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Commentationes Mathematicae Universitatis Carolinae, Vol. 58 (2017), No. 4, 461–464

Persistent URL: http://dml.cz/dmlcz/146990

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SURJIT SINGH KHURANA

Abstract. For a Banach space $E$ and a probability space $(X, A, \lambda)$, a new proof is given that a measure $\mu : A \to E$, with $\mu \ll \lambda$, has RN derivative with respect to $\lambda$ iff there is a compact or a weakly compact $C \subseteq E$ such that $|\mu|_C : A \to [0, \infty]$ is a finite valued countably additive measure. Here we define $|\mu|_C(A) = \sup(\sum_k |\langle\mu(A_k), f_k\rangle|)$ where $\{A_k\}$ is a finite disjoint collection of elements from $A$, each contained in $A$, and $\{f_k\} \subseteq E'$ satisfies $\sup_k |f_k(C)| \leq 1$. Then the result is extended to the case when $E$ is a Frechet space.

Keywords: liftings; lifting topology; weakly compact sets; Radon-Nikodym derivative

Classification: Primary 46B22, 46G05, 46G10, 28A51; Secondary 60B05, 28B05, 28C05

1. Introduction and notations

In this paper $K$ will always denote the field of real or complex numbers (we will call them scalars), $\mathbb{R}$ the real numbers and $\mathbb{N}$ the set of natural numbers. All locally convex space are assumed to be Hausdorff and are over $K$ and notations and results of [8] will be used. Given a locally convex space $E$ with $E'$ its dual, for $x \in E$ and $f \in E'$, we will also write $\langle f, x \rangle = \langle x, f \rangle$ for $f(x)$; for an $A \subseteq E$, $\Gamma(A)$ will denote the absolute convex hull of $A$. Let $(X, A, \lambda)$ be a complete probability space. By a measure we will always mean a countably additive measure. For a measure $\mu$, $|\mu|$ will denote its total variation measure. For measures and vector measures we refer to [2]; see also [4], [5].

In [3], for a Banach space $E$, an interesting characterization is given for a vector measure $\mu : A \to E$ of bounded variation, $\mu \ll \lambda$, to have a derivative. It is proved that $\mu$ has derivative iff there is a compact or weakly compact $C \subseteq E$ such that $|\mu|_C : A \to [0, \infty]$ is a finite valued countably additive measure. Here we define $|\mu|_C(A) = \sup(\sum_k |\langle\mu(A_k), f_k\rangle|)$ where $\{A_k\}$ is a finite disjoint collection of elements from $A$, each contained in $A$, and $\{f_k\} \subseteq E'$ satisfies $\sup_k |f_k(C)| \leq 1$. First we give a new proof of this result and then extend this to Frechet spaces.
2. Main results

**Theorem 1.** Let $A$ be a σ-algebra of subsets of a set $X$, $E$ a Frechet space and $\mu : A \to E$ countably additive measure with $\mu \ll \lambda$. Suppose $\mu$ has finite variation with respect to every continuous semi-norm on $E$. Then $\mu$ has RN derivative relative to $\lambda$ iff there is a compact or weakly compact $C \subset E$ such that $|\mu|_C : A \to [0, \infty]$ is a finite-valued measure. Here we define $|\mu|_C(A) = \sup\{\sum_k |\langle\mu(A_k), f_k\rangle|\}$ where $\{A_k\}$ is a finite disjoint collection of elements from $A$, each contained in $A$, and $\{f_k\} \subset E'$ satisfies $\sup_k |f_k(C)| \leq 1$.

**Proof:** First we consider $E$ to be a Banach and give an entirely different proof than the one given in [3]; we will reduce it to a reflexive Banach subspace of $E$ with a finer topology. Take an absolutely convex weakly compact $C \subset E$ with a countably additive measure $\nu = |\mu|_C \leq 1$. This implies $\mu(A) \subset C$ ([3, Theorem 2.1(1)]). By [1], there is a reflexive Banach space $E_0 \subset E$ such that $C \subset E_0$, $C$ is weakly compact in $E_0$, and the identity mapping $E_0 \to E$ is continuous. Take an $f_0 \in E_0'$ with norm $\leq 1$ and fix $c > 0$. If we consider $C$ as a subset of $E$, $(f_0)_C$ is an affine continuous function on $C$. It is proved in [7, Proposition 3.5, p. 31] that $(E'_C + K)$ is uniformly dense in the space of all continuous affine functions on $C$ (this is proved when $K = \mathbb{R}$ but easily extends to general $K$). Thus there is an $f \in E'$ and $r \in K$ such that $\sup |(f + r - f_0)(C)| \leq c$. Since $C$ is absolutely convex (that implies $0 \in C$), we get $\sup |(f - f_0)(C)| \leq 2c$. Take a decreasing sequence $\{A_n\} \subset A$ such that $A_n \downarrow \emptyset$. Now $f \circ \mu(A_n) \to 0$ and since $\sup |(f - f_0)(C)| \leq 2c$ and $c$ is arbitrarily small, we get $f_0 \circ \mu(A_n) \to 0$ and so $\mu : A \to E_0$ is countably additive. Now we will prove that $\mu : A \to E_0$ is of bounded variation. Take $p > 0$ with $pC \subset B$ (the unit ball of $E_0$), a finite collection $\{f_i\}$ in the closed unit ball of $E'_0$, and disjoint elements $\{A_i\} \subset A$. We have $|pf_i(C)| \leq 1 \forall i$. As explained above, take $\{f'_i\} \subset E'$ with $\sup \{|pf_i - f'_i(C)|\} \leq \frac{1}{2}$ $\forall i$. We get

$$\sum |\langle f_i, \mu(A_i)\rangle| \leq \frac{1}{p} \sum |\langle pf_i - f'_i\rangle \mu(A_i)| + \frac{1}{p} \sum |\langle f'_i\rangle \mu(A_i)| \leq \frac{1}{p} + \frac{1}{p} |\mu|_C(X) \leq \frac{2}{p},$$

This proves $\mu : A \to E_0$ is of bounded variation. Since $E_0$ is reflexive, there is an $h \in L_1(X, E_0)$ with $\mu = h\lambda$. From this it easily follows that $h \in L_1(X, E)$. The converse is same as for Frechet space which we will consider now.

Now we consider the case when $E$ is a Frechet space. Suppose $\mu$ has RN derivative $\frac{d\mu}{d\lambda} = g \in L^1(\lambda, E) = L^1(\lambda) \otimes E$ (the completion in projective tensor product). Thus $g = \sum_i \alpha_i g_i x_i$, $\{g_i\}$, $\{x_i\}$ being null sequences in $L^1(\lambda)$ and $E$ respectively and $\{\alpha_i\} \in \ell_1$ ([8, Theorem 6.4, p. 94]); we can assume that $\int |g_i| d\lambda \leq 1 \forall i$. Let $C$ be the closed, absolutely convex hull of $\{x_i\}$; $C$ is compact. Take a finite, disjoint family $\{A_k\}$ of elements of $A$ and $\{f_k\}$ elements of $E'$ with $\sup |f_k(C)| \leq 1 \forall k$. We have $\sum_k |\langle \mu(A_k), f_k\rangle| \leq \sum_k \sum_i |\alpha_i||g_i||f_k(x_i)| d\lambda \leq \alpha$ where $\alpha = \sum |\alpha_i|$. Thus $|\mu|_C$ is finite-valued. It is a routine verification that $|\mu|_C$ is countably additive ([2, p. 4], [3, Theorem 2.1(5), p. 142]).
Conversely suppose for an absolutely convex weakly compact $C \subset E$, $\nu = |\mu|_C$ is finite-valued. We have $\mu \ll |\mu|_C$. Also it follows from the definition of $|\mu|_C$ that, for an $f \in E'$ with $\sup |f(C)| \leq 1$, we have $|\mu|_C \geq |f \circ \mu|$. Denoting the completion of $|\mu|_C$ by $|\mu|_C$ again, we fix a lifting $\rho_0$ for this measure ([9]) and take the lifting topology $T_0$ on $X$ which has $\{\rho_0(A) : A \in A\}$ as the base of open sets; we can assume this topology to be Hausdorff and denote by $C_b(X)$ all scalar-valued bounded continuous functions on $X$. For each $f \in E'$ there is a $\phi_f \in L_1(|\mu|_C)$ such that $f \circ \mu = \phi_f|\mu|_C$. Put $|f(C)| = p$; we claim $|\phi_f| \leq p$ a.e. $|\mu|_C$. Suppose this is not true. Then there is a $c > 0$ such that $|\mu|_C(A) > 0$ where $A = \{x \in X : |\phi_f(x)| \geq p + c\}$. Now, since $|\frac{1}{p}f(C)| \leq 1$, we have $|\mu|_C(A) \geq \frac{1}{p}|f \circ \mu|(A) = \frac{1}{p} \int_A |\phi_f|d|\mu|_C \geq \frac{1}{p}(p + c)|\mu|_C(A)$ which is a contradiction. Thus there is a unique function in $C_b(X)$ which is equal to $\phi_f$ a.e. $|\mu|_C$; we denote this function also by $\phi_f$.

Define $\phi : X \to K^{E'}$, $(\phi(x))_f = \phi_f(x)$. It is a simple verification that $\phi_{f_1 + f_2} = \phi_{f_1} + \phi_{f_2}$ and $\phi_{r \cdot f} = r \phi_f$ for any $f_1, f_2, f \in E'$ and any $r \in K$. Also $E$, with weak topology, can be considered as a subspace of $K^{E'}$ with product topology. We claim that $\phi(X) \subset C$. If this is not true, by separation theorem ([8, 9.2, p.65]), $\exists x_0 \in X$ and $f \in E'$ such that $p = \sup |f(C)| < Rl(\phi_f(x_0)) = p + 3\eta$ for some $\eta > 0$ (note $C$ is absolutely convex and so $sup(Rl(f(C))) = sup|f(C)|$). Now $\nu(A) > 0$ where $A = \{x \in X : |\phi_f(x)| > p + 2\eta\}$. Since $|\phi_f| \leq p$, we have $p \nu(A) \geq \int_A |\phi_f|d\nu \geq (p + 2\eta)\nu(A)$, a contradiction. By [6, Theorem 2, p.389], $\phi$ is weakly equivalent to a function $\phi_0$ such that $\phi_0(X)$ is contained in a separable weakly compact subset of $E$; thus $\phi_0$ is bounded. Now it is well-known that if a weakly measurable function has a separable range in $E$ then it is strongly measurable ([2, Theorem 2, p.42]; it is proved for a Banach space but thus easily extends to a Frechet space). Now being bounded and strongly measurable, $\phi_0 \in L^1(\nu, E)$. Since $|\mu| \ll \nu$ and $\nu \ll \lambda$, the result follows. \hfill $\square$

Acknowledgment. We are very thankful to the referee for pointing out several typographical errors and also making some very useful suggestions which has improved the paper.

References


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(Received June 13, 2017, revised August 23, 2017)