A_1) \). The cosine function is given by \( C(t) z = \sum_{n \in \mathbb{Z}} \cos(nt) \langle z, z_n \rangle z_n \) with the associated sine function \( S(t) z = t \langle z, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z}} \frac{\sin(nt)}{n} \langle z, z_n \rangle z_n \).

For \( t \in I \) define the operator \( A_2(t) z = a(t) \frac{d^2 z(\tau)}{d\tau^2} \) with domain \( D(A_2(t)) = H^1(\mathbb{R}, \mathbb{C}) \). Set \( A(t) = A_1 + A_2(t) \). It has been proved in [7] that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. A continuous mapping \( x(\cdot) \in C(I, X) \) is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function \( f(\cdot) \in L^1(I, X) \) such that

(2.2) \( f(t) \in F(t, x(t)) \) a.e. \( (I), \)

(2.3) \( x(t) = -\frac{\partial}{\partial s} U(t, 0)x_0 + U(t, 0)y_0 + \int_0^t U(t, s)f(s)ds, \quad t \in I. \)

We shall call \( (x(\cdot), f(\cdot)) \) a trajectory-selection pair of (1.1) if \( f(\cdot) \) verifies (2.2) and \( x(\cdot) \) is defined by (2.3).
In what follows $X$ is a real Banach space and we assume the following hypotheses.

**Hypothesis 2.3.**

(i) There exists an evolution operator $\{U(t, s)\}_{t, s \in I}$ associated to the family $\{A(t)\}_{t \geq 0}$.

(ii) There exist $M, M_0 \geq 0$ such that $|U(t, s)|_{B(X)} \leq M$, $|\frac{\partial}{\partial s}U(t, s)| \leq M_0$, for all $(t, s) \in \Delta$.

(iii) $F(\cdot, \cdot): I \times X \to \mathcal{P}(X)$ has nonempty closed bounded values and, for any $x \in X$, $F(\cdot, x)$ is Lusin measurable on $I$.

(iv) There exists $l(\cdot) \in L^1(I, (0, \infty))$ such that, for each $t \in I$, 

$$d_H(F(t, x_1), F(t, x_2)) \leq l(t)|x_1 - x_2|, \quad \forall x_1, x_2 \in X.$$ 

(v) There exists $q(\cdot) \in L^1(I, (0, \infty))$ such that, for each $t \in I$, we have 

$$F(t, 0) \subset q(t)B.$$ 

Set $m(t) = \int_0^t l(u)du$, $t \in I$. The technical results summarized in the next lemma are essential in the proof of our result. For the proof we refer to [5].

**Lemma 2.4.**

(i) Let $F_i : I \to \mathcal{P}(X)$, $i = 1, 2$ be two Lusin measurable multifunctions and let $\varepsilon_i > 0$, $i = 1, 2$ be such that 

$$H(t) := (F_1(t) + \varepsilon_1B) \cap (F_2(t) + \varepsilon_2B) \neq \emptyset, \quad \forall t \in I.$$ 

Then the multifunction $H : I \to \mathcal{P}(X)$ has a Lusin measurable selection $h : I \to X$.

(ii) Assume that Hypothesis 2.1 is satisfied. Then for any $x(\cdot) : I \to X$ continuous, $u(\cdot) : I \to X$ measurable and $\varepsilon > 0$ we have 

(a) the multifunction $t \to F(t, x(t))$ is Lusin measurable on $I$;

(b) the multifunction $G : I \to \mathcal{P}(X)$ defined by 

$$G(t) := (F(t, x(t)) + \varepsilon B) \cap B_X(u(t), d(u(t), F(t, x(t)))) + \varepsilon$$ 

has a Lusin measurable selection $g : I \to X$.

3. Main result

We are ready now to prove our main result.

**Theorem 3.1.** We assume that Hypothesis 2.3 is satisfied. Then, for every $x_0, y_0 \in X$ the Cauchy problem (1.1) has a solution $x(\cdot) : I \to X$.

**Proof:** Let us note first that, if $z(\cdot) : I \to X$ is continuous, then every Lusin measurable selection $u : I \to X$ of the multifunction $t \to F(t, z(t)) + B$ is Bochner integrable on $I$. More exactly, for any $t \in I$ we have 

$$|u(t)| \leq d_H(F(t, z(t)) + B, 0) \leq d_H(F(t, z(t)), F(t, 0)) + d_H(F(t, 0), 0) + 1 \leq l(t)|z(t)| + q(t) + 1.$$
Let $0 < \varepsilon < 1$, $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$.

Consider $f_0(\cdot) : I \rightarrow X$ an arbitrary Lusin measurable function, Bochner integrable and define

$$x_0(t) = -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)y_0 + \int_0^t U(t,s)f_0(s)ds, \quad t \in I.$$ 

Since $x_0(\cdot)$ is continuous, by Lemma 2.4(ii) there exists a Lusin measurable function $f_1(\cdot) : I \rightarrow X$ satisfying, for $t \in I$,

$$f_1(t) \in (F(t,x_0(t)) + \varepsilon_1B) \cap B(f_0(t), d(f_0(t), F(t,x_0(t))) + \varepsilon_1)$$

Obviously, $f_1(\cdot)$ is Bochner integrable on $I$. Define $x_1(\cdot) : I \rightarrow X$ by

$$x_1(t) = -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)y_0 + \int_0^t U(t,s)f_1(s)ds, \quad t \in I.$$ 

By induction, we construct a sequence $x_n : I \rightarrow X$, $n \geq 2$ given by

$$x_n(t) = -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)y_0 + \int_0^t U(t,s)f_n(s)ds, \quad t \in I,$$

where $f_n(\cdot) : I \rightarrow X$ is a Lusin measurable function satisfying, for $t \in I$,

$$f_n(t) \in (F(t,x_{n-1}(t)) + \varepsilon_nB) \cap B(f_{n-1}(t), d(f_{n-1}(t), F(t,x_{n-1}(t))) + \varepsilon_n).$$

At the same time, as we saw at the beginning of the proof, $f_n(\cdot)$ is also Bochner integrable.

From (3.2) for $n \geq 2$ and $t \in I$, we obtain

$$|f_n(t) - f_{n-1}(t)| \leq d(f_{n-1}(t), F(t,x_{n-1}(t))) + \varepsilon_n$$

$$\leq d(f_{n-1}(t), F(t,x_{n-2}(t)))$$

$$+ d_H(F(t,x_{n-2}(t)), F(t,x_{n-1}(t))) + \varepsilon_n$$

$$\leq \varepsilon_{n-1} + l(t)|x_{n-1}(t) - x_{n-2}(t)| + \varepsilon_n.$$ 

Since $\varepsilon_{n-1} + \varepsilon_n < \varepsilon_{n-2}$ we deduce, for $n \geq 2$, that

$$|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)|x_{n-1}(t) - x_{n-2}(t)|.$$
Denote \( q_0(t) := d(f_0(t), F(t, x_0(t))) \), \( t \in I \). We prove next, by recurrence, that, for \( n \geq 2 \) and \( t \in I \), we have

\[
|x_n(t) - x_{n-1}(t)| \leq \sum_{k=0}^{n-2} \int_0^t \varepsilon_{n-2-k} \frac{M^{k+1}(m(t) - m(u))^k}{k!} du + \varepsilon_0 \int_0^t \frac{M^n(m(t) - m(u))^{n-1}}{(n-1)!} du + \int_0^t \frac{M^n(m(t) - m(u))^{n-1}}{(n-1)!} q_0(u) du.
\]

(3.4)

We start with \( n = 2 \). In view of (3.1), (3.2) and (3.3), for \( t \in I \), one has

\[
|x_2(t) - x_1(t)| \leq \int_0^t |U(t, s)| \cdot |f_2(s) - f_1(s)| ds \\
\leq \int_0^t M[\varepsilon_0 + l(s)|x_1(s) - x_0(s)|] ds \leq \varepsilon_0 Mt \\
+ \int_0^t [Ml(s) \int_0^s |U(s, u)| \cdot |f_1(u) - f_0(u)| du] ds \\
\leq \varepsilon_0 Mt + \int_0^t [M^2l(s) \int_0^s (q_0(u) + \varepsilon_1) du] ds \\
\leq \varepsilon_0 Mt + \int_0^t [M^2(q_0(u) + \varepsilon_1) \int_u^t l(s) ds] du \\
= \varepsilon_0 Mt + \int_0^t M^2(m(t) - m(s))[q_0(s) + \varepsilon_0] ds,
\]

i.e, (3.4) is verified for \( n = 2 \).

Using again (3.2) and (3.3) we have

\[
|x_{n+1}(t) - x_n(t)| \leq \int_0^t |U(t, s)| \cdot |f_{n+1}(s) - f_n(s)| ds \\
\leq \int_0^t M[\varepsilon_{n-1} + l(s)|x_n(s) - x_{n-1}(s)|] ds \\
\leq \varepsilon_{n-1} Mt + \int_0^t l(s) \left( \sum_{k=0}^{n-2} \int_0^s \varepsilon_{n-2-k} \frac{M^{k+2}(m(s) - m(u))^k}{k!} du \\
+ \int_0^s \frac{M^{n+1}(m(s) - m(u))^{n-1}}{(n-1)!} (q_0(u) + \varepsilon_0) du \right) ds \\
= \varepsilon_{n-1} Mt + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \left[ \int_0^s \frac{M^{k+2}(m(s) - m(u))^k}{k!} l(s) du \right] ds
\]
where 

\[
\begin{align*}
\varepsilon_{n-1}M & + \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \int_0^t \frac{M^{k+1}(m(s) - m(u))^{k+1}}{(k+1)!} du \\
& + \int_0^t \frac{M^{n+1}(m(s) - m(u))^{n}}{n!} [q_0(u) + \varepsilon_0] du \\
& = \sum_{k=0}^{n-1} \varepsilon_{n-1-k} \int_0^t \frac{M^{k+1}(m(s) - m(u))^{k}}{k!} du \\
& + \int_0^t \frac{M^{n+1}(m(s) - m(u))^{n}}{n!} [q_0(u) + \varepsilon_0] du,
\end{align*}
\]

and the statement (3.4) is true for \( n + 1 \).

From (3.4) it follows that, for \( n \geq 2 \) and \( t \in I \), one has

\[
|x_n(t) - x_{n-1}(t)| \leq a_n,
\]

where

\[
a_n = \sum_{k=0}^{n-2} \varepsilon_{n-2-k} \frac{M^{k+1}m(T)^{k}}{k!} + \frac{M^n m(T)^{n-1}}{(n-1)!} [\int_0^1 q_0(u) du + \varepsilon_0].
\]

Obviously, the series whose \( n \)-th term is \( a_n \) is convergent. So, from (3.5) we have that \( x_n(\cdot) \) converges uniformly on \( I \) to a continuous function, \( x(\cdot) : I \rightarrow X \).

On the other hand, in view of (3.5) we have

\[
|f_n(t) - f_{n-1}(t)| \leq \varepsilon_{n-2} + l(t)a_{n-1}, \quad t \in I, \ n \geq 3
\]

which implies that the sequence \( f_n(\cdot) \) converges to a Lusin measurable function \( f(\cdot) : I \rightarrow X \).

Since \( x_n(\cdot) \) is bounded and

\[
|f_n(t)| \leq l(t)|x_{n-1}(t)| + q(t) + 1
\]

we infer that \( f(\cdot) \) is also Bochner integrable.

Passing with \( n \rightarrow \infty \) in (3.1) and using Lebesgue dominated convergence theorem we obtain

\[
x(t) = -\frac{\partial}{\partial s} U(t,0)x_0 + U(t,0)y_0 + \int_0^t U(t,s)f(s)ds, \quad t \in I.
\]
On the other hand, from (3.2) we get
\[ f_n(t) \in F(t, x_n(t)) + \varepsilon_n B, \quad t \in I, \quad n \geq 1 \]
and letting \( n \to \infty \) we have
\[ f(t) \in F(t, x(t)), \quad t \in I. \]
and the proof is complete.

References


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