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Approximate tri-quadratic functional equations via Lipschitz conditions

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APPROXIMATE TRI-QUADRATIC FUNCTIONAL EQUATIONS
VIA LIPSCHITZ CONDITIONS

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Abstract. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We introduce a new notion similar to that of the left invariant mean and prove that a family of functions with this property can be approximated by tri-quadratic functions via a Lipschitz norm.

Keywords: tri-quadratic functional equation; Lipschitz space; stability

MSC 2010: 39B82, 39B52

1. INTRODUCTION

A generalized stability problem for the quadratic functional equation

$$\mathcal{Q}(x + y) + \mathcal{Q}(x - y) = 2\mathcal{Q}(x) + 2\mathcal{Q}(y)$$

was proved by Skof in [11] for mappings from a normed space to a Banach space. Czerwik et al. in [1] verified the stability of the quadratic functional equations in Lipschitz spaces. The Lipschitz stability type problems for some functional equations were also studied by Tabor, see [12], [13]. In Lipschitz spaces we investigated the stability of cubic functional equations in [2] and the stability of quartic functional equations in [7] (see also [6], [5]). The stability problem for the quadratic and bi-quadratic functional equation has been studied by many mathematicians under various degrees of generality imposed on the equation or on the underlying space; see, for example, [3], [4], [10], [9] and the references therein. We obtained Lipschitz criteria for bi-quadratic functional equations in Lipschitz spaces in [8].

The algebra of Lipschitz functions on a complete metric space plays a role in noncommutative metric theory similar to that played by the algebra of continuous

functions on a compact space in noncommutative topology. Let \mathcal{H} be an abelian group and \mathcal{W} a real vector space. A function $\mathcal{Q}: \mathcal{H}^3 \rightarrow \mathcal{W}$ is called tri-quadratic if \mathcal{Q} satisfies the system of equations

$$\begin{aligned}\mathcal{Q}(x+y, z, w) + \mathcal{Q}(x-y, z, w) &= 2\mathcal{Q}(x, z, w) + 2\mathcal{Q}(y, z, w), \\ \mathcal{Q}(x, y+z, w) + \mathcal{Q}(x, y-z, w) &= 2\mathcal{Q}(x, y, w) + 2\mathcal{Q}(x, z, w), \\ \mathcal{Q}(x, y, z+w) + \mathcal{Q}(x, y, z-w) &= 2\mathcal{Q}(x, y, z) + 2\mathcal{Q}(x, y, w)\end{aligned}$$

for all $x, y, z \in \mathcal{H}$, that is, \mathcal{Q} is quadratic in each variable. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We prove that a family of functions satisfying tri-symmetric left invariant mean property can be approximated by tri-quadratic functions via a Lipschitz norm.

2. LIPSCHITZ CONDITIONS FOR TRI-QUADRATIC FUNCTIONAL EQUATIONS

In this section, we introduce the notion of tri-symmetric left invariant mean (TSLIM in brief) and prove that a family of functions with TSLIM property can be approximated by tri-quadratic functions via a Lipschitz norm.

A family S of subsets of \mathcal{W} is called linearly invariant if $A + \alpha B \in S$ for $A, B \in S$, $\alpha \in \mathbb{R}$ and $x + A \in S$ for $A \in S$, $x \in \mathcal{W}$. For example, the family of all closed balls with center at zero is a linearly invariant family in a normed vector space. We denote this family by $CB(\mathcal{W})$. Let $\mathcal{L}(\mathcal{W})$ be a linearly invariant family of subsets of \mathcal{W} . By $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ we denote the family of all functions $\mathcal{Q}: \mathcal{H} \rightarrow \mathcal{W}$ such that $\text{Im } \mathcal{Q} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$.

Definition 2.1. The function \mathcal{Q} is called tri-symmetric if

$$\mathcal{Q}(x, y, z) = \mathcal{Q}(y, z, x) = \mathcal{Q}(z, x, y) = \mathcal{Q}(z, y, x) = \mathcal{Q}(x, z, y) = \mathcal{Q}(y, x, z)$$

for all $x, y, z \in \mathcal{H}$.

Definition 2.2. We say that $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ admits a tri-symmetric left invariant mean (briefly TSLIM), if the family $\mathcal{L}(\mathcal{W})$ is linearly invariant and there exists a linear operator $\Gamma: \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \rightarrow \mathcal{W}$ such that

(i) if $\mathcal{Q}_{x,y,z} \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ and $x, y, z \in \mathcal{H}$, then

$$\Gamma[\mathcal{Q}_{x,y,z}] = \Gamma[\mathcal{Q}_{y,z,x}] = \Gamma[\mathcal{Q}_{z,x,y}] = \Gamma[\mathcal{Q}_{z,y,x}] = \Gamma[\mathcal{Q}_{x,z,y}] = \Gamma[\mathcal{Q}_{y,x,z}],$$

(ii) if $\text{Im } \mathcal{Q} \subset B$ for some $B \in \mathcal{L}(\mathcal{W})$, then $\Gamma[\mathcal{Q}] \in B$,

(iii) if $\mathcal{Q} \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ and $a \in \mathcal{H}$, then $\Gamma[\mathcal{Q}^a] = \Gamma[\mathcal{Q}]$, where $\mathcal{Q}^a(x) = \mathcal{Q}(x+a)$.

Definition 2.3. Let $\Delta: \mathcal{H}^3 \times \mathcal{H}^3 \rightarrow \mathcal{L}(\mathcal{W})$ be a set-valued function such that

$$\begin{aligned} & \Delta((x+a, y+b, z+c), (u+a, v+b, w+c)) \\ &= \Delta((a+x, b+y, c+z), (a+u, b+v, c+w)) = \Delta((x, y, z), (u, v, w)) \end{aligned}$$

for all $(a, b, c), (x, y, z), (u, v, w) \in \mathcal{H}^3$. A function $\mathcal{Q}: \mathcal{H}^3 \rightarrow \mathcal{W}$ is said to be Δ -Lipschitz if

$$\mathcal{Q}(x, y, z) - \mathcal{Q}(u, v, w) \in \Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

Let $\mathcal{Q}: \mathcal{H}^3 \rightarrow \mathcal{W}$ be a function. We consider its tri-quadratic difference as follows:

$$T\mathcal{Q}(x, y, z, w) := 2\mathcal{Q}(x, z, w) + 2\mathcal{Q}(y, z, w) - \mathcal{Q}(x+y, z, w) - \mathcal{Q}(x-y, z, w)$$

for all $x, y, z, w \in \mathcal{H}$.

Theorem 2.4. Let \mathcal{H} be an abelian group and let \mathcal{W} be a vector space. Assume that the family $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ admits TSLIM. If $\mathcal{F}: \mathcal{H}^3 \rightarrow \mathcal{W}$ is a function and $T\mathcal{F}(t, \cdot, \cdot, \cdot): \mathcal{H}^3 \rightarrow \mathcal{W}$ is Δ -Lipschitz for every $t \in \mathcal{H}$, then there exists a tri-quadratic function $\mathcal{Q}: \mathcal{H}^3 \rightarrow \mathcal{W}$ such that $\mathcal{F} - \mathcal{Q}$ is $\frac{1}{2}\Delta$ -Lipschitz. Moreover, if $\text{Im } T\mathcal{F} \subset A$ for some $A \in \mathcal{L}(\mathcal{W})$, then $\text{Im}(\mathcal{F} - \mathcal{Q}) \subset \frac{1}{2}A$.

Proof. For every $(x, y, z) \in \mathcal{H}^3$ we define $\varphi_x(\cdot, y, z): \mathcal{H} \rightarrow \mathcal{W}$ by

$$\varphi_x(\cdot, y, z) := \frac{1}{2}\mathcal{F}(\cdot + x, y, z) + \frac{1}{2}\mathcal{F}(\cdot - x, y, z) - \mathcal{F}(\cdot, y, z).$$

We prove that $\text{Im } \varphi_x(\cdot, y, z) \subseteq A$ for some $A \in \mathcal{L}(\mathcal{W})$. We have for $(x, y, z) \in \mathcal{H}^3$,

$$\begin{aligned} \varphi_x(\cdot, y, z) &= \frac{1}{2}\mathcal{F}(\cdot + x, y, z) + \frac{1}{2}\mathcal{F}(\cdot - x, y, z) - \mathcal{F}(\cdot, y, z) - \mathcal{F}(x, y, z) \\ &\quad - \frac{1}{2}\mathcal{F}(\cdot, y, z) - \frac{1}{2}\mathcal{F}(\cdot, y, z) + \mathcal{F}(\cdot, y, z) + \mathcal{F}(0, y, z) \\ &\quad + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z) \\ &= \frac{1}{2}T\mathcal{F}(\cdot, 0, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, x, y, z) + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z). \end{aligned}$$

By assumption, since $T\mathcal{F}(t, \cdot, \cdot, \cdot)$ is Δ -Lipschitz for every $t \in \mathcal{H}$, $\text{Im } \varphi_x(\cdot, y, z) \subseteq A$, where $A := \frac{1}{2}\Delta((0, y, z), (x, y, z)) + \mathcal{F}(x, y, z) - \mathcal{F}(0, y, z)$. The family $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ admits TSLIM, so there exists a linear operator $\Gamma: \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \rightarrow \mathcal{W}$ such that

$$(i) \quad \Gamma[\varphi_x(\cdot, y, z)] = \Gamma[\varphi_y(\cdot, z, x)] = \Gamma[\varphi_z(\cdot, x, y)] = \Gamma[\varphi_y(\cdot, x, z)] = \Gamma[\varphi_x(\cdot, z, y)] = \Gamma[\varphi_z(\cdot, y, x)] \text{ for every } (x, y, z) \in \mathcal{H}^3$$

- (ii) $\Gamma[\varphi_x(\cdot, y, z)] \in A$ for some $A \in \mathcal{L}(\mathcal{W})$ and every $(x, y, z) \in \mathcal{H}^3$,
- (iii) if $u \in \mathcal{H}$ and $\varphi_x^u(\cdot, y, z): \mathcal{H} \rightarrow \mathcal{W}$ defined by $\varphi_x^u(\cdot, y, z) := \varphi_x(\cdot + u, y, z)$ for every $(x, y, z) \in \mathcal{H}^3$, then $\varphi_x^u(\cdot, y, z) \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ and $\Gamma[\varphi_x^u(\cdot, y, z)] = \Gamma[\varphi_x(\cdot, y, z)]$.

Define the function $\mathcal{Q}: \mathcal{H}^3 \rightarrow \mathcal{W}$ by $\mathcal{Q}(x, y, z) := \Gamma[\varphi_x(\cdot, y, z)]$. In view of property (i) of Γ , \mathcal{Q} is tri-symmetric. We prove that $\mathcal{F} - \mathcal{Q}$ is $\frac{1}{2}\Delta$ -Lipschitz. Since $T\mathcal{F}(t, \cdot, \cdot, \cdot)$ is Δ -Lipschitz for $t \in \mathcal{H}$,

$$(2.1) \quad T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w) \in \Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$ and so

$$\text{Im} \left(\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w) \right) \subseteq \frac{1}{2}\Delta((x, y, z), (u, v, w)).$$

In view of property (ii) of Γ , we find that

$$\Gamma \left[\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w) \right] \in \frac{1}{2}\Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$. Note that $\mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ contains constant functions. Property (ii) of Γ entails that for a constant function $C: \mathcal{H} \rightarrow \mathcal{W}$, $\Gamma[C] = C$. For every $(x, y, z) \in \mathcal{H}^3$ we define the constant function $C_{x,y,z}: \mathcal{H} \rightarrow \mathcal{W}$ by $C_{x,y,z}(\cdot) := \mathcal{F}(x, y, z)$. We see that

$$\begin{aligned} & (\mathcal{F}(x, y, z) - \mathcal{Q}(x, y, z)) - (\mathcal{F}(u, v, w) - \mathcal{Q}(u, v, w)) \\ &= (\Gamma[C_{x,y,z}(\cdot)] - \Gamma[\varphi_x(\cdot, y, z)]) - (\Gamma[C_{u,v,w}(\cdot)] - \Gamma[\varphi_u(\cdot, v, w)]) \\ &= \Gamma[C_{x,y,z}(\cdot) - \varphi_x(\cdot, y, z)] - \Gamma[C_{u,v,w}(\cdot) - \varphi_u(\cdot, v, w)] \\ &= \Gamma \left[\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w) \right] \end{aligned}$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$. This shows that

$$(\mathcal{F}(x, y, z) - \mathcal{Q}(x, y, z)) - (\mathcal{F}(u, v, w) - \mathcal{Q}(u, v, w)) \in \frac{1}{2}\Delta((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$, i.e., $\mathcal{F} - \mathcal{Q}$ is a $\frac{1}{2}\Delta$ -Lipschitz function. Applying property (iii) of Γ and the definition of Γ , we find that

$$(2.2) \quad \begin{aligned} 2\mathcal{Q}(x, z, w) + 2\mathcal{Q}(y, z, w) &= 2\Gamma[\varphi_x(\cdot, z, w)] + 2\Gamma[\varphi_y(\cdot, z, w)] \\ &= \Gamma[\varphi_x^y(\cdot, z, w)] + \Gamma[\varphi_x^{-y}(\cdot, z, w)] + 2\Gamma[\varphi_y(\cdot, z, w)]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(2.3) \quad & \Gamma[\varphi_x^y(\cdot, z, w)] + \Gamma[\varphi_x^{-y}(\cdot, z, w)] + 2\Gamma[\varphi_y(\cdot, z, w)] \\
&= \Gamma\left[\frac{1}{2}\mathcal{F}(\cdot + x + y, z, w) + \frac{1}{2}\mathcal{F}(\cdot - x + y, z, w) - \mathcal{F}(\cdot + y, z, w)\right] \\
&\quad + \Gamma\left[\frac{1}{2}\mathcal{F}(\cdot + x - y, z, w) + \frac{1}{2}\mathcal{F}(\cdot - x - y, z, w) - \mathcal{F}(\cdot - y, z, w)\right] \\
&\quad + \Gamma[\mathcal{F}(\cdot + y, z, w) + \mathcal{F}(\cdot - y, z, w) - 2\mathcal{F}(\cdot, z, w)] \\
&= \Gamma[\varphi_{x+y}(\cdot, z, w)] + \Gamma[\varphi_{x-y}(\cdot, z, w)] \\
&= \mathcal{Q}(x + y, z, w) + \mathcal{Q}(x - y, z, w).
\end{aligned}$$

From (2.2) and (2.3) it follows that \mathcal{Q} is quadratic in its first variable. Since \mathcal{Q} is tri-symmetric, \mathcal{Q} is quadratic in its second and third variables and hence \mathcal{Q} is tri-quadratic. Moreover, if $\text{Im } T\mathcal{F} \subset A$, then

$$\text{Im}\left(\frac{1}{2}T\mathcal{F}(\cdot, x, y, z)\right) \subset \text{Im}\left(\frac{1}{2}T\mathcal{F}\right) \subset \frac{1}{2}A.$$

In other words, $\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ for all $(x, y, z) \in \mathcal{H}^3$. Thus, property (ii) of Γ implies

$$\mathcal{F}(x, y, z) - \mathcal{Q}(x, y, z) = \Gamma\left[\frac{1}{2}T\mathcal{F}(\cdot, x, y, z)\right] \in \frac{1}{2}A$$

for all $(x, y, z) \in \mathcal{H}^3$. Therefore, $\text{Im}(\mathcal{F} - \mathcal{Q}) \subset \frac{1}{2}A$. □

Definition 2.5. Let (\mathcal{H}^3, ϱ) be a metric group and \mathcal{W} a normed space. A function $m_{\mathcal{F}}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a module of continuity of $\mathcal{F}: \mathcal{H}^3 \rightarrow \mathcal{W}$ if $\varrho((x, y, z), (u, v, w)) \leq \delta$ implies $\|\mathcal{F}(x, y, z) - \mathcal{F}(u, v, w)\| \leq m_{\mathcal{F}}(\delta)$ for every $\delta > 0$ and $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

Definition 2.6. A function $\mathcal{F}: \mathcal{H}^3 \rightarrow \mathcal{W}$ is called a Lipschitz function of order $\alpha > 0$ if there exists a constant $L > 0$ such that

$$(2.4) \quad \|\mathcal{F}(x, y, z) - \mathcal{F}(u, v, w)\| \leq L\varrho^\alpha((x, y, z), (u, v, w))$$

for every $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

For a metric group (\mathcal{H}^3, ϱ) , a normed space \mathcal{W} , and $\alpha \in (0, 1]$, let $\text{Lip}_\alpha(\mathcal{H}^3, \mathcal{W})$ be the Lipschitz space consisting of all bounded Lipschitz functions of order $\alpha > 0$ with the norm

$$\|\mathcal{F}\|_\alpha := \|\mathcal{F}\|_{\text{sup}} + P_\alpha(\mathcal{F}),$$

where $\|\cdot\|_{\text{sup}}$ is the supremum norm and

$$P_\alpha(\mathcal{F}) = \sup \left\{ \frac{\|\mathcal{F}(x, y, z) - \mathcal{F}(u, v, w)\|}{\varrho^\alpha((x, y, z), (u, v, w))} : (x, y, z), (u, v, w) \in \mathcal{H}^3, \right. \\ \left. (x, y, z) \neq (u, v, w) \right\}.$$

Definition 2.7. Consider an abelian group $(\mathcal{H}^3, +)$ with a metric ϱ invariant under translation, i.e., satisfying the condition

$$\varrho((x+a, y+b, z+c), (u+a, v+b, w+c)) \\ = \varrho((a+x, b+y, c+z), (a+u, b+v, c+w)) = \varrho((x, y, z), (u, v, w))$$

for all $(a, b, c), (x, y, z), (u, v, w) \in \mathcal{H}^3$. A metric D on $\mathcal{H}^3 \times \mathcal{H}$ is called a metric pair if it is invariant under translation and the following condition holds:

$$D((x, y, z, a), (u, v, w, a)) = D((a, x, y, z), (a, u, v, w)) = \varrho((x, y, z), (u, v, w))$$

for all $a \in \mathcal{H}, (x, y, z), (u, v, w) \in \mathcal{H}^3$.

Theorem 2.8. Let $(\mathcal{H}^3, +, \varrho, D)$ be a metric pair, \mathcal{W} a normed space such that $\mathcal{M}(\mathcal{H}, CB(\mathcal{W}))$ admits TSLIM, and $\mathcal{F}: \mathcal{H}^3 \rightarrow \mathcal{W}$ a function. If $T\mathcal{F} \in \text{Lip}_\alpha(\mathcal{H} \times \mathcal{H}^3, \mathcal{W})$, then there exists a tri-quadratic function \mathcal{Q} such that

$$\|\mathcal{F} - \mathcal{Q}\|_\alpha \leq \frac{1}{2} \|T\mathcal{F}\|_\alpha.$$

Proof. Assume that $m_{T\mathcal{F}}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the module of continuity of $T\mathcal{F}$ with the metric pair D . Define the set-valued function $\Delta: \mathcal{H}^3 \times \mathcal{H}^3 \rightarrow CB(\mathcal{W})$ by

$$\Delta((x, y, z), (u, v, w)) := \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta) B(0, 1),$$

where $B(0, 1)$ is the closed unit ball with center at zero. The following inequality shows that $T\mathcal{F}(t, \cdot, \cdot, \cdot)$ is Δ -Lipschitz:

$$\|T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w)\| \leq \inf_{D((t, x, y, z), (t, u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta) \\ = \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta)$$

for all $t \in \mathcal{H}, (x, y, z), (u, v, w) \in \mathcal{H}^3$. Thus, there exists a tri-quadratic function \mathcal{Q} such that $\mathcal{F} - \mathcal{Q}$ is $\frac{1}{2}\Delta$ -Lipschitz by Theorem 2.4. Hence,

$$\|(\mathcal{F} - \mathcal{Q})(x, y, z) - (\mathcal{F} - \mathcal{Q})(u, v, w)\| \leq \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} \frac{1}{2} m_{T\mathcal{F}}(\delta),$$

which entails that $m_{\mathcal{F}-\mathcal{Q}} = \frac{1}{2}m_{T\mathcal{F}}$. Moreover, $\|T\mathcal{F}\|_{\text{sup}} < \infty$ and clearly $\text{Im } T\mathcal{F} \subset \|T\mathcal{F}\|_{\text{sup}}B(0, 1)$. Using the last part of Theorem 2.4 we get

$$(2.5) \quad \|\mathcal{F} - \mathcal{Q}\|_{\text{sup}} \leq \frac{1}{2}\|T\mathcal{F}\|_{\text{sup}}.$$

Define the function $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\omega(t) := P_\alpha(T\mathcal{F})t^\alpha$. In view of $T\mathcal{F} \in \text{Lip}_\alpha(\mathcal{H} \times \mathcal{H}^3, \mathcal{W})$, we have

$$\|T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w)\| \leq \omega(D((t, x, y, z), (t, u, v, w))),$$

which ensures that ω is the module of continuity of the function $T\mathcal{F}$ and consequently $m_{\mathcal{F}-\mathcal{Q}} = \frac{1}{2}\omega$. Then,

$$\begin{aligned} \|(\mathcal{F} - \mathcal{Q})(x, y, z) - (\mathcal{F} - \mathcal{Q})(u, v, w)\| &\leq \frac{1}{2}\omega(\varrho((x, y, z), (u, v, w))) \\ &= \frac{1}{2}P_\alpha(T\mathcal{F})\varrho^\alpha((x, y, z), (u, v, w)). \end{aligned}$$

This inequality implies that $\mathcal{F} - \mathcal{Q}$ is a Lipschitz function of order α and $P_\alpha(\mathcal{F} - \mathcal{Q}) \leq \frac{1}{2}P_\alpha(T\mathcal{F})$. From inequality (2.5) it follows that

$$\begin{aligned} \|\mathcal{F} - \mathcal{Q}\|_\alpha &= \|\mathcal{F} - \mathcal{Q}\|_{\text{sup}} + P_\alpha(\mathcal{F} - \mathcal{Q}) \\ &\leq \frac{1}{2}\|T\mathcal{F}\|_{\text{sup}} + \frac{1}{2}P_\alpha(T\mathcal{F}) = \frac{1}{2}\|T\mathcal{F}\|_\alpha. \end{aligned}$$

□

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