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ON THE HOMOTOPY TRANSFER OF $A_{\infty}$ STRUCTURES

JAKUB KOPŘIVA

Dedicated to the memory of Martin Doubek

Abstract. The present article is devoted to the study of transfers for $A_{\infty}$ structures, their maps and homotopies, as developed in [7]. In particular, we supply the proofs of claims formulated therein and provide their extension by comparing them with the former approach based on the homological perturbation lemma.

1. Introduction

The notion of strongly homotopy associative or $A_{\infty}$ algebras is a generalization of the concept of differential graded algebras. These algebras were introduced by J. Stasheff with the aim of a characterization of (de)looping and bar construction in the category of topological spaces. Since then they found many applications ranging from algebraic topology and operads to quantum theories in theoretical physics.

We consider the following situation: let $(V, \partial_V)$ and $(W, \partial_W)$ be two chain complexes of modules, and $f: (V, \partial_V) \to (W, \partial_W)$ and $g: (W, \partial_W) \to (V, \partial_V)$ two mappings of chain complexes such that $gf$ is homotopic to the identity map on $V$ and $(V, \partial_V)$ is equipped with $A_{\infty}$ algebra structure. Then a natural question arises — can $A_{\infty}$ structure be transferred to $(W, \partial_W)$ and secondly, what is its explicit form in terms of $A_{\infty}$ algebra structure on $(V, \partial_V)$ and in which sense is it unique?

While the existence of a transfer follows from general model structure considerations, an unconditional and elaborate answer producing explicit formulas for the transferred objects was formulated in [7]. The present article contributes to the problem of transfer of $A_{\infty}$ structures. Its modest aim is to supply detailed proofs of many claims omitted in the original article [7], thereby facilitating complete subtle proofs to a reader interested in this topic. This exposition also extends the results of the aforementioned article in several ways, and sheds a light on its relationship with the homological perturbation lemma.
The content of our article goes as follows. In the Section 2, we recall a well-known correspondence between $A_\infty$ algebras and codifferentials on reduced tensor coalgebras. This allows us to simplify the proofs in Section 3 considerably. The Section 3 is devoted to the problem of homotopy transfer of $A_\infty$ algebras. We first derive the formulas introduced in [7], and then give their self-contained proofs. Here we achieve a substantial simplification of all proofs due to the reduction of sign factors. We also comment on another remark in [7], namely, the relationship between the homological perturbation lemma and homotopy transfer of $A_\infty$ algebras. We prove that on certain assumptions the explicit formulas in [7] do coincide with those coming from the homological perturbation lemma.

We shall work in the category of $\mathbb{Z}$-graded modules over an arbitrary commutative unital ring $R$, and their graded $R$-homomorphisms.

We first briefly recall the concepts of $A_\infty$ algebra, $A_\infty$ morphism of $A_\infty$ algebras and $A_\infty$ homotopy of $A_\infty$ morphisms, cf. [7], [4].

**Definition 1.1.** Let $(V, \partial_V)$ be a chain complex of modules indexed by $\mathbb{Z}$, i.e. $(V, \partial_V)$ is a $\mathbb{Z}$-graded modules $V = \bigoplus_{i=-\infty}^{\infty} V_i$ with $\partial_V(V_i) \subset V_{i-1}$ and $\partial_V \circ \partial_V = 0$. Let $\mu_n : V^\otimes n \to V$ be a collection of linear mappings of degree $n - 2$ ($n \geq 2$), satisfying

$$
\partial_V \mu_n = \sum_{i=1}^{n} (-1)^i \mu_n (1^\otimes i-1 \otimes \partial_V \otimes 1^\otimes n-i)
$$

(1)

for all $n \geq 2$ and $A(n) = \{ k, \ell \in \mathbb{N} | k + \ell = n + 1, k, \ell \geq 2, 1 \leq i \leq k \}$. The structure $(V, \partial_V, \mu_2, \mu_3, \ldots)$ is called $A_\infty$ algebra.

Throughout the article, we use the Koszul sign convention. This means that for $U, V$ a $W$ graded modules and $f : U \to V, g : U \to V, h : V \to W$ and $i : V \to W$ linear maps of degrees $|f|, |g|, |h|$ and $|i|$, respectively, holds

$$(h \otimes i)(f \otimes g) = (-1)^{|f||i|} h f \otimes ig.$$  

Similarly for $u_1, u_2 \in U$ of degree $|u_1|$ and $|u_2|$, respectively, holds

$$(f \otimes g)(u_1 \otimes u_2) = (-1)^{|u_1||g|} f(u_1) \otimes g(u_2).$$  

**Definition 1.2.** Let $(V, \partial_V, \mu_2, \ldots)$ and $(W, \partial_W, \nu_2, \ldots)$ be $A_\infty$ algebras. Then the set $\{ f_n : V^\otimes n \to W, |f_n| = n - 1 \}_{n \geq 1}$ is called $A_\infty$ morphism if

$$
\partial_W f_n + \sum_{B(n)} (-1)^{\partial(r_1, \ldots, r_k)} \nu_k (f_{r_1} \otimes \cdots \otimes f_{r_k})
$$

$$
= f_1 \mu_n - \sum_{i=1}^{n} (-1)^n f_n (1^\otimes i-1 \otimes \partial_V \otimes 1^\otimes n-i)
$$

$$
- \sum_{A(n)} (-1)^{i(\ell+1)+n} f_k (1^\otimes i-1 \otimes \mu_\ell \otimes 1^\otimes k-i)
$$

(2)
holds for all \( n \geq 1 \) with \( B(n) = \{ k, r_1, \ldots, r_k \in \mathbb{N} \mid k \geq 2, r_1, \ldots, r_k \geq 1, r_1 + \cdots + r_k = n \} \) and \( \vartheta(r_1, \ldots, r_k) = \sum_{1 \leq i < j \leq k} r_i(r_j + 1) \).

Morphisms of \( A_\infty \) algebras can be composed: for \((U, \partial U, \varrho_2, \ldots), (V, \partial V, \mu_2, \ldots)\) and \((W, \partial W, \nu_2, \ldots)\) \( A_\infty \) algebras, \( \{ f_n : U^{\otimes n} \to V \}_{n \geq 1} \) and \( \{ g_n : V^{\otimes n} \to W \}_{n \geq 1} \) \( A_\infty \) morphisms, their composition \( \{(gf)_n : U^{\otimes n} \to W \}_{n \geq 1} \) is defined as

\[
(gf)_n = g_1f_n + \sum_{B(n)} (-1)^{\vartheta(r_1, \ldots, r_k)}g_k(f_{r_1} \otimes \cdots \otimes f_{r_k}).
\]

**Definition 1.3.** Let \( \{ f_n : V^{\otimes n} \to W \}_{n \geq 1} \) and \( \{ g_n : V^{\otimes n} \to W \}_{n \geq 1} \) be morphisms between \( A_\infty \) algebras \((V, \partial V, \mu_2, \ldots)\) and \((W, \partial W, \nu_2, \ldots)\). The set of linear mappings \( \{ h_n : V^{\otimes n} \to W, |h_n| = n \}_{n \geq 1} \) is an \( A_\infty \) homotopy between \( A_\infty \) morphisms \( \{ f_n : V^{\otimes n} \to W \}_{n \geq 1} \) and \( \{ g_n : V^{\otimes n} \to W \}_{n \geq 1} \) provided

\[
f_n - g_n = h_1\mu_n - \sum_{i=1}^{n} (-1)^n h_n (1^{\otimes i-1} \otimes \partial V \otimes 1^{\otimes n-i})
- \sum_{A(n)} (-1)^{i(\ell+1)+n} h_k (1^{\otimes i-1} \otimes \mu_\ell \otimes 1^{\otimes k-i}) + \delta_W h_n
+ \sum_{B(n)} \sum_{1 \leq i \leq k} (-1)^{\vartheta(r_1, \ldots, r_k)} \nu_k
\times (f_{r_1} \otimes \cdots \otimes f_{r_{i-1}} \otimes h_{r_i} \otimes g_{r_{i+1}} \otimes \cdots \otimes g_{r_k}),
\]

is true for all \( n \geq 1 \) with \( B(n) = \{ k, r_1, \ldots, r_k \in \mathbb{N} \mid k \geq 2, r_1, \ldots, r_k \geq 1, r_1 + \cdots + r_k = n \} \).

2. Reduced tensor coalgebras

In the present section we introduce a bijective correspondence between \( A_\infty \) algebras and codifferentials on reduced tensor coalgebras, cf. [4]. We retain the notation \( V = \bigoplus_{i=-\infty}^{\infty} V_i \) for \( \mathbb{Z} \)-graded modules as well as

(A) \( A(n) = \{ k, \ell \in \mathbb{N} \mid k + \ell = n + 1, k, \ell \geq 2, 1 \leq i \leq k \} \),

(B) \( B(n) = \{ k, r_1, \ldots, r_k \in \mathbb{N} \mid k \geq 2, r_1, \ldots, r_k \geq 1, r_1 + \cdots + r_k = n \} \)

for \( n \in \mathbb{N} \), and \( A(1) = A(2) = B(1) = \emptyset \). We use a few natural variations on this notation, e.g. \( A'(n) = \{ k', \ell' \in \mathbb{N} \mid k' + \ell' = n + 1, k', \ell' \geq 2, 1 \leq i' \leq k' \} \).

2.1. Codifferentials on tensor coalgebras

**Definition 2.1.** Let \( \overline{TV} = \bigoplus_{n=1}^{\infty} V^{\otimes n} \), where the elements in \( V^{\otimes i} \) have degree (or homogeneity) \( i \), and let the mapping \( C : \overline{TV} \to \overline{TV} \otimes \overline{TV} \) be defined in such a way that \( C : v \mapsto 0 \) for \( v \in V^{\otimes 1} = V \) and

\[
C : v_1 \otimes \cdots \otimes v_n \mapsto \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n),
\]

for \( n \geq 2 \) and \( v_1, \ldots, v_n \in V \). The pair \( (\overline{TV}, C) \) is called the reduced tensor coalgebra.
Definition 2.2. A linear mapping \( \delta : \mathcal{T}V \to \mathcal{T}V \) of degree \(-1\) is called coderivation if \( C \circ \delta = (\delta \otimes 1 + 1 \otimes \delta) \circ C \). Moreover, if \( \delta \) satisfies \( \delta \circ \delta = 0 \), it is called codifferential.

Remark 2.3. We notice that \( C \) is coassociative, \((1 \otimes C) \circ C = (C \otimes 1) \circ C\). For all \( v \in \mathcal{T}V \) holds \( C(v) = 0 \) if and only if \( v \) is of homogeneity 1. For all maps \( \varphi : V^{\otimes n} \to \mathcal{T}W, n \geq 1 \), holds \( C_{\mathcal{T}W} \circ \varphi = 0 \) if and only if \( \varphi(V^{\otimes n}) \subseteq W \). For all \( v = v_1 \otimes \cdots \otimes v_n \in \mathcal{T}V \) and \( w = w_1 \otimes \cdots \otimes w_m \in \mathcal{T}V \), we have

\[
C(v \otimes w) = \sum_{i=1}^{n-1} (v_{1,i} \otimes (v_{i+1,n} \otimes w)) + (v) \otimes (w) + \sum_{i=1}^{m-1} (v \otimes w_{1,i}) \otimes (w_{i+1,m}),
\]

with \( v_{i,j} = v_i \otimes \cdots \otimes v_j, i \leq j, i,j \in \{1,\ldots,n\} \), and analogously for \( w_{i,j} \). This little calculation expresses a fact that \( \mathcal{T}V \) is a bialgebra which is, as a conilpotent coalgebra, cogenerated by \( V \).

Lemma 2.4. Let \( E : \mathcal{T}V \to \mathcal{T}W \) be a linear mapping for which there exist \( \{e_n : V^{\otimes n} \to W\}_{n \geq 1} \) with \( E|_{V^{\otimes n}} = e_n + \sum_{B(n)} e_{r_1} \otimes \cdots \otimes e_{r_k} \), and \( B(n) \) given in [B]. Then

\[
(6) \quad C_{\mathcal{T}W} \circ E|_{V^{\otimes n}} = \sum_{i=1}^{n-1} (E|_{V^{\otimes i}}) \otimes (E|_{V^{\otimes n-i}}).
\]

Proof. Obviously, we can write \( E|_{V^{\otimes n}} = e_n + \sum_{i=1}^{n-1} e_i \otimes E|_{V^{\otimes n-i}} \). The proof is by induction on \( n \): the claim holds for \( n = 1 \) and we assume it is true for all natural numbers less than \( n \). Then

\[
C_{\mathcal{T}W} \circ E|_{V^{\otimes n}} = C_{\mathcal{T}W} \circ (e_n + \sum_{i=1}^{n-1} e_i \otimes E|_{V^{\otimes n-i}})
\]

\[
= C_{\mathcal{T}W} \circ \left( \sum_{i=1}^{n-1} e_i \otimes E|_{V^{\otimes n-i}} \right)
\]

\[
= \sum_{i=1}^{n-1} (e_i) \otimes (E|_{V^{\otimes n-i}}) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1-i} (e_i \otimes E|_{V^{\otimes j}}) \otimes (E|_{V^{\otimes n-i-j}})
\]

\[
= \sum_{i=1}^{n-1} (e_i) \otimes (E|_{V^{\otimes n-i}}) + \sum_{j=1}^{n-1} \sum_{\ell=2}^{n-1-j} (e_j \otimes E|_{V^{\otimes \ell-j}}) \otimes (E|_{V^{\otimes n-\ell}})
\]

\[
= (e_1) \otimes (E|_{V^{\otimes n-1}}) + \sum_{j=1}^{n-1} \sum_{\ell=2}^{n-1-j} (e_j \otimes E|_{V^{\otimes \ell-j}}) \otimes (E|_{V^{\otimes n-\ell}}),
\]

and the proof follows by induction hypothesis from \( E|_{V^{\otimes \ell}} = e_\ell + \sum_{i=1}^{\ell-1} e_i \otimes E|_{V^{\otimes \ell-i}} \).

Theorem 2.5. Let \( E : \mathcal{T}V \to \mathcal{T}W \) and \( G : \mathcal{T}V \to \mathcal{T}W \) be linear mappings for which there exist linear mappings \( \{e_n : V^{\otimes n} \to W\}_{n \geq 1}, \{g_n : V^{\otimes n} \to W\}_{n \geq 1} \) such that \( E|_{V^{\otimes n}} = e_n + \sum_{B(n)} e_{r_1} \otimes \cdots \otimes e_{r_k} \) and \( G|_{V^{\otimes n}} = g_n + \sum_{B(n)} g_{r_1} \otimes \cdots \otimes g_{r_k} \).
with $B(n)$ given in [B]. Given a linear mapping $F : TV \to TW$, the following conditions are equivalent:

1. $C_{TW} \circ F = (E \otimes F + F \otimes G) \circ C_{TV}$,
2. there exist linear mappings $\{f_n : V \otimes^n \to W\}_{n \geq 1}$ such that

$$F|_{V \otimes^n} = f_n + \sum_{B(n)} \sum_{1 \leq i \leq k} e_{r_1} \otimes \cdots \otimes e_{r_{i-1}} \otimes f_{r_i} \otimes g_{r_{i+1}} \otimes \cdots \otimes g_{r_k}.$$ 

**Proof.** $(2) \Rightarrow (1)$: We have $F|_{V \otimes^n} = f_n + \sum_{i=1}^{n-1} E|_{V \otimes^i} \otimes f_{n-i} + \sum_{i=1}^{n-1} f_i \otimes G|_{V \otimes^{n-i}} + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} E|_{V \otimes^j} \otimes f_i \otimes G|_{V \otimes^{n-i-j}}$ for all $n \geq 1$. We now verify $(1)$ by expanding both sides:

$$(E \otimes F + F \otimes G) \circ C_{TV}|_{V \otimes^n} = (E \otimes F + F \otimes G) \circ \sum_{i=1}^{n-1} (1^{\otimes^{n-i}}) \otimes (1^{\otimes^i})$$

$$= \sum_{i=1}^{n-1} \left[(E|_{V \otimes^{n-i}}) \otimes (F|_{V \otimes^i}) + (F|_{V \otimes^{n-i}}) \otimes (G|_{V \otimes^i})\right],$$

and by Lemma 2.4 we get

$$C_{TW} \circ \left(\sum_{i=1}^{n-1} E|_{V \otimes^i} \otimes f_{n-i}\right)$$

$$= \sum_{i=1}^{n-1} (E|_{V \otimes^{n-i}}) \otimes (f_i) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (E|_{V \otimes^{n-i-j}}) \otimes (E|_{V \otimes^j} \otimes f_i),$$

$$C_{TW} \circ \left(\sum_{i=1}^{n-1} f_i \otimes G|_{V \otimes^{n-i}}\right)$$

$$= \sum_{i=1}^{n-1} (f_i) \otimes (G|_{V \otimes^{n-i}}) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (f_i \otimes G|_{V \otimes^j}) \otimes (G|_{V \otimes^{n-i-j}}),$$

$$C_{TW} \circ \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} E|_{V \otimes^j} \otimes f_i \otimes G|_{V \otimes^{n-i-j}}\right)$$

$$= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} (E|_{V \otimes^{n-i-j}}) \otimes (f_i \otimes G|_{V \otimes^j}) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} (E|_{V \otimes^j} \otimes f_i) \otimes (G|_{V \otimes^{n-i-j}})$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} \sum_{k=1}^{n-i-j-1} (E|_{V \otimes^{n-i-j-k}}) \otimes (E|_{V \otimes^j} \otimes f_i \otimes G|_{V \otimes^k})$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} \sum_{k=1}^{n-i-j-1} (E|_{V \otimes^j} \otimes f_i \otimes G|_{V \otimes^k}) \otimes (G|_{V \otimes^{n-i-j-k}}).$$
The summation in the variables $i + j$ and $i + j + k$, respectively, yields

\[
C_{TW} \circ \left( \sum_{i=1}^{n-1} E_{V^{\otimes i}} \otimes f_{n-i} \right)
\]

\[
= \sum_{i=1}^{n-1} (E_{V^{\otimes n-i}}) \otimes (f_i) + \sum_{\ell=2}^{n-1} \sum_{j=1}^{\ell-1} (E_{V^{\otimes \ell-j}}) \otimes (E_{V^{\otimes \ell-j}}) \otimes (f_j)
\]

\[
C_{TW} \circ \left( \sum_{i=1}^{n-1} f_i \otimes G_{V^{\otimes n-i}} \right)
\]

\[
= \sum_{i=1}^{n-1} (f_i) \otimes (G_{V^{\otimes n-i}}) + \sum_{\ell=2}^{n-1} \sum_{j=1}^{\ell-1} (f_j \otimes G_{V^{\otimes \ell-j}}) \otimes (G_{V^{\otimes \ell-j}})
\]

\[
C_{TW} \circ \left( \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} E_{V^{\otimes j}} \otimes f_i \otimes G_{V^{\otimes n-i-j}} \right)
\]

\[
= \sum_{\ell=2}^{n-1} \sum_{j=1}^{\ell-1} (E_{V^{\otimes \ell-j}}) \otimes (f_j \otimes G_{V^{\otimes \ell-j}})
\]

\[
+ \sum_{\ell=2}^{n-1} \sum_{j=1}^{\ell-1} (E_{V^{\otimes \ell-j}}) \otimes (f_j \otimes G_{V^{\otimes \ell-j}}) \otimes (G_{V^{\otimes \ell-j}})
\]

\[
+ \sum_{\ell=3}^{n-1} \sum_{j=1}^{\ell-1} \sum_{i=1}^{j-1} (E_{V^{\otimes n-i}}) \otimes (E_{V^{\otimes \ell-j}}) \otimes (f_i \otimes G_{V^{\otimes \ell-j}}) \otimes (G_{V^{\otimes \ell-j}})
\]

\[
+ \sum_{\ell=3}^{n-1} \sum_{j=1}^{\ell-1} \sum_{i=1}^{j-1} (E_{V^{\otimes \ell-j}}) \otimes (f_i \otimes G_{V^{\otimes \ell-j}}) \otimes (G_{V^{\otimes \ell-j}})
\]

Taking all terms of the form $(E_{V^{\otimes n-i}}) \otimes \star$ and $\star \otimes (G_{V^{\otimes n-i}})$ results in

\[
C_{TW} \circ F_{V^{\otimes n}} = \sum_{i=1}^{n-1} \left[ (E_{V^{\otimes n-i}}) \otimes (F_{V^{\otimes i}}) + (F_{V^{\otimes n-i}}) \otimes (G_{V^{\otimes i}}) \right]
\]

and the implication is proved. Notice that we also proved, on the assumption

\[
F_{V^{\otimes m}} = f_n + \sum_{B(m)} \sum_{1 \leq i \leq k} e_{r_1} \otimes \cdots \otimes e_{r_i-1} \otimes f_{r_i} \otimes g_{r_{i+1}} \otimes \cdots \otimes g_k \quad \text{for } n > m \geq 1,
\]

that

\[
C_{TW} \circ \left( \sum_{i=1}^{n-1} E_{V^{\otimes i}} \otimes f_{n-i} + \sum_{i=1}^{n-1} f_i \otimes G_{V^{\otimes n-i}} + \sum_{i=1}^{n-i-1} \sum_{j=1}^{n-i-1} E_{V^{\otimes j}} \otimes f_i \otimes G_{V^{\otimes n-i-j}} \right)
\]

\[
= \sum_{i=1}^{n-1} \left[ (E_{V^{\otimes n-i}}) \otimes (F_{V^{\otimes i}}) + (F_{V^{\otimes n-i}}) \otimes (G_{V^{\otimes i}}) \right].
\]

(1) $\Rightarrow$ (2): The proof is again by induction. For all $v \in V$ holds $C_{TW} \circ F(v) = 0$, which gives $F(V) \subseteq W$ and so there exists a linear mapping $f_1 : V \to W$ such that
Then the following conditions are equivalent:

A linear mapping $\delta : \mathcal{T}V \to \mathcal{T}V$ of degree $-1$ fulfills $C \circ \delta = (\delta \otimes 1_V + 1_V \otimes \delta) \circ C$ if and only if there exist a set of maps $\{\delta_n : V \otimes^n \to V\}_{n \geq 1}$ of degree $-1$ such that $\delta|_V = \delta_1$ and for $n \geq 2$ holds $\delta|_{V \otimes^n} = \delta_n + \sum_{i=1}^{n} 1_V^{\otimes i-1} \otimes \delta_1 \otimes 1_V \otimes 1_V^{\otimes n-i} + \sum_{A(n)} 1_V^{\otimes i-1} \otimes \delta_\ell \otimes 1_V^{\otimes k-i}$, where $A(n)$ is given by $[A]$. 

**Proof.** In Theorem 2.5 we take $E = G = 1_V$, where $e_1 = g_1 = 1_V$ and $e_n = g_n = 0$ for $n \geq 2$. 

**Lemma 2.7.** Let $\delta : \mathcal{T}V \to \mathcal{T}V$ be a linear map of degree $-1$ such that $\delta|_V = \delta_1$ and for $n \geq 2$ holds $\delta|_{V \otimes^n} = \delta_n + \sum_{i=1}^{n} 1_V^{\otimes i-1} \otimes \delta_1 \otimes 1_V \otimes 1_V^{\otimes n-i} + \sum_{A(n)} 1_V^{\otimes i-1} \otimes \delta_\ell \otimes 1_V^{\otimes k-i}$. Then the following conditions are equivalent:

1. $\delta \circ \delta = 0$,
2. $\delta_1 \circ \delta_1 = 0$ and for all $n \geq 2$ we have

$$\delta_1(\delta_n) + \sum_{i=1}^{n} \delta_n(1_V^{\otimes i-1} \otimes \delta_1 \otimes 1_V^{\otimes n-i}) + \sum_{A(n)} \delta_k(1_V^{\otimes i-1} \otimes \delta_\ell \otimes 1_V^{\otimes k-i}) = 0,$$

where $A(n)$ is given by $[A]$. 

**Proof.** (1) $\Rightarrow$ (2): The proof goes by induction. By assumption we have for $v \in V$ $\delta(\delta_1(v)) = 0$, so $\delta_1 : V \to V$ implies $\delta_1(\delta_1(v)) = 0$. Now assume $[7]$ is true for all natural numbers less than $n$. Then

$$\delta^2|_{V \otimes^n} = \delta_1(\delta_n) + \sum_{i=1}^{n} \delta|_{V \otimes^n}(1_V^{\otimes i-1} \otimes \delta_1 \otimes 1_V^{\otimes n-i})$$

$$+ \sum_{A(n)} \delta|_{V \otimes^k}(1_V^{\otimes i-1} \otimes \delta_\ell \otimes 1_V^{\otimes k-i}).$$
Schematically, this means

\[ \delta^2|_{V^\otimes n} = \delta_1(\delta_n) + \sum_{i=1}^{n} \delta_n(1_{V^i} \otimes \delta_1 \otimes 1_{V^{n-i}}) \]

\[ + \sum_{A(n)} \delta_k(1_{V^i} \otimes \delta_\ell \otimes 1_{V^{k-i}}) + \sum_{b+d+1} 1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e} \]

\[ + \sum_{b,c,d} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e}) \]

where the last row is a consequence of the Koszul sign convention:

\[ (1_{V^a} \otimes \delta_b \otimes 1_{V^{a+c+1+e}})(1_{V^a+b+c} \otimes \delta_d \otimes 1_{V^e}) = 1_{V^a} \otimes \delta_b \otimes 1_{V^c} \otimes \delta_d \otimes 1_{V^e} , \]

\[ (1_{V^a} \otimes \delta_b \otimes 1_{V^c} \otimes 1_{V^d} \otimes 1_{V^e}) \]

with \(|\delta_n| = -1\) for all \(n \in \mathbb{N}\). The term \( \sum_{A(n)} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e}) \)

\[ \sum_{b,c,d} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e}) = 1_{V^a} \otimes \delta_b \otimes 1_{V^c} \otimes \delta_d \otimes 1_{V^e} \]

We have \(a + b + c + d + e = n\), choose arbitrary \(a, e \geq 0, 1 \leq a + e < n\) and sum over all \(b, c, d\) such that \(0 \leq b, d \leq n - a - e\) and \(1 \leq c \leq n - a - e\) such that \(b + c + d = n - a - e\):

\[ \sum_{b,c,d} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e}) = \delta_1(\delta_{n'}) + \sum_{i=1}^{n'} \delta_{n'}(1_{V^i} \otimes \delta_1 \otimes 1_{V^{n'-i}}) \]

\[ + \sum_{A(n')} \delta_k(1_{V^i} \otimes \delta_\ell \otimes 1_{V^{k-i}}) , \]

where \(n' = n - a - e\). By induction hypothesis, the last display is equal to 0, and we have

\[ \sum_{a,c,d} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d} \otimes 1_{V^e}) = \sum_{a,c,d} 1_{V^a} \otimes \left( \sum_{b,c,d} \delta_{b+d+1}(1_{V^b} \otimes \delta_c \otimes 1_{V^d}) \right) \]

\[ \otimes 1_{V^e} = \sum_{a,e} 1_{V^a} \otimes 0 \otimes 1_{V^e} = 0 . \]

Consequently, (7) is true for \(n\) and

\[ \delta_1(\delta_n) + \sum_{i=1}^{n} \delta_n(1_{V^i} \otimes \delta_1 \otimes 1_{V^{n-i}}) + \sum_{A(n)} \delta_k(1_{V^i} \otimes \delta_\ell \otimes 1_{V^{k-i}}) = \delta^2|_{V^\otimes n} = 0 . \]

(2) \(\Rightarrow\) (1): The second implication can be easily deduced from the first one. \(\square\)

2.2. Morphisms and homotopies.

**Definition 2.8.** Let \(\delta^V\) be a codifferential on \((TV, C)\) and \(\delta^W\) be a codifferential on \((TW, C)\). A linear mapping \(F: (TV, C, \delta^V) \to (TW, C, \delta^W)\) of degree 0 is called morphism provided \(C_{TW} \circ F = (F \otimes F) \circ C_{TV}\) and \(\delta^W \circ F = F \circ \delta^V\).
Lemma 2.9. Let $F : (TV, \delta^V) \to (TW, \delta^W)$ be a linear map of degree 0. Then the following claims are equivalent:

1. $C_{TW} \circ F = (F \otimes F) \circ C_{TV}$,
2. there is a set of linear mappings $\{f_n : V^\otimes n \to W\}_{n \geq 1}$ of degree 0 such that $F|_{V^\otimes n} = f_n + \sum_{B(n)} f_{r_1} \otimes \cdots \otimes f_{r_k}$, with $B(n)$ given in (B).

Proof. (2) $\Rightarrow$ (1): A consequence of Lemma 2.4

(1) $\Rightarrow$ (2): The proof goes by induction. For $v \in V$ we have $C(v) = 0$, which implies $0 = (F \otimes F) \circ C_{TV} = C_{TW} \circ F$ and so $F(v) \in W$.
Assuming the claim is true for all natural numbers less than $n$,

$$(F \otimes F) \circ C|_{V^\otimes n} = (F \otimes F) \circ \sum_{i=1}^{n-1} (1^\otimes i) \otimes (1^\otimes n-i) = \sum_{i=1}^{n-1} (F|_{V^\otimes i}) \otimes (F|_{V^\otimes n-i})$$

and by induction hypothesis $F|_{V^\otimes m} = f_m + \sum_{B(m)} f_{r_1} \otimes \cdots \otimes f_{r_k}$ for all $n > m \geq 1$.

Lemma 2.4 gives

$$\sum_{i=1}^{n-1} (F|_{V^\otimes i}) \otimes (F|_{V^\otimes n-i}) = C_{TW} \circ \left( \sum_{i=1}^{n-1} f_i \otimes F|_{V^\otimes n-i} \right)$$

and because $C_{TW}$ is linear, $F|_{V^\otimes n}$ differs from $\sum_{i=1}^{n-1} f_i \otimes F|_{V^\otimes n-i}$ by a linear map $f_n : V^\otimes n \to W$. Then $F|_{V^\otimes n}$ is of the required form and the proof is complete. \ 

\[\square\]

Lemma 2.10. Let $F : (TV, \delta^V) \to (TW, \delta^W)$ be a linear map of degree 0 such that $F|_{V^\otimes n} = f_n + \sum_{B(n)} f_{r_1} \otimes \cdots \otimes f_{r_k}$, with all $\{f_n : V^\otimes n \to W\}_{n \geq 1}$ linear of degree 0. Then the following are equivalent:

1. $\delta^W \circ F = F \circ \delta^V$,
2. for all $n \geq 1$ holds

$$\delta^W_1(f_n) + \sum_{B(n)} \delta^W_k(f_{r_1} \otimes \cdots \otimes f_{r_k}) = f_1(\delta^V_n)$$

$$+ \sum_{i=1}^{n} f_n(1_V^\otimes i-1 \otimes \delta^V_1 \otimes 1_V^\otimes n-i) + \sum_{A(n)} f_k(1_V^\otimes i-1 \otimes \delta^V_k \otimes 1_V^\otimes k-i).$$

Proof. (1) $\Rightarrow$ (2): The proof goes by induction. The restriction to $V$, $\delta^W \circ F|_V = F \circ \delta^V|_V$, corresponds to $\delta^W \circ f_1 = f_1 \circ \delta^V$. We now assume (8) applies to all natural numbers less than $n$. We expand both sides of (8),

$$\delta^W \circ F|_{V^\otimes n}$$

$$= \delta^W_1(f_n) + \sum_{B(n)} \sum_{a,b} f_{r_1} \otimes \cdots \otimes f_{r_a} \otimes \delta^W_b\left(f_{r_{a+1}} \otimes \cdots \otimes f_{r_{a+b}}\right) \otimes f_{r_{a+b+1}} \otimes \cdots \otimes f_{r_k},$$

$$F \circ \delta^V|_{V^\otimes n}$$

$$= f_1(\delta^V_1) + \sum_{B(n)} \sum_{j,\ell} f_{r_1} \otimes \cdots \otimes f_{r_{j-1}} \otimes f_{r_j}(1_V^\otimes j \otimes \delta^V_\ell \otimes 1_V^\otimes r_{j-1}) \otimes f_{r_{j+1}} \otimes \cdots \otimes f_{r_k}$$
and compare the terms of same homogeneities. We fix $j \geq 1$ and $r_1, \ldots, r_j \geq 1$, $r_1 + \cdots + r_j < n$ and $0 \leq m \leq j$, and focus on terms of the form $f_{r_1} \star \cdots \star f_{r_{i-1}} \otimes \star f_{r_i} \cdots \otimes f_{r_j}$, where $\star$ is an expression of the form $\delta^W_*(f_* \cdots \otimes f_*)$ or $f_* (\mathbb{1}_V^\otimes \otimes \delta^V_1 \otimes \mathbb{1}_V^\otimes)$.

Terms on the right hand side of the form $f_{r_1} \otimes \cdots \otimes f_{r_{i-1}} \otimes \delta^W_*(f_* \cdots \otimes f_*) \otimes f_{r_i} \cdots \otimes f_{r_j}$ correspond to

$$f_{r_1} \otimes \cdots \otimes f_{r_m} \otimes \left( \delta^W_1(f_{r'}^n) + \sum_{B(n')} \delta^W_k \left( f_{r_1}^n \otimes \cdots \otimes f_{r_k}^n \right) \right) \otimes f_{r_{m+1}} \otimes \cdots \otimes f_{r_j},$$

while the terms of the form $f_{r_1} \otimes \cdots \otimes f_{r_{i-1}} \otimes f_* (\mathbb{1}_V^\otimes \otimes \delta^V_1 \otimes \mathbb{1}_V^\otimes) \otimes f_{r_i} \cdots \otimes f_{r_j}$ correspond to

$$f_{r_1} \otimes \cdots \otimes f_{r_m} \otimes \left( f_1(\delta^V_{n'}) + \sum_{i=1}^{n'} f_{n'} \left( \mathbb{1}_V^\otimes \delta^V_1 \otimes \mathbb{1}_V^\otimes \right) \right) \otimes f_{r_{m+1}} \otimes \cdots \otimes f_{r_j},$$

with $n' = n - r_1 + \cdots + r_j$. Because $n' < n$, they fulfill the equality (8) and hence are equal. Subtracting from both sides all elements of homogeneity greater than 1, we arrive at

$$\delta^W_1(f_n) + \sum_{B(n)} \delta^W_k(f_{r_1} \otimes \cdots \otimes f_{r_k})$$

$$= f_1(\delta^V_n) + \sum_{i=1}^{n} f_{n} \left( \mathbb{1}_V^\otimes \delta^V_1 \otimes \mathbb{1}_V^\otimes \right) + \sum_{A(n')} f_k \left( \mathbb{1}_V^\otimes \delta^V_1 \otimes \mathbb{1}_V^\otimes \right).$$

However, this equality is true by (8) for $n$.

(2) $\Rightarrow$ (1): This implication can be again reduced to the previous one.

**Definition 2.11.** Let $\delta^V$ be a codifferential on $(\mathcal{T}V, C)$ and $\delta^W$ be a codifferential on $(\mathcal{T}W, C)$. Let $F$: $(\mathcal{T}V, C, \delta^V) \to (\mathcal{T}W, C, \delta^W)$ and $G$: $(\mathcal{T}V, C, \delta^V) \to (\mathcal{T}W, C, \delta^W)$ be morphisms. $F$ and $G$ are homotopy equivalent provided there exist linear maps $H: \mathcal{T}V \to \mathcal{T}W$ of degree 1 such that $C_{\mathcal{T}W} \circ H = (F \otimes H + H \otimes G) \circ C_{\mathcal{T}V}$ and $F - G = H\delta^W + \delta^W H$. The map $H$ is a homotopy between $F$ and $G$.

**Remark 2.12.** Theorem 2.5 implies that $H: \mathcal{T}V \to \mathcal{T}W$ of degree 1 fulfills $C_{\mathcal{T}W} \circ H = (F \otimes H + H \otimes G) \circ C_{\mathcal{T}V}$ if and only if there is a set of maps $\{h_n: V^{\otimes n} \to W\}_{n \geq 1}$ of degree 1 such that $H|_{V^{\otimes n}} = h_n + \sum_{B(n), r_i > 0} f_{r_1} \otimes \cdots \otimes f_{r_{i-1}} \otimes h_i \otimes g_{r_{i+1}} \otimes \cdots \otimes g_{r_k}$.

**Theorem 2.13.** We retain the assumptions of Definition 2.11 and in addition assume the existence of the set of linear maps $\{e_n: V^{\otimes n} \to W\}_{n \geq 1}$, $\{g_n: V^{\otimes n} \to W\}_{n \geq 1}$ of even degree $d$ such that $E|_{V^{\otimes n}} = e_n + \sum_{B(n)} e_{r_1} \otimes \cdots \otimes e_{r_k}$ and $G|_{V^{\otimes n}} =$
Then the following assertions are equivalent:

1. \( E - G = F\delta^V + \delta^W F \),
2. \( e_n - g_n = f_1(\delta^V) + \sum_{i=1}^n f_n(1^\otimes_i \otimes \delta^V \otimes 1^\otimes_{n-i}) + \sum_{A(n)} f_k(1^\otimes_i \otimes \delta^V \otimes 1^\otimes_{k-i}) + \delta^W_1(f_n) + \sum_{B(n), r_i > 0} \delta_k^W (e_{r_1} \otimes \cdots \otimes e_{r_{i-1}} \otimes f_{r_i} \otimes g_{r_{i+1}} \otimes \cdots \otimes g_{r_k}) \)

for all \( n \geq 1 \).

**Proof.** The proof can be done along the same lines as the proofs of Lemma 2.7 and Lemma 2.10.

### 2.3. Codifferentials and \( A_\infty \) algebras.

**Definition 2.14.** For \( V \) graded we define \( sV \) in such a way that \((sV)_i = V_{i-1}\).

The graded modules \( V \) and \( sV \) are canonically isomorphic: \( s : V \to sV \) is a linear map of degree 1 called suspension, \( \omega : sV \to V \) is a linear map of degree \(-1\) called desuspension.

**Remark 2.15.** We have \( s^\otimes n \otimes \omega^\otimes n = (-1)^{n(n-1)/2} \) by the Koszul sign convention.

**Theorem 2.16.** The following claims are equivalent:

1. \( \{\mu_n : V^\otimes n \to V ; |\mu_n| = n - 2\}_{n \geq 1} \) is \( A_\infty \) structure on \( V \).
2. The linear maps \( \delta_n = s \circ \mu_n \circ \omega^\otimes n \) are of degree \(-1\), and are the components of a codifferential on \( \overline{T}sV \) in the sense of Theorem 2.6.

**Proof.** (2) \(\Rightarrow\) (1): \( \delta_n = s \circ \mu_n \circ \omega^\otimes n \) are the components of a codifferential, and so we have for all \( n \geq 1 \)

\[
\delta_1(\delta_n) + \sum_{i=1}^n \delta_n(1^\otimes_i \otimes \delta_1 \otimes 1^\otimes_{n-i}) + \sum_{A(n)} \delta_k(1^\otimes_i \otimes \delta_1 \otimes 1^\otimes_{k-i}) = 0.
\]

This can be rewritten, by Koszul sign convention, as

\[
\delta_1(\delta_n) = s \circ \mu_1 \circ \omega \circ s \circ \mu_n \circ \omega^\otimes n = s \circ \mu_1(\mu_n) \circ \omega^\otimes n,
\]

\[
\sum_{i=1}^n \delta_n(1^\otimes_i \otimes \delta_1 \otimes 1^\otimes_{n-i}) = \sum_{i=1}^n s \circ \mu_n \circ \omega^\otimes n(1^\otimes_i \otimes s \circ \mu_1 \circ \omega \otimes 1^\otimes_{n-i})
\]

\[
= \sum_{i=1}^n (-1)^{n-i} s \circ \mu_n(\omega^\otimes i-1 \otimes \mu_1 \circ \omega \otimes \omega^\otimes n-i)
\]

\[
= \sum_{i=1}^n (-1)^{n-i}(-1)^{i-1} s \circ \mu_n(1^\otimes_i \otimes \mu_1 \otimes 1^\otimes_{n-i}) \circ \omega^\otimes n,
\]
The following claims are equivalent:

\[ \sum_{A(n)} \delta_k \left( \frac{1}{V} \otimes (1) \otimes \frac{1}{V} \right) = \sum_{A(n)} s \circ \mu_k \circ \omega \otimes (1) \otimes s \circ \mu_\ell \circ \omega \otimes (1) \cdot \frac{1}{V} \otimes (1) \otimes \frac{1}{V} \]

\[ = \sum_{A(n)} (-1)^{k-i} s \circ \mu_k \left( \omega \otimes (1) \otimes \mu_\ell \otimes \omega \otimes (1) \right) \]

\[ = \sum_{A(n)} (-1)^{k-i} (-1)^{\ell(i-1)} s \circ \mu_k \left( \frac{1}{V} \otimes (1) \otimes \mu_\ell \otimes \frac{1}{V} \otimes (1) \right) \circ \omega \otimes n. \]

The mappings \( s \) and \( \omega \) are linear, hence

\[ s \circ (\mu_1 (\mu_n) + \sum_{i=1}^{n-1} (-1)^{i-1} \mu_i \otimes \mu_n) + \sum_{i=1}^{n} (-1)^{i+1} \mu_k \left( \frac{1}{V} \otimes (1) \otimes \mu_\ell \otimes \frac{1}{V} \otimes (1) \right) \circ \omega \otimes n = 0. \]

(1) \( \Rightarrow \) (2): This can be easily reduced to the proof of the previous implication. \( \square \)

**Theorem 2.17.** The following claims are equivalent:

1. \( \{ \varphi_n : V ^ {\otimes n} \to W ; |\varphi_n| = n - 1 \}_{n \geq 1} \) is \( A_\infty \) morphism from \((V, \mu)\) to \((W, \nu)\),
2. the mappings \( f_n = s_W \circ \varphi_n \circ \omega \otimes n \)

are of degree 0, and are the components of \( A_\infty \) morphism from \((\bar{T}sV, \delta^V)\) to \((\bar{T}sW, \delta^W)\) in the sense of Lemma 2.9. The codifferentials are given by \( A_\infty \) structures on \( V \) and \( W \), respectively, via Theorem 2.16.

The following claims are equivalent:

1. \( \{ h_n : V ^ {\otimes n} \to W ; |h_n| = n \}_{n \geq 1} \) is \( A_\infty \) homotopy between \( A_\infty \) morphisms \( \varphi \)

with components \( \{ \varphi_n : V ^ {\otimes n} \to W ; |\varphi_n| = n - 1 \}_{n \geq 1} \) and \( \psi \) with components \( \{ \psi_n : V ^ {\otimes n} \to W ; |\psi_n| = n - 1 \}_{n \geq 1} \), respectively, from \((V, \mu)\) to \((W, \nu)\),
2. \( h_n = s_W \circ h_n \circ \omega \otimes n \)

are of degree 1, and are the components of \( A_\infty \) homotopy between morphisms \( F \) and \( G \) from \((\bar{T}sV, \delta^V)\) to \((\bar{T}sW, \delta^W)\), where \( F \) corresponds to \( \varphi \) and \( G \) corresponds to \( \psi \) in the sense of the first equivalence in the theorem. The codifferentials are given by \( A_\infty \) structures on \( V \) and \( W \), respectively, as in Theorem 2.16.

**Proof.** The proof goes along the same lines as in Theorem 2.16. \( \square \)

### 3. Homotopy Transfer of \( A_\infty \) Algebras

The starting point for the present section are the chain complexes \((V, \partial_V)\) and \((W, \partial_W)\), \( f : V \to W \), \( g : W \to V \) their morphisms such that \( gf \) is homotopy equivalent to \( 1_V \) by a homotopy \( h \). Let \((V, \partial_V)\) be equipped with \( A_\infty \) algebra structure, which means that there is a set of multilinear maps \( \mu = (\mu_2, \mu_3, \ldots) \) satisfying the relations [1]. We would like to induce \( A_\infty \) structure \((W, \partial_W, \nu_2, \nu_3, \ldots)\) on \((W, \partial_W)\) by transferring \((V, \partial_V, \mu_2, \mu_3, \ldots)\), as well as the morphisms of \( A_\infty \)
algebras $\psi = (g, \psi_2, \psi_3, \ldots)$ from $(W, \partial W, \nu)$ to $(V, \partial V, \mu)$ and $\varphi = (f, \varphi_2, \varphi_3, \ldots)$ acting in the opposite direction such that their composition $\psi \varphi$ is $A_\infty$ homotopy equivalent with the identity map via $H = (h, H_2, H_3, \ldots)$.

The strategy to solve this problem, cf. [7], suggests to construct the set of maps $\{p_n : V^\otimes n \to V\}_{n \geq 2}$ of degree $n - 2$ called $p$-kernels, and the set of maps $\{q_n : V^\otimes n \to V\}_{n \geq 1}$ of degree $n - 1$ called $q$-kernels in such a way that $\nu_n, \varphi_n, \psi_n$ and $H_n$ defined by

(9) $\nu_n := f \circ p_n \circ g^\otimes n, \quad \varphi_n := f \circ q_n, \quad \psi_n := h \circ p_n \circ g^\otimes n, \quad H_n = h \circ q_n,$

fulfill the transfer problem of $A_\infty$ algebra as discussed in the previous paragraph.

We shall first introduce the $p$-kernels and based on them we introduce the $q$-kernels later on. Apart from (A) a (B), we shall rely on the notation (cf., [7])

(C) $C(n) = \{k, i, r_1, \ldots, r_i \in \mathbb{N} | 2 \leq k \leq n, 1 \leq i \leq k, r_1, \ldots, r_i \geq 1, \quad r_1 + \cdots + r_i + k - i = n\},$

for $n \in \mathbb{N}$, and

(\vartheta) $\vartheta(u_1, \ldots, u_k) = \sum_{1 \leq i < j \leq k} u_i(u_j + 1),$

for arbitrary $u_1, \ldots, u_k, k \in \mathbb{N}$.

3.1. $p$-kernels.

Lemma 3.1. The $p$-kernels together with $\partial W$ constitute an $A_\infty$ structure on $(W, \partial W)$ via (9) if and only if for all $n \geq 2$ holds

$$f \circ \left( \partial_V p_n - \sum_{u=1}^{n} (-1)^n p_n(1_V^\otimes u - 1 \otimes \partial_V \otimes 1_V^\otimes n-u) \right)$$

$$- \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k(1_V^\otimes i-1 \otimes gf \circ p_\ell \otimes 1_V^\otimes k-i) \otimes g^\otimes n = 0.$$

Proof. $(W, \partial W, \nu_2, \ldots)$ is an $A_\infty$ algebra if we have for all $n \geq 1$

$$\partial_W \nu_n - \sum_{u=1}^{n} (-1)^n \nu_n(1_W^\otimes u-1 \otimes \partial_W \otimes 1_W^\otimes n-u)$$

$$- \sum_{A(n)} (-1)^{i(\ell+1)+n} \nu_k(1_W^\otimes i-1 \otimes \nu_\ell \otimes 1_W^\otimes k-i) = 0.$$
This is true for \( n = 1 \), because \((W, \partial_W)\) is the chain complex \((f \circ \partial_V = \partial_W \circ f\) and analogously for \( g \)). Now expand \( \nu_n \) following \([9]\):

\[
\partial_W \nu_n = \sum_{u=1}^{n} (-1)^n \nu_n (1^\otimes_{W}^{u-1} \otimes \partial_W \otimes 1^\otimes_{W}^{n-u}) - \sum_{A(n)} (-1)^{i(\ell+1)+n} \nu_k (1^\otimes_{W}^{i-1} \otimes \nu_l \otimes 1^\otimes_{W}^{k-i})
\]

\[
= \partial_W (f \circ p_n \circ g^\otimes n) - \sum_{u=1}^{n} (-1)^n (f \circ p_n \circ g^\otimes n)(1^\otimes_{W}^{u-1} \otimes \partial_W \otimes 1^\otimes_{W}^{n-u})
\]

\[
- \sum_{A(n)} (-1)^{i(\ell+1)+n} (f \circ p_k \circ g^\otimes k) (1^\otimes_{W}^{i-1} \otimes (f \circ p_\ell \circ g^\otimes \ell) \otimes 1^\otimes_{W}^{k-i}).
\]

Because both \( f \) and \( g \) are linear maps of degree 0, this equals to

\[
f \circ (\partial_V \circ p_n) \circ g^\otimes n = f \circ \left( \sum_{u=1}^{n} (-1)^n p_n (g^\otimes_{u-1} \otimes g \circ \partial_W \otimes g^\otimes_{n-u}) \right)
\]

\[
- f \circ \left( \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (g^\otimes_{i-1} \otimes g f \circ p_\ell \circ g^\otimes \ell \otimes g^\otimes k-i) \right),
\]

which is

\[
f \circ (\partial_V \circ p_n) \circ g^\otimes n = f \circ \left( \sum_{u=1}^{n} (-1)^n p_n (1^\otimes_{V}^{u-1} \otimes \partial_V \otimes 1^\otimes_{V}^{n-u}) \right) \circ g^\otimes n
\]

\[
- f \circ \left( \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (1^\otimes_{V}^{i-1} \otimes g f \circ p_\ell \otimes 1^\otimes_{V}^{k-i}) \right) \circ g^\otimes n = 0.
\]

Lemma 3.2. Let us assume that \( p \)-kernels induce the transfer of \( A_\infty \) algebra as formulated above, and they fulfill \((n \geq 2)\)

\[
\partial_V p_n - \sum_{u=1}^{n} (-1)^n p_n (1^\otimes_{V}^{u-1} \otimes \partial_V \otimes 1^\otimes_{V}^{n-u})
\]

\[
- \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (1^\otimes_{V}^{i-1} \otimes g f \circ p_\ell \otimes 1^\otimes_{V}^{k-i}) = 0.
\]

Then

\[
p_n \circ g^\otimes n = \left( \sum_{B(n)} (-1)^{\theta(r_1, \ldots, r_k)} \mu_k (h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k}) \right) \circ g^\otimes n,
\]

where we define \( h \circ p_1 = 1_V \).

Proof. According to \([10]\) these \( p \)-kernels induce \( A_\infty \) structure on \((W, \partial_W)\) by Lemma 3.1. It remains to verify that they give \( A_\infty \) morphism from \((V, \partial_V, \mu)\) to
\((W, \partial_W, \nu)\), i.e.

\[
\partial_V \psi_n + \sum_{B(n)} (-1)^{\vartheta(r_1, \ldots, r_k)} \mu_k(\psi_{r_1} \otimes \cdots \otimes \psi_{r_k})
\]

\[
= \psi_1 \nu_n - \sum_{u=1}^n (-1)^n \psi_n(1_W^{\otimes u-1} \otimes \partial_W \otimes 1_W^{\otimes n-u})
\]

\[
- \sum_{A(n)} (-1)^{i(n+1)} \psi_k(1_W^{\otimes i-1} \otimes \nu_\ell \otimes 1_W^{\otimes k-i}),
\]

which by (9) can be formulated as

\[
\partial_V h \circ p_n \circ g^{\otimes n} + \left(\sum_{B(n)} (-1)^{\vartheta(r_1, \ldots, r_k)} \mu_k(h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k})\right) \circ g^{\otimes n}
\]

\[
= g f \circ p_n \circ g^{\otimes n} - h \circ \left(\sum_{u=1}^n (-1)^n p_n(1_V^{\otimes u-1} \otimes \partial_V \otimes 1_V^{\otimes n-u})\right) \circ g^{\otimes n}
\]

\[
- h \circ \left(\sum_{A(n)} (-1)^{i(n+1)} + p_k(1_V^{\otimes i-1} \otimes g f \circ p_\ell \otimes 1_V^{\otimes k-i})\right) \circ g^{\otimes n}.
\]

Due to \(gf - 1_V = \partial_V h + h \partial_V\), we have

\[
\left(\sum_{B(n)} (-1)^{\vartheta(r_1, \ldots, r_k)} \mu_k(h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k})\right) \circ g^{\otimes n}
\]

\[
= p_n \circ g^{\otimes n} - h \circ \left(- \partial_V p_n + \sum_{u=1}^n (-1)^n p_n(1_V^{\otimes u-1} \otimes \partial_V \otimes 1_V^{\otimes n-u})\right) \circ g^{\otimes n}
\]

\[
- h \circ \left(\sum_{A(n)} (-1)^{i(n+1)} + p_k(1_V^{\otimes i-1} \otimes g f \circ p_\ell \otimes 1_V^{\otimes k-i})\right) \circ g^{\otimes n}.
\]

By assumption \(10\), we obtain

\[
h \circ \left(- \partial_V p_n + \sum_{u=1}^n (-1)^n p_n(1_V^{\otimes u-1} \otimes \partial_V \otimes 1_V^{\otimes n-u})\right) \circ g^{\otimes n}
\]

\[
+ h \circ \left(\sum_{A(n)} (-1)^{i(n+1)} + p_k(1_V^{\otimes i-1} \otimes g f \circ p_\ell \otimes 1_V^{\otimes k-i})\right) \circ g^{\otimes n} = 0,
\]

which reduces to

\[
\left(p_n - \sum_{B(n)} (-1)^{\vartheta(r_1, \ldots, r_k)} \mu_k(h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k})\right) \circ g^{\otimes n} = 0.
\]
Theorem 3.6. The assumption of Lemma 3.2 can be weakened to
\[
\partial_V p_n \circ g^\otimes n - \sum_{u=1}^{n} (-1)^n p_n (1^\otimes u^{-1} \otimes \partial_V \otimes 1^\otimes n-u) \circ g^\otimes n - \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (1^\otimes i-1 \otimes g f \circ p_\ell \otimes 1^\otimes k-i) \circ g^\otimes n = 0,
\]
(11)
where (11) is fulfilled if the \( p \)-kernels define a monomorphism. In the situation of interest is \( f \), however, assumed to be an epimorphism.

Definition 3.4 (\( p \)-kernels, \[2\]). We define for each \( n \geq 2 \):
\[
p_n = \sum_{B(n)} (-1)^{\vartheta(r_1,\ldots,r_k)} \mu_k (h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k}),
\]
(12)
where \( h \circ p_1 = 1_V \), with \( B(n) \) given in \[B\] and \( \vartheta(r_1,\ldots,r_k) \) given in \[\vartheta\].

Remark 3.5. For \( p \)-kernels there exists a non-inductive explicit expression. Each term in the \( p \)-kernel can be represented by a rooted plane tree, and there is a function which associates to a rooted plane tree a sign corresponding to our inductive definition.

Theorem 3.6. The \( p \)-kernels introduced in \[2\] satisfy
\[
\partial_V p_n - \sum_{u=1}^{n} (-1)^n p_n (1^\otimes u^{-1} \otimes \partial_V \otimes 1^\otimes n-u)
- \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (1^\otimes i-1 \otimes g f \circ p_\ell \otimes 1^\otimes k-i) = 0,
\]
for all \( n \geq 2 \).

Proof. Let us first simplify our situation by passing to the suspension \( T_s V \) with the induced codifferential \( \delta \). Because \( s \) and \( \omega \) are by Definition 2.14 isomorphisms, (10) is true if and only if
\[
s \circ \left( \partial_V p_n - \sum_{u=1}^{n} (-1)^n p_n (1^\otimes u^{-1} \otimes \partial_V \otimes 1^\otimes n-u) \right) \circ \omega^\otimes n
= s \circ \left( \sum_{A(n)} (-1)^{i(\ell+1)+n} p_k (1^\otimes i-1 \otimes g f \circ p_\ell \otimes 1^\otimes k-i) \right) \circ \omega^\otimes n.
\]
Introducing \( \hat{p}_m = s \circ p_m \circ \omega^\otimes m, \hat{g} = s \circ g \circ \omega \) and \( \hat{f} = s \circ f \circ \omega \) \((|\hat{p}_m| = -1, |\hat{g}| = |\hat{f}| = 0)\), we have
\[
\delta_1 \hat{p}_n + \sum_{u=1}^{n} \hat{p}_n (1^\otimes u^{-1} \otimes \delta_1 \otimes 1^\otimes n-u) + \sum_{A(n)} \hat{p}_k (1^\otimes i-1 \otimes \hat{g} \hat{f} \circ \hat{p}_\ell \otimes 1^\otimes k-i) = 0.
\]
The proof of the last claim goes by induction. The case \( n = 2 \) corresponds to
\[
\delta_1 \delta_2 + \delta_2 (1_V \otimes \delta_1) + \delta_2 (\delta_1 \otimes 1_V) = 0,
\]
which is certainly true because \( \{ \delta_n : V^\otimes n \rightarrow V \}_{n \geq 1} \) are the components of the codifferential on \( TsV \) (cf., [7] for \( n = 2 \) in Lemma 2.7).

By induction hypothesis, we assume the claim is true for all natural numbers less than \( n \). The proof is naturally divided into three steps:

I. We shall first expand the term \( \delta_1 \hat{p}_n \): we have \( \hat{p}_n = s \circ p_n \circ \omega^\otimes n \), so by Definition 3.4

\[
\hat{p}_n = s \circ \left( \sum_{B(n)} (-1)^{\theta(r_1, \ldots, r_k) + \theta_1} \mu_k(h \circ p_{r_1} \otimes \cdots \otimes h \circ p_{r_k}) \right) \circ \omega^\otimes n
\]

\[
= \sum_{B(n)} (-1)^{\theta(r_1, \ldots, r_k) + \theta_1} s \circ \mu_k(h \circ s \circ h \circ p_{r_1} \circ \omega^\otimes r_1 \otimes \cdots \otimes h \circ s \circ h \circ p_{r_k} \circ \omega^\otimes r_k)
\]

with \( \sigma = \sum_{1 < j < k} r_i |r_j + 1| \). However, \( |s \circ h \circ p_{r_i} \circ \omega^\otimes r_i| = 1 + 1 + (r_i - 2) - r_i = 0 \), so the last display equals to

\[
\sum_{B(n)} s \circ \mu_k \circ \omega^\otimes k(s \circ h \circ \omega \circ s \circ p_{r_1} \circ \omega^\otimes r_1 \otimes \cdots \otimes s \circ h \circ \omega \circ s \circ p_{r_k} \circ \omega^\otimes r_k).
\]

Consequently,

\[
\hat{p}_n = \sum_{B(n)} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_k}) \quad \hat{h} = s \circ h \circ \omega \quad (|h| = 1),
\]

and so

\[
\delta_1 \hat{p}_n = \sum_{B(n)} \delta_1 \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_k})
\]

\[
= - \sum_{B(n)} \left( \sum_{i=1}^k \delta_k(1^\otimes i-1 \otimes \delta_1 \otimes 1^\otimes (k-i)) (h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_k}) \right)
\]

\[
- \sum_{B(n)} \left( \sum_{A(k)} \delta_{k'}(1^\otimes i-1 \otimes \delta_k \otimes 1^\otimes k'i-i)) (h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_k}) \right).
\]

The last summation can be rewritten as

\[
\sum_{B(n)} \left( \sum_{A(k)} \delta_{k'}(1^\otimes i-1 \otimes \delta_k \otimes 1^\otimes k'i-i)) (h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_k}) \right)
\]

\[
= \sum_{B(n)} \sum_{A(k)} \delta_{k'}(h \circ \hat{p}_{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i+1}} \otimes \cdots \otimes h \circ \hat{p}_{r_k})
\]

\[
= \sum_{B(n), r_i > 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}),
\]
where the last equality comes from the summation over all \( r_1, \ldots, r_{i+\ell} \) with \( r_i + \cdots + r_{i+\ell} \) fixed. We conclude

\[
\delta_1 \hat{p}_m = - \sum_{B(n), r_i > 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes (\hat{\delta} + 1_V) \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}) \\
- \sum_{B(n), r_i = 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \delta_1 \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}).
\]

**II.** We shall apply the induction hypothesis to \( \delta_1 \hat{p}_n \). We remind the formal equality \( h \circ \hat{p}_1 = 1_V \) and also \( gf - 1_V = \partial_V h + h \partial_V \) equivalent to \( \delta_1 h + 1_V = \hat{g} f - \hat{h} \delta_1 \). Then

\[
\delta_1 \hat{p}_n = \sum_{B(n), r_i > 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes (\hat{h} \delta_1 - \hat{g} f) \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}) \\
- \sum_{B(n), r_i = 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \delta_1 \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}).
\]

(14)

The second part of the first term on the right hand side (14) equals

\[
- \sum_{B(n)} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{g} f \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})
\]

\[
= \sum_{B(n)} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})(1_V \otimes s \otimes \hat{g} f \circ \hat{p}_{r_i} \otimes 1_V^{n-s-r_i}),
\]

where \( s = \sum_{j<i} r_j \). The second term in (14) equals

\[
- \sum_{B(n), r_i = 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \delta_1 \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})
\]

\[
= - \sum_{B(n), r_i = 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})(1_V \otimes s \otimes \delta_1 \otimes 1_V^{n-s-r_i}).
\]

By induction hypothesis, we have for all \( m < n \)

\[
\delta_1 \hat{p}_m = - \sum_{u=1}^m \hat{p}_m (1_V \otimes u-1 \otimes \delta_1 \otimes 1_V^{m-u}) - \sum_{A(m)} \hat{p}_k (1_V \otimes i-1 \otimes \hat{g} f \circ \hat{p}_k \otimes 1_V^{k-i}).
\]

Finally, the first part of the first term (14) equals

\[
\sum_{B(n), r_i > 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})
\]

\[
= - \sum_{B(n), r_i > 1} \delta_k(h \circ \hat{p}_{r_1} \otimes \cdots \otimes (\hat{\delta} + 1_V) \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}) \\
- \sum_{A(r_i)} \hat{p}_k (1_V \otimes i-1 \otimes \hat{g} f \circ \hat{p}_k \otimes 1_V^{k-i}) \cdots \otimes \hat{h} \circ \hat{p}_{r_k}.
\]
III. Now we pair up the contributions appearing in the previous step: the right hand side of (14) can be rewritten as

\[(P1) \quad \sum_{B(n), r_i > 1} \delta_k \left( h \circ \hat{p}_{r_1} \otimes \cdots \otimes \sum_{u=1}^{r_1} \hat{h} \circ \hat{p}_{r_u} \left( 1^\otimes u - 1 \otimes \delta_1 \otimes 1^\otimes r_i - u \right) \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \right) \]

\[(P2) \quad \sum_{B(n), r_i > 1} \delta_k \left( h \circ \hat{p}_{r_1} \otimes \cdots \otimes \sum_{A(r_i)} \hat{h} \circ \hat{p}_k \left( 1^\otimes i - 1 \otimes \hat{g} \circ \hat{p}_\ell \otimes 1^\otimes k - i \right) \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \right) \]

\[(P3) \quad \sum_{B(n)} \delta_k \left( h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_1 \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \right) \left( 1^\otimes s \otimes \delta_1 \otimes 1^\otimes n - s - r_i \right) \]

\[(P4) \quad \sum_{B(n), r_i = 1} \delta_k \left( h \circ \hat{p}_{r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_i} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \right) \left( 1^\otimes s \otimes \delta_1 \otimes 1^\otimes n - s - r_i \right), \]

with \( s = \sum_{j<i} r_j \), and we get

\[(P1) + (P4) = - \sum_{u=1}^{n} \hat{p}_n \left( 1^\otimes u - 1 \otimes \delta_1 \otimes 1^\otimes n - u \right), \]

\[(P2) + (P3) = - \sum_{A(n)} \hat{p}_k \left( 1^\otimes i - 1 \otimes \hat{g} \circ \hat{p}_\ell \otimes 1^\otimes k - i \right). \]

\[\square\]

Remark 3.7. Theorem 3.6 implies that the \( p \)-kernels in [7] fulfill (11).

3.2. \( q \)-kernels.

Lemma 3.8. The \( q \)-kernels constitute \( A_\infty \) morphism \( \varphi = (f, \varphi_2, \varphi_3, \ldots) \), \( \varphi_n = f \circ q_n \) and \( \nu_n = f \circ p_n \circ g^\otimes n \), from \( (V, \partial_V, \mu_2, \mu_3, \ldots) \) to \( (W, \partial_W, \nu_2, \nu_3, \ldots) \) if and only if for all \( n \geq 2 \):

\[ f \circ \left( \partial_V q_n + \sum_{u=1}^{n} (-1)^n q_n \left( 1^\otimes u - 1 \otimes \partial_V \otimes 1^\otimes n - u \right) \right) \]

\[ + \sum_{B(n)} (-1)^{\delta(r_1, \ldots, r_k)} p_k \left( g f \circ q_{r_1} \otimes \cdots \otimes g f \circ q_{r_k} \right) \]

\[ + \sum_{A(n)} (-1)^{i(\ell+1) + n} q_k \left( 1^\otimes i - 1 \otimes \mu_\ell \otimes 1^\otimes n - k \right) - q_1 \mu_n \right) = 0. \]

Proof. The proof easily follows from the explicit expansion of \( A_\infty \) morphism \( \varphi = (f, \varphi_2, \varphi_3, \ldots) \), which maps \( (V, \partial_V, \mu_2, \mu_3, \ldots) \) to \( (W, \partial_W, \nu_2, \nu_3, \ldots) \) for \( \varphi_n = f \circ q_n \) and \( \nu_n = f \circ p_n \circ g^\otimes n \) (cf. [9]). \( \square \)
Lemma 3.9. Let the $q$-kernels fulfill

\[
\begin{align*}
\partial V q_n + & \sum_{u=1}^{n} (-1)^n q_n \left( 1_V^{\otimes u-1} \otimes \partial V \otimes 1_V^{\otimes n-u} \right) \\
& + \sum_{B(n)} (-1)^{\vartheta(r_1,\ldots,r_k)} p_k (gf \circ q_{r_1} \otimes \cdots \otimes gf \circ q_{r_k}) \\
& + \sum_{A(n)} (-1)^{i+1} n q_k \left( 1_V^{\otimes i-1} \otimes \mu \otimes 1_V^{\otimes n-k} \right) = 0.
\end{align*}
\]

for all $n \geq 2$. Then we have

\[
q_n = \sum_{C(n)} (-1)^{n+r_i+\vartheta(r_1,\ldots,r_i)} \mu_k \left( (\psi\varphi)_{r_1} \otimes \cdots \otimes (\psi\varphi)_{r_{i-1}} \otimes h \circ q_{r_i} \otimes 1_V^{\otimes k-i} \right),
\]

for all $n \geq 2$, where $A_\infty$ morphisms $\varphi$ and $\psi$ are given by $p$-kernels and $q$-kernels, \[\text{\cite{[13]}}\]. We also used the notation $C(n)$ as in \[\text{\cite{[C]}}\] and $\vartheta(r_1,\ldots,r_k)$ as in \[\text{\cite{[\vartheta]}}\].

**Proof.** Assuming \[\text{\cite{[15]}}\], the set of $q$-kernels constitutes by Lemma \[\text{\cite{3.8}}\] $A_\infty$ morphism $\varphi = (f, \varphi_2, \varphi_3, \ldots)$ from $(V, \partial V, \mu_2, \mu_3, \ldots)$ to $(W, \partial W, \nu_2, \nu_3, \ldots)$. We also demand the set of maps $H_n = h \circ q_n$ gives $A_\infty$ homotopy $H = (h, H_2, H_3, \ldots)$ between $\psi\varphi$ and $1$. This is equivalent by Definition \[\text{\cite{1.3}}\] to

\[
\begin{align*}
\partial V H_n - & \sum_{u=1}^{n} (-1)^n H_n \left( 1_V^{\otimes u-1} \otimes \partial V \otimes 1_V^{\otimes n-u} \right) \\
& + \sum_{C(n)} (-1)^{n+r_i+\vartheta(r_1,\ldots,r_i)} \mu_k \left( (\psi\varphi)_{r_1} \otimes \cdots \otimes (\psi\varphi)_{r_{i-1}} \otimes H_{r_i} \otimes 1_V^{\otimes k-i} \right) + H_1 \mu_n \\
= & \sum_{A(n)} (-1)^{i+1} n H_k \left( 1_V^{\otimes i-1} \otimes \mu \otimes 1_V^{\otimes n-k} \right) + (\psi\varphi)_n - (1)_n
\end{align*}
\]

for all $n \geq 2$. According to \[\text{\cite{[3]}}\], we have

\[
(\psi\varphi)_m = \psi_1 \varphi_m + \sum_{B(m)} (-1)^{\vartheta(r_1,\ldots,r_k)} \psi_k (\varphi_{r_1} \otimes \cdots \otimes \varphi_{r_k}),
\]

and so we can write the composition of $A_\infty$ morphisms in terms of $p$-kernels and $q$-kernels:

\[
(\psi\varphi)_m = gf \circ q_m + \sum_{B(m)} (-1)^{\vartheta(r_1,\ldots,r_k)} h \circ p_k (gf \circ q_{r_1} \otimes \cdots \otimes gf \circ q_{r_k}).
\]
By Definition 1.3, the $A_\infty$ homotopy $H = (h, H_2, H_3, \ldots)$ can be rewritten in terms of $p$-kernels and $q$-kernels (we use again $\partial_V h = gf - 1_V - h\partial_V$ and $(1)_n = 0$):

\[
gf \circ q_n - q_n - h\partial_V q_n - \sum_{u=1}^{n} (-1)^n h \circ q_n (1_V^{\otimes u-1} \otimes \partial_V \otimes 1_V^{\otimes n-u}) + \sum_{C(n)} (-1)^{n+r_1+\vartheta(r_1,\ldots,r_i)} \mu_k((\psi\varphi)_{r_1} \otimes \cdots \otimes (\psi\varphi)_{r_{i-1}} \otimes h \circ q_{r_i} \otimes 1_V^{\otimes k-i}) + h \circ q_1\mu_n
\]

\[
= \sum_{A(n)} (-1)^{i(\ell+1)+n} h \circ q_k (1_V^{\otimes i-1} \otimes \mu_\ell \otimes 1_V^{\otimes n-k}) + gf \circ q_n + \sum_{B(n)} (-1)^{\vartheta(r_1,\ldots,r_k)} h \circ p_k(gf \circ q_{r_1} \otimes \cdots \otimes gf \circ q_{r_k}).
\]

We subtract from both sides of the last display $gf \circ q_n$, and by (15) conclude

\[
-h\partial_V q_n - \sum_{u=1}^{n} (-1)^n h \circ q_n (1_V^{\otimes u-1} \otimes \partial_V \otimes 1_V^{\otimes n-u}) + h \circ q_1\mu_n
\]

\[
= \sum_{A(n)} (-1)^{i(\ell+1)+n} h \circ q_k (1_V^{\otimes i-1} \otimes \mu_\ell \otimes 1_V^{\otimes n-k}) + \sum_{B(n)} (-1)^{\vartheta(r_1,\ldots,r_k)} h \circ p_k(gf \circ q_{r_1} \otimes \cdots \otimes gf \circ q_{r_k}),
\]

which finally results in

\[
q_n = \sum_{C(n)} (-1)^{n+r_1+\vartheta(r_1,\ldots,r_i)} \mu_k((\psi\varphi)_{r_1} \otimes \cdots \otimes (\psi\varphi)_{r_{i-1}} \otimes h \circ q_{r_i} \otimes 1_V^{\otimes k-i}).
\]

\[\square\]

Remark 3.10. The assumption (15) is fulfilled as soon as the $q$-kernels give a $A_\infty$ morphism $\varphi = (f, \varphi_2, \varphi_3, \ldots)$ from $(V, \partial_V, \mu_2, \mu_3, \ldots)$ to $(W, \partial_W, \nu_2, \nu_3, \ldots)$ and $f$ is a monomorphism.

Definition 3.11 ($q$-kernels, [7]). Let $n \geq 2$ and define $q_1 := 1_V$. We define $q$-kernels inductively by

\[
q_n = \sum_{C(n)} (-1)^{n+r_1+\vartheta(r_1,\ldots,r_i)} \mu_k((\psi\varphi)_{r_1} \otimes \cdots \otimes (\psi\varphi)_{r_{i-1}} \otimes h \circ q_{r_i} \otimes 1_V^{\otimes k-i}),
\]

where $(\psi\varphi)_m = gf \circ q_m + \sum_{B(m)} (-1)^{\vartheta(r_1,\ldots,r_k)} h \circ p_k(gf \circ q_{r_1} \otimes \cdots \otimes gf \circ q_{r_k})$ (cf., (16)). $p$-kernels were introduced in 3.4 with $C(n)$ given in [C] and $\vartheta(u_1,\ldots,u_k)$ in (q).

Remark 3.12. There is an explicit description of the $q$-kernels in terms of rooted plane trees, but it is much more complicated when compared to the analogous description for the $p$-kernels.
We shall now prove that the \( q \)-kernels introduced in Definition 3.11 satisfy (15). Let us consider again the suspension \( TSV \) with the induced codifferential \( \delta \) such that \( \delta_1 = s \circ \partial_V \circ \omega \) and \( \delta_n = s \circ \mu_n \circ \omega^{\otimes n}, n \geq 2 \). Then \( \hat{\psi}_m = s \circ \psi_m \circ \omega^{\otimes m} \) and \( \hat{\varphi}_m = s \circ \varphi_m \circ \omega^{\otimes m} \) for \( m \geq 2 \) \( (|\hat{\psi}_m| = |\varphi_m| = |\hat{\psi}_m| = 0) \), and (15) is equivalent to
\[
\delta_1 \hat{\psi}_n + \sum_{B(n)} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
\[
= \sum_{u=1}^n \hat{q}_u (1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes n-u}) + \sum_{A(m)} \hat{q}_{\ell} (1_V^{\otimes \ell-1} \otimes \delta_{\ell} \otimes 1_V^{\otimes n-k}) + \hat{q}_1 \delta_n .
\]
In the following two lemmas we prove that \( \hat{\psi} \hat{\varphi} \) is an \( A_\infty \) morphism.

**Lemma 3.13.** Let us assume (15) is true for all \( n \leq m \). Then the \( p \)-kernels in Definition 3.4 and the \( q \)-kernels in Definition 3.11 fulfill
\[
\delta_1 (\psi \varphi)_m = \sum_{u=1}^m (\psi \varphi)_m (1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes m-u})
\]
\[
+ \sum_{A(m)} (\hat{\psi} \hat{\varphi})_k (1_V^{\otimes \ell-1} \otimes \delta_{\ell} \otimes 1_V^{\otimes n-k}) + (\hat{\psi} \hat{\varphi})_1 \delta_m
\]
\[
- \sum_{B(m)} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
for all \( m \geq 2 \).

**Proof.** We shall first expand the composition of morphisms in the suspended form as in (16), and also use the homotopy \( \hat{h} \) between \( \hat{g} \hat{f} \) and \( 1_V \):
\[
\delta_1 (\psi \varphi)_m = \hat{g} \hat{f} \circ \delta_1 q_m + \sum_{B(m)} \delta_1 \hat{h} \circ \hat{p}_k \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
(17)
\[
= \hat{g} \hat{f} \circ \delta_1 q_m + \sum_{B(m)} (\hat{g} \hat{f} - 1_V - \hat{h} \delta_1) \circ \hat{p}_k \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}).
\]

By Theorem 3.6
\[
\sum_{B(m)} \hat{h} \circ \delta_1 \hat{p}_k \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
(18)
\[
= \sum_{B(m)} \sum_{u=1}^k \hat{h} \circ \hat{p}_u (1_V^{\otimes \ell-1} \otimes \delta_{\ell} \otimes 1_V^{\otimes k-k'}) \circ \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
(19)
and as \( \hat{g}, \hat{f} \) and \( \hat{q}_m \) are of degree 0, we have
\[
(18) = \sum_{B(m)} \sum_{u=1}^k \hat{h} \circ \hat{p}_u (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \delta_1 q_u) \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}.
\]
By (15) for \( n \leq m \), we expand the terms of the form \( \delta_1 \hat{q}_\bullet \) as

\[
- \sum_{B(m)u=1}^{k} \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_u} \left( 1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes u-v} \right) \cdots \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \\
- \sum_{B(m)u=1}^{k} \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_k \cdot \left( 1_V^{\otimes i'-1} \otimes \delta_{i'} \otimes 1_V^{\otimes r_u-k'} \right) \cdots \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \\
- \sum_{B(m)u=1}^{k} \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_1 \delta_{r_u} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \\
+ \sum_{B(m)u=1}^{k} \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_k \cdot \left( 1_V^{\otimes j'-1} \otimes \delta_{j'} \otimes 1_V^{\otimes r_u-j'} \right) \cdots \hat{g} \hat{f} \circ \hat{q}_{r_k} \right).
\]

In the second contribution (19), which equals to

\[
- \sum_{B(m)A'(k)} \hat{p}_{k'} \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_r \right) \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_1+\ell'-1} \right) \\
\cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right),
\]

we sum over all inner positions of \( \hat{p}_{k'} \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \) and get

\[
(19) = - \sum_{B(m)u=1}^{k} \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_r \right) \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_1+\ell'-1} \right) \\
\cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right).
\]

Up to a sign, this is the same expression as the expression on the fourth line of the expansion (18). We substitute into (17) for \( \sum_{B(m)} \hat{h} \circ \delta_1 \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \) the combination (18) + (19) and also substitute for \( \delta_1 \hat{q}_m \) according to (15):

\[
\delta_1 (\hat{q}_\varphi)_m = - \sum_{B(m)} \hat{g} \hat{f} \circ \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right) \\
+ \sum_{u=1}^{m} \hat{g} \hat{f} \circ \hat{q}_m \left( 1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes m-u} \right) \\
+ \sum_{A(m)} \hat{g} \hat{f} \circ \hat{q}_k \left( 1_V^{\otimes i-1} \otimes \delta_i \otimes 1_V^{\otimes n-k} \right) + \hat{g} \hat{f} \circ \hat{q}_1 \delta_m \\
+ \sum_{B(m)} \left( \hat{g} \hat{f} - 1_V \right) \circ \hat{p}_k \left( \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k} \right).
\]
+ \sum_{B(m)} \sum_{u=1}^{k} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_u} (1_{V}^{\otimes v-1} \otimes \delta_1 \otimes 1_{V}^{\otimes r_u-v})) \\
\otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}) \\
+ \sum_{B(m)} \sum_{u=1}^{k} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{k'} (1_{V}^{\otimes v-1} \otimes \delta_{v'} \otimes 1_{V}^{\otimes r_u-k'})) \\
\otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}) \\
+ \sum_{B(m)} \sum_{u=1}^{k} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{1} \delta_{r_u} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}).

This completes the proof. \hfill \square

Lemma 3.14. The $p$-kernels in Definition 3.4 and the $q$-kernels in Definition 3.11 fulfill
\[
\sum_{B(m)} \delta_k ((\psi \varphi)_{r_1} \otimes \cdots \otimes (\psi \varphi)_{r_k}) = \sum_{B(m)} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
for all $m \geq 2$.

Proof. By (13), we have
\[
\sum_{B(m)} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}) = \sum_{B(m)} \sum_{B'(k)} \delta_{r'} (h \circ \hat{p}_{r'_1} \otimes \cdots \otimes h \circ \hat{p}_{r'_{k'}})
\times (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}).
\]
Taking into account that $\hat{g}, \hat{f}$ and $\hat{q}_m$ are of degree 0, the last display equals to
\[
\sum_{B(m)} \sum_{B'(k)} \delta_{r'} (h \circ \hat{p}_{r'_1} (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_{r'_1}}) \\
\otimes \cdots \otimes h \circ \hat{p}_{r'_{k'}} (\hat{g} \hat{f} \circ \hat{q}_{r_{r'_{k'-1}}} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_{r'_{k'}}}))
\]
and the summation over the terms $\delta_{k} (\ast_{r_1} \otimes \cdots \otimes \ast_{r_k})$ in all possible indices ($\ast_j$ denoting a map $V^{\otimes j} \to V$) gives
\[
\sum_{B(m)} \delta_k \left( (\hat{g} \hat{f} \circ \hat{q}_{r_1} + \sum_{B'(r_1)} \hat{p}_{k'} (\hat{g} \hat{f} \circ \hat{q}_{r'_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r'_{r'_1}})) \\
\otimes \cdots \otimes (\hat{g} \hat{f} \circ \hat{q}_{r_k} + \sum_{B'(r_k)} \hat{p}_{k'} (\hat{g} \hat{f} \circ \hat{q}_{r'_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r'_{r_k}}))\right).
\]
However this is already (16) composed with the suspension, and the proof is complete. \hfill \square

Because the formula for the $q$-kernels in Lemma 3.9 was based on the assumption (15), we have to prove that it is fulfilled by the $q$-kernels in Definition 3.11.
Theorem 3.15. The \( p \)-kernels in Definition 3.4 and the \( q \)-kernels in Definition 3.11 fulfill (15), i.e.

\[
\partial_V q_n + \sum_{u=1}^{n} (-1)^u q_n (1_V^{\otimes u-1} \otimes \delta_V \otimes 1_V^{\otimes n-u})
+ \sum_{B(n)} (-1)^{\delta(r_1, \ldots, r_k)} p_k (g \mathcal{f} \circ q_{r_1} \otimes \cdots \otimes g \mathcal{f} \circ q_{r_k})
+ \sum_{A(n)} (-1)^{(\ell+1)+n} q_k (1_V^{\otimes i-1} \otimes \mu_{\ell} \otimes 1_V^{\otimes n-k}) - q_1 \mu_n = 0.
\]

This means that the objects introduced in (9) solve the problem of the transfer of \( A_\infty \) structure.

Proof. We shall prove an equivalent assertion:

\[
\delta_1 \hat{q}_n + \sum_{B(n)} \hat{p}_k (\hat{g} \mathcal{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \mathcal{f} \circ \hat{q}_{r_k})
= \sum_{u=1}^{n} \hat{q}_n (1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes n-u}) + \sum_{A(n)} \hat{q}_k (1_V^{\otimes i-1} \otimes \delta_\ell \otimes 1_V^{\otimes n-k}) + \hat{q}_1 \delta_n,
\]

with suspended \( q \)-kernels given by Definition 3.11

(20) \[ \hat{q}_n = \sum_{C(n)} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \].

The proof goes by induction on \( n \): for \( n = 2 \), we have by (7) (for \( n = 2 \)) and (20):

\[
\delta_1 \hat{q}_2 = \delta_1 (\delta_2 (\hat{g} \mathcal{f} \otimes \hat{h}) + \delta_2 (\hat{h} \otimes 1_V))
= - \delta_2 (\delta_1 \otimes 1_V) (\hat{g} \mathcal{f} \otimes \hat{h}) - \delta_2 (1_V \otimes \delta_1) (\hat{g} \mathcal{f} \otimes \hat{h})
- \delta_2 (\delta_1 \otimes 1_V) (\hat{h} \otimes 1_V) - \delta_2 (1_V \otimes \delta_1) (\hat{h} \otimes 1_V).
\]

By the Koszul sign convention

\[
\delta_2 (\delta_1 \otimes 1_V) (\hat{g} \mathcal{f} \otimes \hat{h}) = (-1)^{\delta_1 \mid \hat{h}} \delta_2 (\hat{g} \mathcal{f} \otimes \hat{h}) (\delta_1 \otimes 1_V),
\]

\[
\delta_2 (1_V \otimes \delta_1) (\hat{g} \mathcal{f} \otimes \hat{h}) = \delta_2 (\hat{g} \mathcal{f} \otimes \hat{g} \mathcal{f} - 1_V - \hat{h} \delta_1)
= \delta_2 (\hat{g} \mathcal{f} \otimes \hat{g} \mathcal{f}) - \delta_2 (\hat{g} \mathcal{f} \otimes 1_V) - \delta_2 (\hat{g} \mathcal{f} \otimes \hat{h}) (1_V \otimes \delta_1),
\]

\[
\delta_2 (\delta_1 \otimes 1_V) (\hat{h} \otimes 1_V) = \delta_2 (\hat{g} \mathcal{f} \otimes 1_V - \hat{h} \delta_1 \otimes 1_V)
= \delta_2 (\hat{g} \mathcal{f} \otimes 1_V) - \delta_2 (1_V \otimes 1_V - \delta_2 (\hat{h} \otimes 1_V) (\delta_1 \otimes 1_V),
\]

\[
\delta_2 (1_V \otimes \delta_1) (\hat{h} \otimes 1_V) = (-1)^{\delta_1 \mid \hat{h}} \delta_2 (\hat{h} \otimes 1_V) (1_V \otimes \delta_1),
\]
where \((-1)^{|\delta_1||\hat{h}|} = -1\) is a consequence of \(|\hat{h}| = |\delta_1| = 1\), and so

\[
\delta_1 \hat{q}_2 = \delta_2(g \hat{f} \otimes \hat{h})(\delta_1 \otimes 1_V) + \delta_2(\hat{h} \otimes 1_V)(\delta_1 \otimes 1_V) + \delta_2(\hat{g} \hat{f} \otimes \hat{h})(1_V \otimes \delta_1) \\
+ \delta_2(\hat{h} \otimes 1_V)(1_V \otimes \delta_1) + \delta_2(1_V \otimes 1_V) - \delta_2(\hat{g} \hat{f} \otimes \hat{g} \hat{f}) \\
= \hat{q}_2(\delta_1 \otimes 1_V) + \hat{q}_2(1_V \otimes \delta_1) + \hat{q}_1 \delta_2 - \hat{p}_2(\hat{g} \hat{f} \otimes \hat{g} \hat{f}).
\]

The induction step is divided into three steps:

I. We first expand the term \(\delta_1 \hat{q}_n\): by (20)

\[
\delta_1 \hat{q}_n = \sum_{C(n)} \delta_1 \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

\[
= - \sum_{C(n)} \left( \sum_{u=1}^k \delta_k(1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes k-u}) \right) \\
\times (((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
- \sum_{C(n)} \left( \sum_{A'(k)} \delta_{k'}(1_V^{\otimes i'-1} \otimes \delta_{l'} \otimes 1_V^{\otimes k'-i'}) \right) \\
\times (((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

The first summation can be rewritten as

\[
- \sum_{C(n)} \left( \sum_{u=1}^k \delta_k(1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes k-u}) \right) ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

(Q1.1)

\[
= - \sum_{C(n)} \sum_{u=1}^{i-1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_1(\hat{\psi} \hat{\varphi})_{r_u} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

(Q1.2)

\[
- \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \delta_1 \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
- (-1)^{|\delta_1||\hat{h}|} \sum_{C(n)} \sum_{u=i+1}^k \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-u-i})
\]

(Q1.3)

while the second as

\[
- \sum_{C(n)} \left( \sum_{A'(k)} \delta_{k'}(1_V^{\otimes i'-1} \otimes \delta_{l'} \otimes 1_V^{\otimes k'-i'}) \right) ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]
\[
(Q2.1) \quad = - \sum_{C(n)} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_*) \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]
\[
(Q2.2) \quad = - \sum_{C(n)} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_*) \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i} \otimes 1_V^{\otimes k-j})
\]
\[
(Q2.3) \quad = -(1)^{\delta_{ij}} \sum_{C(n)} \sum_{A'(k-i)} \delta_{i'}(\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

The summation over all indices in \((Q2.1)\) terms of the form \(\delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})\) leads to
\[
(Q2.1) = - \sum_{C(n)} \sum_{u=1}^{i-1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

Analogously, the summation over all indices in \((Q2.2)\) terms of the form \(\delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})\) gives
\[
(Q2.2) = - \sum_{C(n)} \sum_{r_i > 1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

**II. By Lemma 3.14**
\[
(Q2.1) = - \sum_{C(n)} \sum_{u=1}^{i-1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]

and Lemma 3.13 for \((\hat{\psi} \hat{\varphi})_m\) (Definition 3.11) and definition of \(C(n)\) in (C) imply that \(m\) is strictly less than \(n\), so that assumptions of Lemma 3.13 are fulfilled by our induction hypothesis gives
\[
(Q1.1) + (Q2.1) =
\]
\[
(Q1.1 + 2.1a) \quad = \sum_{C(n)} \sum_{u=1}^{i-1} \sum_{r_u=1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (-1)^{\hat{h}||\delta_1||} (\hat{\psi} \hat{\varphi})_{r_u} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]
\[
+ \sum_{C(n)} \sum_{u=1}^{i-1} \sum_{r_u=1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes (-1)^{\hat{h}||\delta_1||} (\hat{\psi} \hat{\varphi})_{r_u} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]
\[
(Q1.1 + 2.1b) \quad = \sum_{C(n)} \sum_{u=1}^{i-1} \sum_{r_u=1} \delta_k((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes \delta_*(\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i})
\]
where the first five terms come from (Q1.2) by application of Lemma 3.13, and the fifth one cancels out when combined with (Q2.1). Recall that we have $|\delta\ell| = -1$ for all $\ell$, and so $(-1)^{|\hat{h}||\delta\ell|} = -1$ as well as

\begin{align*}
(Q1.2) + (Q2.2) &= \sum_{C(n), r_i > 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes (-\delta_1 \hat{h} - 1_V)\hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
&\quad + \sum_{C(n), r_i = 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes -\delta_1 \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
&\quad + \sum_{C(n), r_i > 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes (\hat{h}\delta_1 - \hat{g}\hat{f})\hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
&\quad + \sum_{C(n), r_i = 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes (\hat{h}\delta_1 - \hat{g}\hat{f} + 1_V)\hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) .
\end{align*}

Thanks to the induction hypothesis we substitute for $\delta_1 \hat{q}_s$ and the last display turns into

\begin{align*}
&- \sum_{C(n), r_i > 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes \sum_{B'(r_i)} \hat{h} \circ \hat{p}_{r'}(\hat{g}\hat{f} \circ \hat{q}_{r'_i} \otimes \cdots \otimes \hat{g}\hat{f} \circ \hat{q}_{r'_{i-1}}) \otimes 1_V^{\otimes k-i}) \\
&- \sum_{C(n), r_i > 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes \hat{g}\hat{f} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i}) \\
&+ \sum_{C(n), r_i > 1} \delta_k((\hat{\psi}\hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi}\hat{\varphi})_{r_{i-1}} \otimes \sum_{u=1}^{r_i} \hat{q}_{r_i}(1_V^{\otimes u-1} \otimes \delta_1 \otimes 1_V^{\otimes (r_i-u)} \otimes 1_V^{\otimes k-i}) .
\end{align*}
\( (Q1.2 + 2.2b) \)
\[
\sum_{C(n), r_i > 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \sum_{A'}(r_i) \hat{q}^i_k(1_{V}^{\otimes r_i-1} \otimes \delta_{r^i} \otimes 1_{V}^{\otimes r_i-1} + \hat{q}_1 \delta_{r_i} \otimes 1_{V}^{\otimes k-i})
\]

\( (Q1.2 + 2.2c) \)
\[
\sum_{C(n), r_i = 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes h \circ \hat{q}_{r_i} \delta_1 \otimes 1_{V}^{\otimes k-i})
\]
\[
\sum_{C(n), r_i = 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes (-\hat{g} \hat{f} + 1_{V}) \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i})
\]

The non-numbered terms (first, second and sixth) can be further simplified. We notice
\[
- \sum_{C(n), r_i > 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \sum_{B'}(r_i) \hat{h} \circ \hat{p}_{r'} (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_{i-1}}) \otimes 1_{V}^{\otimes k-i})
\]
\[
- \sum_{C(n), r_i > 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes \hat{g} \hat{f} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i})
\]
\[
+ \sum_{C(n), r_i = 1} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes (-\hat{g} \hat{f} + 1_{V}) \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i})
\]
\[
= - \sum_{C(n)} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes (\hat{\psi} \hat{\varphi})_{r_i} \otimes 1_{V}^{\otimes k-i})
\]
\[
+ \sum_{C(n)} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_{i-1}} \otimes 1_{V}^{\otimes k-i+1})
\]
\[
= - \sum_{B(n)} \delta_k ((\hat{\psi} \hat{\varphi})_{r_1} \otimes \cdots \otimes (\hat{\psi} \hat{\varphi})_{r_k}) + \delta_n (1_{V}^{\otimes n}).
\]

By Lemma 3.14 this expression equals to
\( (Q1.2 + 2.2d) \)
\[
- \sum_{B(n)} \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}) + \hat{q}_1 \delta_n.
\]

**III.** In the last step we pair various contributions together: the first step can be written as
\[
\delta_1 \hat{q}_n = (Q1.1) + (Q1.2) + (Q1.3) + (Q2.1) + (Q2.2) + (Q2.3),
\]
while the second step as
\[
(Q1.1) + (Q2.1) = (Q1.1 + 2.1a) + (Q1.1 + 2.1b) + (Q1.1 + 2.1c) + (Q1.1 + 2.1d)
\]
and
\[
(Q1.2) + (Q2.2) = (Q1.2 + 2.2a) + (Q1.2 + 2.2b) + (Q1.2 + 2.2c) + (Q1.2 + 2.2d).
\]
Taken altogether,
\[
\begin{align*}
(Q1.3) + (Q1.1 + 2.1a) + (Q1.1 + 2.1b) + (Q1.2 + 2.2a) + (Q1.2 + 2.2c) \\
= \sum_{u=1}^{n} q_u (1^\otimes u-1 \otimes \delta_1 \otimes 1^\otimes n-u),
\end{align*}
\]
\[
(Q2.3) + (Q1.1 + 2.1c) + (Q1.1 + 2.1d) + (Q1.2 + 2.2b) \\
= \sum_{A(n)} q_k (1^\otimes i-1 \otimes \delta_i \otimes 1^\otimes n-k),
\]
\[
(Q1.2 + 2.2d) = - \sum_{B(n)} \hat{q}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}) + \hat{q}_1 \delta_n.
\]

The proof is complete. \hfill \Box

4. Homotopy Transfer and the Homological Perturbation Lemma

In the present section we discuss a motivation to find explicit formulas for the transfer of \(A_\infty\) algebra structure presented in an apparently arbitrary form in \([9]\).

In the following, we recall the homological perturbation lemma and show that it gives a recipe to search for the transfer problem exactly in the form \([9]\). This is the approach with which we develop and formalize \([7, \text{Remark 4}]\).

**Lemma 4.1** (Homological perturbation lemma, \([\Pi]\)). Let \((V, \partial_V)\) and \((W, \partial_W)\) be chain complexes together with quasi-isomorphisms \(f : V \to W\) and \(g : W \to V\) such that \(gf - 1_V = \partial_V h + h \partial_V\) for a linear map \(h : V \to V\). Let \(\mu : V \to V\) be a linear map of the same degree as \(\partial_V\) such that \((\partial_V + \mu)^2 = 0\) and the linear map \(1_V - \mu h\) is invertible \((\mu\) is called in this context perturbation\().\) We define
\[
\nu = \partial_W + fAg, \quad \psi = g + hAg, \quad \varphi = f + fAh, \quad H = h + hAh,
\]
where \(A = (1_V - \mu h)^{-1}\mu\). Then \((V, \partial_V + \mu)\) and \((W, \nu)\) are chain complexes and \(\varphi : V \to V, \psi : W \to W\) their quasi-isomorphisms with \(\psi \varphi - 1_V = (\partial_V + \mu)H + H(\partial_V + \mu)\).

In our case, on \((V, \partial_V)\) we have an additional \(A_\infty\) structure given by a collection of multilinear maps \(\mu = (\mu_2, \mu_3, \ldots)\) fulfilling certain axioms. In order to regard \(\mu\) as a perturbation, we have to pass to the (suspended) tensor algebra generated by \(V\). Let us consider \(\overline{TS}V\) with a coderivation \(\delta_V\) and \(\overline{TS}W\) with a coderivation \(\delta_W\), \(\overline{G}\) and \(\overline{H}\) a homotopy between \(\overline{G}\) \(\overline{F}\) and the identity on \(\overline{TS}V\). Here \(\delta_V\) is given by components \(\{s \circ \partial_V \circ \omega : sV \to sV\} \cup \{0 : sV^\otimes n \to sV\}_{n \geq 2}\) in the sense of Theorem 2.6 and it is codifferential by Lemma 2.7 because \(\partial_V\) is a differential on \(V\). Analogous conclusions do apply to \(\delta_W\). The map \(\overline{F} : (\overline{TS}V, \delta_V) \to (\overline{TS}W, \delta_W)\) is given by components \(\{s \circ f \circ \omega : sV \to sW\} \cup \{0 : (sV)^\otimes n \to sW\}_{n \geq 2}\) (Lemma 2.9).

By Lemma 2.10, \(\overline{F}\) is a morphism \((f\) is a map of chain complexes\), i.e. \(\overline{F}|_{(sV)^\otimes n} = \hat{f}^\otimes \) for \(\hat{f} = s \circ f \circ \omega\). Analogous conclusions apply to \(\overline{G}\) as well. Homotopy \(\overline{H} : \overline{TS}V \to \overline{TS}V\) is a map given by \(\{\hat{g} \hat{f} : sV \to sV\} \cup \{0 : (sV)^\otimes n \to sV\}_{n \geq 2}\) on the left, \(\{s \circ h \circ \omega : sV \to sV\} \cup \{0 : (sV)^\otimes n \to sV\}_{n \geq 2}\) in the middle and
We know that \( \delta \) is fulfilled and we can write
\[
\delta_{\mu}\big|_{(sV)^{\otimes n}} = \delta_n + \sum_{A(n)} 1^{i-1}_V \otimes \delta_\ell \otimes 1^{n-k}_V
\]
so that \( \hat{H}\big((sV)^{\otimes n}\big) \subseteq (sV)^{\otimes n} \) for all \( n \geq 1 \), and also \( \delta_{\mu}|_{sV} = 0 \) implies
\[
\delta_{\mu}|_{(sV)^{\otimes n}} = \delta_n + \sum_{A(n)} 1^{i-1}_V \otimes \delta_\ell \otimes 1^{n-k}_V
\]
for all \( n \geq 2 \) with \( A(n) \) as in \([A]\). Consequently, for all \( n \geq 2 \) holds \( \delta_{\mu}|(sV)^{\otimes n} \subseteq sV \oplus \cdots \oplus (sV)^{\otimes n-1} \), and its iteration results in \( (\delta_{\mu}\hat{H})^{n-1}\big((sV)^{\otimes n}\big) \subseteq sV, \)
\[
(\delta_{\mu}\hat{H})^{n}\big((sV)^{\otimes n}\big) = 0.
\]

By previous discussion and in accordance with Remark 2.3, \([1]\),
\[
(1 - \delta_{\mu}\hat{H})^{-1}|_{sV \oplus \cdots \oplus (sV)^{\otimes n}} = 1 + \sum_{i=1}^{n-1} (\delta_{\mu}\hat{H})^i,
\]
which means that \( 1 - \delta_{\mu}\hat{H} \) is invertible. Now all assumptions of Lemma 4.1 are fulfilled and we can write
\[
\delta_W + \delta_{\nu} = \delta_W + \hat{F}\left(\delta_{\mu}\sum_{n \geq 0}(\hat{H}\delta_{\mu})^n\right)\hat{G}, \quad \psi = \hat{G} + \hat{H}\left(\delta_{\mu}\sum_{n \geq 0}(\hat{H}\delta_{\mu})^n\right)\hat{G},
\]
\[
\varphi = \hat{F} + \hat{F}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right), \quad \hat{H} = \hat{H} + \hat{H}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right).
\]
Here we see immediately the motivation for \([9]\): \( \delta_{\mu}\sum_{n \geq 0}(\hat{H}\delta_{\mu})^n \) corresponds to the \( \hat{p} \)-kernels and \( \sum_{n \geq 1}(\delta_{\mu}\hat{H})^n \) corresponds to the \( \hat{q} \)-kernels. For our purposes it is more convenient to write
\[
\delta_W + \delta_{\nu} = \delta_W + \hat{F}\delta_{\mu}\hat{G} + \hat{F}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right)\delta_{\mu}\hat{G},
\]
\[
\psi = \hat{G} + \hat{H}\delta_{\mu}\hat{G} + \hat{H}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right)\delta_{\mu}\hat{G},
\]
\[
\varphi = \hat{F} + \hat{F}\delta_{\mu}\hat{H} + \hat{F}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right)\delta_{\mu}\hat{H},
\]
\[
\hat{H} = \hat{H} + \hat{H}\delta_{\mu}\hat{H} + \hat{H}\left(\sum_{n \geq 1}(\delta_{\mu}\hat{H})^n\right)\delta_{\mu}\hat{H}.
\]
There is a drawback related to these formulas, however: by a direct inspection we see that $\delta_W + \delta_\nu$ is not a coderivation in the sense of Theorem 2.6. $\hat{\phi}$ and $\hat{\psi}$ do not define a morphism in the sense of Lemma 2.9 and $\hat{H}$ does not fulfill the first part of morphism definition in the sense of Theorem 2.5.

In what follows we prove that on the additional assumptions (see [7, Remark 4]):

$$\hat{f} \hat{g} = 1, \quad \hat{f} \hat{h} = 0, \quad \hat{h} \hat{g} = 0, \quad \hat{h} \hat{h} = 0,$$

the homological perturbation lemma gives the results compatible with Section 3.

**Lemma 4.2.** Let us assume the formulas in (23) are satisfied. Then

1. $\hat{q}_n \circ \hat{g}^{\otimes n} = 0$ for $n \geq 2$,
2. $\hat{q}_{i+1+j} \circ ((\hat{g}\hat{f})^{\otimes i} \otimes \hat{h} \otimes 1_{\hat{V}^{j+1}}) = 0$ for all $i, j \geq 0, i + j \geq 1$.

**Proof.** (1): The proof goes by induction. By definition $\hat{q}_2 = \delta_2(\hat{g}\hat{f} \otimes \hat{h}) + \delta_2(\hat{h} \otimes 1_{\hat{V}})$ for $n = 2$, so that $\hat{q}_2 \otimes \hat{g}^{\otimes 2} = \delta_2(\hat{g}\hat{f}\hat{g} \otimes \hat{h}\hat{g}) + \delta_2(\hat{h}\hat{g} \otimes \hat{g})$ and the claim follows from (23).

We assume the assertion is true for all natural numbers less than $n \in \mathbb{N}$ ($n \geq 2$). By definition

$$\hat{q}_n \circ \hat{g}^{\otimes n} = \sum_{C(n)} \delta_k ([\hat{\psi}\hat{\phi}]_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\phi}]_{r_{i-1}} \circ \hat{g}^{\otimes r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \circ \hat{g}^{\otimes r_i} \otimes \hat{g}^{\otimes k-i}),$$

where

$$[\hat{\psi}\hat{\phi}]_m = \hat{g}\hat{f} \circ \hat{q}_m + \sum_{B(m)} \hat{h} \circ \hat{p}_k (\hat{g}\hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}\hat{f} \circ \hat{q}_{r_k})$$

with $[\hat{\psi}\hat{\phi}]_1 = \hat{g}\hat{f}$. In the case $r_i > 1$, the composition $\hat{h} \circ \hat{q}_{r_i} \circ \hat{g}^{\otimes r_i}$ is trivial by the induction hypothesis. If $r_i = 1$, $\hat{h} \circ \hat{q}_1 \circ \hat{g} = \hat{h}\hat{g}$ is trivial by (23).

(2): The proof is by induction on $n = i+1+j$. For $n = 2$ we prove

$$\hat{q}_2(\hat{h} \otimes 1_{\hat{V}}) = 0, \quad \hat{q}_2(\hat{g}\hat{f} \otimes \hat{h}) = 0.$$

As we know $\hat{q}_2(\hat{h} \otimes 1_{\hat{V}}) = (\hat{g}\hat{f})^{\otimes i} \otimes \hat{h} \otimes 1_{\hat{V}^{j+1}}$ and $\hat{q}_2(\hat{g}\hat{f} \otimes \hat{h}) = \delta_2(\hat{g}\hat{f}\hat{g} \otimes \hat{h}\hat{h}) + \delta_2(\hat{h}\hat{g} \otimes \hat{h})$, the claim follows thanks to (23).

Let the claim hold for $m \geq 2$ and all natural numbers less than $n$, we prove it is true for $n$. First of all, for $n > i' + j' + 1 \geq 2$ we have

$$[\hat{\psi}\hat{\phi}]_{i'+1+j'} \circ ((\hat{g}\hat{f})^{\otimes i'} \otimes \hat{h} \otimes 1_{\hat{V}^{j'+1}}) = 0$$

and also $\hat{g}\hat{f} \circ \hat{q}_1 \circ \hat{h} = \hat{g}\hat{f} \circ \hat{h} = 0$. By definition

$$[\hat{\psi}\hat{\phi}]_{i'+1+j'} \circ ((\hat{g}\hat{f})^{\otimes i'} \otimes \hat{h} \otimes 1_{\hat{V}^{j'+1}}) = \hat{g}\hat{f} \circ \hat{q}_{i'+1+j'} \circ ((\hat{g}\hat{f})^{\otimes i'} \otimes \hat{h} \otimes 1_{\hat{V}^{j'+1}})$$

$$+ \sum_{B(i'+1+j')} \hat{h} \circ \hat{p}_k (\hat{g}\hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}\hat{f} \circ \hat{q}_{r_k}) \circ ((\hat{g}\hat{f})^{\otimes i'} \otimes \hat{h} \otimes 1_{\hat{V}^{j'+1}}),$$
and by induction hypothesis $\hat{q}_{i'+1+j'} \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0$. The last summation can be conveniently rewritten as

$$
\sum_{B(i'+1+j')} \hat{h} \circ \hat{p}_k(\hat{g}\hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}\hat{f} \circ \hat{q}_{r_k}) \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'})
$$

$$
= \sum_{B(i'+1+j')} \hat{h} \circ \hat{p}_k(\hat{g}\hat{f} \circ \hat{q}_{r_1} \circ (\hat{g}\hat{f})^\otimes r_1 \otimes \cdots \otimes \hat{g}\hat{f} \circ \hat{q}_{r_k}) \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'})
$$

and the induction implies $\hat{q}_{r_u} \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0$ for $r_u > 1$. We already showed $\hat{g}\hat{f} \circ \hat{q}_{r_u} \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = \hat{g}\hat{f} \circ \hat{q}_1 \circ \hat{h} = 0$ for $r_u = 1$.

We now return back to the main thread of the proof and show $\hat{q}_n \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0$. We consider $k, i', r_1, \ldots, r_{i'-1}, r_{i'}$ in $C(n)$ given by (1), and compute

$$
\delta_k([\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i'-1}} \otimes \hat{h} \circ \hat{q}_{r_{i'}} \otimes 1_{V'}^{\otimes k-i'}) \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'})
$$

After substitution for $[\hat{\psi}\hat{\varphi}]$, there are the following three possibilities for indices $i$ and $i'$:

1. $i < r_1 + \cdots + r_{i'-1}$: Then there exist $1 \leq u < i$ such that $[\hat{\psi}\hat{\varphi}]_{r_u} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'})$. For $r_u \geq 2$ we already proved $[\hat{\psi}\hat{\varphi}]_{r_u} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0$, for $r_u = 1$ we have $[\hat{\psi}\hat{\varphi}]_{r_u} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = \hat{g}\hat{f} \circ \hat{h} = 0$.

2. $r_1 + \cdots + r_{i'-1} \leq i < r_1 + \cdots + r_{i'}$: In the tensor product there is a term of the form $\hat{h} \circ \hat{q}_{r_{i'}} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'})$, which is by the induction hypothesis 0 for $r_{i'} > 1$. If $r_{i'} = 1$, then $\hat{h} \circ \hat{q}_{r_{i'}} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = \hat{h} \circ \hat{f}$ equals to 0 by (1) of the lemma. If $r_{i'} = 1$, then $\hat{h} \circ \hat{q}_{r_{i'}} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = \hat{h} \circ \hat{g}\hat{f}$ equals to zero again by (1).

3. $r_1 + \cdots + r_{i'} \leq i$: In this case we get in the tensor product the term of the form $\hat{h} \circ \hat{q}_{r_{i'}} \circ ((\hat{g}\hat{f})^\otimes r_{i'} = \hat{h} \circ \hat{q}_{r_{i'}} \circ \hat{g}^\otimes r_{i'} \circ \hat{f}^\otimes r_{i'}$, which is trivial for $r_{i'} \geq 2$ by (1) of the lemma. If $r_{i'} = 1$, then $\hat{h} \circ \hat{q}_{r_{i'}} \circ ((\hat{g}\hat{f})^\otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = \hat{h} \circ \hat{g}\hat{f}$ equals to zero again by (1).

Because $k, i', r_1, \ldots, r_{i'-1}, r_{i'}$ in $C(n)$ was chosen arbitrarily, we get

$$
\sum_{C(n)} \delta_k([\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i'-1}} \otimes \hat{h} \circ \hat{q}_{r_{i'}} \otimes 1_{V'}^{\otimes k-i'}) \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0,
$$

and so finally $\hat{q}_n \circ ((\hat{g}\hat{f})^\otimes i' \otimes \hat{h} \otimes 1_{V'}^{\otimes j'}) = 0$. 

\[\square\]

**Remark 4.3.** We easily observe:

1. For all $n \geq 2$ and for linear mappings $\{a_n : (sV)^{\otimes n} \to sV\}_{n \geq 1}$,

$$
\sum_{B(n)} a_{r_1} \otimes \cdots \otimes a_{r_k} = \sum_{B(n), r_k > 1} a_{r_1} \otimes \cdots \otimes a_{r_k} + \sum_{u=1}^{n-3} \sum_{B(n-u), r_k > 1} a_{r_1} \otimes \cdots \otimes a_{r_k} \otimes a_1^{\otimes u}
$$

$$
+ \sum_{u=2}^{n-1} a_u \otimes a_1^{\otimes n-u} + a_1^{\otimes n},
$$
where $B(n)$ given as in $[B]$,

(2) For all $n \geq 2$, we have

$$[\hat{\psi}\hat{\phi}]_n \circ \hat{g}^\otimes n = \hat{h} \circ \hat{p}_n \circ \hat{g}^\otimes n,$$

and if $\hat{h} \circ \hat{p}_1 = 1_V$ (Definition 3.4) the formula is true for $n = 1$ as well.

(3) For all $n \geq 1$ and $0 \leq u \leq n - 1$:

$$[\hat{\psi}\hat{\phi}]_n \circ ((\hat{f} \hat{g})^\otimes u \otimes \hat{h} \otimes 1_V^\otimes n-1-u) = 0.$$

**Lemma 4.4.** Let us assume (23) is true for $n \geq 2$. Then

$$\hat{p}_n \circ \hat{g}^\otimes n = \delta_n \circ \hat{g}^\otimes n + \sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^\otimes n-i-u \right) \circ \hat{g}^\otimes n. \quad (25)$$

**Proof.** The proof goes by induction on $n$. As for $n = 2$ we have $\hat{p}_2 = \delta_2$, hence the claim follows.

We now assume the assertion holds for all natural numbers greater than 1 and less than $n$. Let us consider $2 \leq m < n$ and $k, i, r_1, \ldots, r_{i-1}, r_i$ as given in $C(m)$, so that

$$\delta_k ([\hat{\psi}\hat{\phi}]_{r_1} \circ \cdots \circ [\hat{\psi}\hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_1} \otimes 1_V^\otimes k-i) \circ (\hat{g}^\otimes u \otimes \delta_i \otimes \hat{g}^\otimes m-1-u) = 0$$

whenever $u < r_1 + \cdots + r_{i-1}$ or $r_1 + \cdots + r_i \leq u$ because $\hat{h} \circ \hat{q}_{r_1} \circ \hat{g}^{r_1} = 0$ for all $r_i \geq 1$ by Lemma 4.2.

We fix $n - 1 \geq k \geq 2, k \geq i \geq 1$ and $r_1, \ldots, r_{i-1} \geq 1$ as in $[C]$. As follows from the previous observation, all terms in

$$\sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^\otimes n-i-u \right) \circ \hat{g}^\otimes n$$

are of the form $\delta_k ([\hat{\psi}\hat{\phi}]_{r_1} \circ \hat{g}^{r_1} \cdots \circ [\hat{\psi}\hat{\phi}]_{r_{i-1}} \circ \hat{g}^{r_{i-1}} \otimes \star \otimes \hat{g}^{k-i})$ with $\star$ representing a mapping $(sV)^{\otimes \star} \to sV$ (the $\hat{q}$-kernels are given by $(20)$. We can rewrite them in the form

$$\delta_k ([\hat{\psi}\hat{\phi}]_{r_1} \circ \hat{g}^{r_1} \cdots \circ [\hat{\psi}\hat{\phi}]_{r_{i-1}} \circ \hat{g}^{r_{i-1}} \otimes$$

$$\otimes \left[ \delta_{n'} \circ \hat{g}^{r_1} + \sum_{i=2}^{n'-1} \hat{q}_{n'-i+1} \circ \left( \sum_{u=0}^{n'-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^\otimes n'-i-u \right) \circ \hat{g}^{n'} \right] \otimes \hat{g}^{k-i} \right),$$

where $n' = n + i - k - (r_1 + \cdots + r_{i-1})$, $n' > 1$. Applying the second point of Remark 4.3 to $[\hat{\psi}\hat{\phi}]_* \circ \hat{g}^{\otimes \star}$, the inducing hypothesis reduces the last display to

$$\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ \hat{g}^{r_1} \cdots \circ \hat{h} \circ \hat{p}_{r_{i-1}} \circ \hat{g}^{r_{i-1}} \circ \hat{h} \circ \hat{p}_{n'} \circ \hat{g}^{n'} \circ \hat{g}^{k-i}) \quad (26)$$
(we write $\hat{g} = h \circ \hat{p}_1 \circ \hat{g}$). By the first point of Remark 4.3, 
\[
\sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{g}^\otimes n
\]
\[
= \sum_{C(n), r_i > 1} \delta_k \left( h \circ \hat{p}_{r_1} \circ g^\otimes r_1 \otimes \cdots \otimes h \circ \hat{p}_{r_i} \circ g^\otimes r_i \otimes \hat{g}^\otimes k-i \right)
\]
\[
= \sum_{B(n), k \neq n} \delta_k \left( h \circ \hat{p}_{r_1} \circ g^\otimes r_1 \otimes \cdots \otimes h \circ \hat{p}_{r_k} \circ g^\otimes r_k \right),
\]
so that for each term in the sum there exists $u, r_u > 1$ (they are of the form of terms in \([26]\) with $n' > 1$). Adding the remaining term $\delta_n \circ \hat{g}^\otimes n$ and using the formula \([13]\) for the $\hat{p}$-kernels, the proof concludes.

**Lemma 4.5.** Let us assume \([23]\), and also 
\[
\hat{q}_n = \delta_n \circ \hat{H}_{(sV)^\otimes n} + \sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}_{(sV)^\otimes n}
\]
to be true for all $2 \leq m \leq n$. Then 
\[
\left[ \hat{\psi} \hat{\phi} \right]_{n} - h \circ \hat{p}_n \circ (\hat{g} \hat{f})^\otimes n = \left[ \hat{\psi} \hat{\phi} \right]_1 \circ \delta_n \circ \hat{H}_{(sV)^\otimes n}
\]
\[
+ \sum_{i=2}^{n-1} \left[ \hat{\psi} \hat{\phi} \right]_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}_{(sV)^\otimes n}.
\]

**Proof.** By \([24]\), we have for all $m \geq 2$
\[
\left[ \hat{\psi} \hat{\phi} \right]_m = \hat{g} \hat{f} \circ \hat{q}_m + \sum_{B(m)} h \circ \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}),
\]
(and $\left[ \hat{\psi} \hat{\phi} \right]_1 = \hat{g} \hat{f}$). We can split 
\[
\left[ \hat{\psi} \hat{\phi} \right]_1 \circ \delta_n \circ \hat{H}_{(sV)^\otimes n} + \sum_{i=2}^{n-1} \left[ \hat{\psi} \hat{\phi} \right]_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}_{(sV)^\otimes n}
\]
in two components and write 
\[
\hat{g} \hat{f} \circ \hat{q}_1 \circ \delta_n \circ \hat{H}_{(sV)^\otimes n}
\]
\[
+ \sum_{i=2}^{n-1} \hat{g} \hat{f} \circ \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}_{(sV)^\otimes n}
\]
\[
+ \sum_{i=2}^{n-1} \sum_{B(n-i+1)} h \circ \hat{p}_k (\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]
\[
\circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}_{(sV)^\otimes n}.
\]
Because $\hat{q}_n = 1_V$, we have $[28] = \hat{g}f \circ \hat{q}_n$ thanks to $[27]$. As for the second component $[29]$, consider $k, r_1, \ldots, r_k \in B(n-i+1)$ for some $i \geq 2$ with $B(n-i+1)$ as in $[B]$. Then

$$\hat{h} \circ \hat{p}_k(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k}) \circ \left(\sum_{u=0}^{n-i} 1_V^\otimes u \otimes \delta_i \otimes 1_V^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}$$

$$= \sum_{u=1}^{k} \hat{h} \circ \hat{p}_k \left(\hat{g}f \circ \hat{q}_{r_1} \otimes (\hat{g}f)^\otimes r_1 \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_{u-1}} \circ (\hat{g}f)^\otimes r_{u-1} \right) \otimes \left(\sum_{v=0}^{r_{u-1}-1} 1_V^\otimes v \otimes \delta_i \otimes 1_V^\otimes r_{u-1}-v \right) \circ \hat{H}|_{(sV)^\otimes r_{u-1}+i} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k} \right).$$

The reason for the appearance of such terms is that when $r_* \geq 2$ and $\hat{h}$ were in any other $\hat{q}$-kernel than $\delta_i$, we would get $\hat{q}_{r_*} \circ ((gf)^\otimes h \otimes 1_V^\otimes 1)$ which is trivial by Lemma 4.2. If $r_* = 1$, we get $\hat{g}f \circ \hat{q}_1 \circ \hat{h} = 0$ because $\hat{q}_1 = 1_V$ and $\hat{f} \hat{h} = 0$ by $[23]$. Thus we have for $i \geq 2$:

$$\sum_{B(n-i+1)} \hat{h} \circ \hat{p}_k(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k}) \circ \left(\sum_{u=0}^{n-i} 1_V^\otimes u \otimes \delta_i \otimes 1_V^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}$$

$$= \sum_{B(n-i+1)} \hat{h} \circ \hat{p}_k(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k})$$

$$+ \sum_{u=1}^{n-i-1} \sum_{B(n-i+1-u)} \hat{h} \circ \hat{p}_{u+k}(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k})$$

$$+ \sum_{r_1=1}^{n-1} \hat{h} \circ \hat{p}_{n-r_1+1}((\hat{g}f \circ \hat{q}_{1})^\otimes r_1 \otimes \hat{g}f \circ \hat{q}_{r_1})$$

where $\hat{g}f \circ \hat{q}_{r_1} = \hat{g}f \circ \hat{q}_{r_1} \circ \left(\sum_{v=0}^{r_1-1} 1_V^\otimes v \otimes \delta_i \otimes 1_V^\otimes r_1-1-v \right) \circ \hat{H}|_{(sV)^\otimes r_1+1+i}$. We notice $\hat{q}_m \circ (\hat{g}f)^\otimes m \neq 0$ if and only if $m = 1$ (cf., Lemma 4.2). Therefore, we expand the second contribution into

$$[29] = \sum_{i=2}^{n-1} \sum_{B(n-i+1)} \hat{h} \circ \hat{p}_k(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k})$$

$$+ \sum_{i=2}^{n-1} \sum_{u=1}^{n-i-1} \sum_{B(n-i+1-u)} \hat{h} \circ \hat{p}_{u+k}(\hat{g}f \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g}f \circ \hat{q}_{r_k})$$

$$+ \sum_{i=2}^{n-1} \sum_{r_1=1}^{n-i-1} \hat{h} \circ \hat{p}_{n-r_1+1}((\hat{g}f \circ \hat{q}_{1})^\otimes r_1 \otimes \hat{g}f \circ \hat{q}_{r_1})$$
and for fixed \(k, i, r_2, \ldots, r_i\) sum up all terms of the form \(\hat{h} \circ \hat{p}_k((\hat{g} \hat{f} \circ \hat{q}_1)^{\otimes k-i} \otimes \hat{g} \hat{f} \circ \hat{q}_{r_2} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_i})\) in \((29)\):

\[
\hat{h} \circ \hat{p}_k((\hat{g} \hat{f} \circ \hat{q}_1)^{\otimes k-i} \otimes \sum_{r_1=1}^{r'} \hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \hat{g} \hat{f} \circ \hat{q}_{r_2} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_i}),
\]

where \(r' = n - k + i - (r_2 + \cdots + r_i)\).

We recall \(\hat{g} \hat{f} \circ \hat{q}_{r_1} = \hat{g} \hat{f} \circ \hat{q}_{r_1} \circ \left(\sum_{u=0}^{r_1-1} \mathbf{1}_V^{\otimes u} \otimes \delta_{r'-r_1+1} \otimes \mathbf{1}_V^{\otimes r_1-1-u}\right) \circ \hat{H}(sV)^{\otimes r_1+1}\) and use \((27)\) to get

\[(30) \quad \hat{h} \circ \hat{p}_k((\hat{g} \hat{f} \circ \hat{q}_1)^{\otimes k-i} \otimes \hat{g} \hat{f} \circ \hat{q}_{r'} \otimes \hat{g} \hat{f} \circ \hat{q}_{r_2} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_i}).\]

Clearly \(r' > 1\), and the summation over all terms in \((29)\) leads to

\[
(29) = \sum_{B(n), r_1 > 1} \hat{h} \circ \hat{p}_k(\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]

\[
+ \sum_{u=1}^{n-3} \sum_{B(n-u), r_1 > 1} \hat{h} \circ \hat{p}_k((\hat{g} \hat{f} \circ \hat{q}_1)^{\otimes u} \otimes \hat{g} \hat{f} \circ \hat{q}_{r_2} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k})
\]

\[
+ \sum_{r=2}^{n-1} \hat{h} \circ \hat{p}_{n-r+1}((\hat{g} \hat{f} \circ \hat{q}_1)^{\otimes n-r} \otimes \hat{g} \hat{f} \circ \hat{q}_r).
\]

We conclude

\[
(29) = \sum_{B(n), k \neq n} \hat{h} \circ \hat{p}_k(\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}),
\]

because all terms are as those in \((30)\) and there is always at least one \(u\) such that \(r_1 > 1\) (this is equivalent to \(r' > 1\) in \((30)\)). Recall we started with

\[
\left[[\hat{\psi} \hat{\varphi}]_1 \circ \delta_n \circ \hat{H}|_{(sV)^{\otimes n}} + \sum_{i=2}^{n-1} [\hat{\psi} \hat{\varphi}]_{n-i+1} \circ \left(\sum_{u=0}^{n-i} \mathbf{1}_V^{\otimes u} \otimes \delta_i \otimes \mathbf{1}_V^{\otimes n-i-u}\right) \circ \hat{H}|_{(sV)^{\otimes n}}\right] = (28) + (29)
\]

and showed

\[
(28) + (29) = \hat{g} \hat{f} \circ \hat{q}_n + \sum_{B(n), k \neq n} \hat{h} \circ \hat{p}_k(\hat{g} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{g} \hat{f} \circ \hat{q}_{r_k}).
\]

Taking into account the definition of \([\hat{\psi} \hat{\varphi}]_n\) in \((24)\), the desired conclusion follows immediately.

**Lemma 4.6.** Let us assume \((23)\) to be true. Then for all \(n \geq 2\)

\[
(31) \quad \hat{q}_n = \delta_n \circ \hat{H}|_{(sV)^{\otimes n}} + \sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left(\sum_{u=0}^{n-i} \mathbf{1}_V^{\otimes u} \otimes \delta_i \otimes \mathbf{1}_V^{\otimes n-i-u}\right) \circ \hat{H}|_{(sV)^{\otimes n}}.
\]
**Proof.** The proof is by the induction hypothesis on \(n\). For \(n = 2\), by (31) we have
\[
\delta_2(\hat{g}\hat{f} \otimes \hat{h}) + \delta_2(\hat{h} \otimes 1_V) = \delta_2 \circ (\hat{g}\hat{f} \otimes \hat{h} + \hat{h} \otimes 1_V) \quad \text{which is certainly true.}
\]
We assume the claim is true for all natural numbers greater than 1 and strictly less than \(n\). Let us consider \(2 \leq j < n\) and \(k, i, r_1, \ldots, r_{i-1}, r_i\) as given in \(C(n-j+1)\). The same reasoning as in Lemma 1.5 leads to (32)
\[
\delta_k\left([\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{a}_{r_i} \otimes 1_V^{k-i}\right)
\sim \left(\sum_{u=0}^{n-j} 1_V^{\otimes u} \otimes \delta_j \otimes 1_V^{\otimes n-j-u}\right) \circ \hat{H}_{|_{(sV)^{\otimes n}}}
\]
\[
= \delta_k\left([\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{a}_{r_i} \otimes 1_V^{k-i}\right)
\]
\[
+ \delta_k\left(\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g}\hat{f})^{\otimes r_1} \otimes [\hat{\psi}\hat{\varphi}]_{r_2} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \circ \hat{h} \circ \hat{a}_{r_i} \otimes 1_V^{k-i}\right)
\]
\[
+ \delta_k\left(\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g}\hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{i-2}} \circ (\hat{g}\hat{f})^{\otimes r_{i-2}} \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \circ \hat{h} \circ \hat{a}_{r_i} \otimes 1_V^{k-i}\right)
\]
\[
+ \delta_k\left(\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g}\hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{i-1}} \circ (\hat{g}\hat{f})^{\otimes r_{i-1}} \circ \hat{h} \circ \hat{a}_{r_i} \otimes 1_V^{k-i}\right)
\]
\[
+ \delta_k\left(\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g}\hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{i-1}} \circ (\hat{g}\hat{f})^{\otimes r_{i-1}} \circ \hat{h} \circ \hat{a}_{r_i} \circ \hat{H}_{|_{(sV)^{\otimes k-i}}}\right).
\]

Hereby we expanded a general summand in the definition of \(\hat{q}_{n-j+1}\) as in (20), where
\[
[\hat{\psi}\hat{\varphi}]_{r_2} = [\hat{\psi}\hat{\varphi}]_{r_1} \left(\sum_{u=0}^{r_2-1} 1_V^{\otimes u} \otimes \delta_j \otimes 1_V^{\otimes r_2-1-u}\right) \circ \hat{H}_{|_{(sV)^{\otimes r_2-1+j}}}.
\]
\[
\hat{h} \circ \hat{q}_{r_1} = \hat{h} \circ \hat{q}_{r_1} \left(\sum_{u=0}^{r_2-1} 1_V^{\otimes u} \otimes \delta_j \otimes 1_V^{\otimes r_2-1-u}\right) \circ \hat{H}_{|_{(sV)^{\otimes r_1-1+j}}}.
\]
\[
\hat{h} \circ \hat{q}_{r_1} = \hat{h} \circ \hat{q}_{r_1} \left(\sum_{u=0}^{r_2-1} 1_V^{\otimes u} \otimes \delta_j \otimes 1_V^{\otimes r_1-1-u}\right) \circ (\hat{g}\hat{f})^{\otimes r_1-1+j}.
\]

In the previous formulas there are no signs whatsoever, because \(\hat{h} \circ \hat{q}_{r_1}\) pass through the terms of degree 0, and \(\hat{h}\) in \(\hat{H}\) and \(\delta_j\) are of degree 1 and \(-1\), respectively, so that their sign contributions cancel out.

In the next few steps we show how the terms are organized:

**I.** Let us choose \(k, i, r_1, \ldots, r_i\) given in \(C(n)\) such that \(r_i > 1\), and sum up all terms of the form \(\delta_k(\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g}\hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{i-1}} \circ (\hat{g}\hat{f})^{\otimes r_{i-1}} \otimes \hat{h} \circ \hat{q}_r \otimes 1_V^{\otimes k-i})\) out of the summation
\[
\delta_n \circ \hat{H}_{|_{(sV)^{\otimes n}}} + \sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left(\sum_{u=0}^{n-i} 1_V^{\otimes u} \otimes \delta_i \otimes 1_V^{\otimes n-i-u}\right) \circ \hat{H}_{|_{(sV)^{\otimes n}}}
\]
for all allowable $r$. We get

$$\delta_k \left( h \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i-1}} \circ (\hat{g} \hat{f})^{\otimes r_{i-1}} \otimes \sum_{r=1}^{r_i} \hat{h} \circ \hat{q}_r \otimes 1_{V^{k-r}} \right),$$

where

$$\hat{h} \circ \hat{q}_r = \hat{h} \circ \hat{q}_r \circ \left( \sum_{u=0}^{r-1} 1_{V^u} \otimes \delta_{r_i-r+1} \otimes 1_{V^{r_i-1-u}} \right) \circ \hat{H} |_{(\theta V)^{\otimes r_i}}.$$ 

Because $r_i < n$, we get by the induction hypothesis

$$(33) \quad \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i-1}} \circ (\hat{g} \hat{f})^{\otimes r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}}).$$

If $r_{i-1} > 1$, we sum up all terms of the form $\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_r \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}})$:

$$\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes \sum_{r=1}^{r_i-1} [\hat{\psi} \hat{\phi}]_r \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}}),$$

with

$$[\hat{\psi} \hat{\phi}]_r = [\hat{\psi} \hat{\phi}]_r \circ \left( \sum_{u=0}^{r-1} 1_{V^u} \otimes \delta_{r_i-r+1} \otimes 1_{V^{r_i-1-u}} \right) \circ \hat{H} |_{(\theta V)^{\otimes r_{i-1}}}.$$ 

Because $r_i < n$, by the induction hypothesis is Lemma 4.5 fulfilled and the last display reduces to

$$\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i-2}} \circ (\hat{g} \hat{f})^{\otimes r_{i-2}} \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}}).$$

The sum of the last display and $[33]$ results in

$$\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i-2}} \circ (\hat{g} \hat{f})^{\otimes r_{i-2}} \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}}),$$

which is the same expression as for $r_{i-1} = 1$ because $[\hat{\psi} \hat{\phi}]_{r_{i-1}} = \hat{h} \circ \hat{p}_{r_{i-1}} \circ (\hat{g} \hat{f})^{r_{i-1}}$ in this case. Repeating this procedure, we arrive at $\delta_k ([\hat{\psi} \hat{\phi}]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}})$.

We summarize the previous considerations: for $k, i, r_1, \ldots, r_i$ as in $C(n)$ such that $r_i > 1$, we have

$$\delta_k ([\hat{\psi} \hat{\phi}]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}})$$

$$= \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{r_1} \otimes \cdots \otimes h \circ \hat{p}_{r_{i-1}} \circ (\hat{g} \hat{f})^{r_{i-1}} \otimes \sum_{r=1}^{r_i} \hat{h} \circ \hat{q}_r \otimes 1_{V^{k-i}})$$

$$+ \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{r_1} \otimes \cdots \otimes \sum_{r=1}^{r_i-1} [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}})$$

and

$$\delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{r_1} \otimes \cdots \otimes \sum_{r=1}^{r_i} [\hat{\psi} \hat{\phi}]_{r_i} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V^{k-i}}).$$
\[+ \cdots + \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{r_1} \otimes \sum_{r=1}^{r_2} [\hat{\psi} \hat{\phi}]_r \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_v ^{\otimes k-i})\]

\[= \delta_k \left( \sum_{r=1}^{r_1} [\hat{\psi} \hat{\phi}]_r \otimes [\hat{\psi} \hat{\phi}]_{r_2} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_u} \otimes (\hat{g} \hat{f})^{\otimes i-u+1} \otimes \hat{h} \otimes 1_v ^{\otimes k-i} \right)\]

\[+ \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \sum_{r=1}^{r_2} [\hat{\psi} \hat{\phi}]_r \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_v ^{\otimes k-i})\]

\[+ \cdots + \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{u-1}} \circ (\hat{g} \hat{f})^{\otimes r_{u-1}} \otimes \sum_{r=1}^{r_u} [\hat{\psi} \hat{\phi}]_r \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_v ^{\otimes k-i})\]

\[= \delta_k \left( \sum_{r=1}^{r_1} [\hat{\psi} \hat{\phi}]_r \otimes [\hat{\psi} \hat{\phi}]_{r_2} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_u} \otimes (\hat{g} \hat{f})^{\otimes i-u+1} \otimes \hat{h} \otimes 1_v ^{\otimes k-i} \right)\]

\[+ \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{u-1}} \circ (\hat{g} \hat{f})^{\otimes r_{u-1}} \otimes \sum_{r=1}^{r_u} [\hat{\psi} \hat{\phi}]_r \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_v ^{\otimes k-i})\]

II. Let us choose \(k, i, r_1, \ldots, r_i\) as in \(C(n)\) such that \(i > 1, r_i = 1\), and there exists \(1 \leq u \leq i - 1\) such that \(r_u > 1\) and \(r_{u+1} = \cdots = r_{i-1} = 1\). Then

\[\delta_k ([\hat{\psi} \hat{\phi}]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_u} \otimes (\hat{g} \hat{f})^{\otimes i-u+1} \otimes \hat{h} \otimes 1_v ^{\otimes k-i})\]

\[= \delta_k \left( \sum_{r=1}^{r_1} [\hat{\psi} \hat{\phi}]_r \otimes [\hat{\psi} \hat{\phi}]_{r_2} \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_u} \otimes (\hat{g} \hat{f})^{\otimes i-u+1} \otimes \hat{h} \otimes 1_v ^{\otimes k-i} \right)\]

\[+ \cdots + \delta_k (\hat{h} \circ \hat{p}_{r_1} \circ (\hat{g} \hat{f})^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_{u-1}} \circ (\hat{g} \hat{f})^{\otimes r_{u-1}} \otimes \sum_{r=1}^{r_u} [\hat{\psi} \hat{\phi}]_r \otimes \cdots \otimes [\hat{\psi} \hat{\phi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_v ^{\otimes k-i})\]

\[\otimes (\hat{g} \hat{f})^{\otimes i-u+1} \otimes \hat{h} \otimes 1_v ^{\otimes k-i})\]

\[(35)\]

with

\[\overline{h} \circ \hat{q}_r = \hat{h} \circ \hat{q}_r \circ \left( \sum_{v=0}^{r-1} 1_v ^{\otimes v} \otimes \delta_{r_{u-r+1}} \otimes 1_v ^{\otimes r_{u-1-v}} \right) \circ (\hat{g} \hat{f})^{\otimes r_u}.\]

By Lemma 4.4

\[\sum_{r=1}^{r_u} \overline{h} \circ \hat{q}_r = \hat{h} \circ \hat{p}_{r_u} \circ (\hat{g} \hat{f})^{\otimes r_u},\]

which can be justified in the same way as in the first step I.

We expand all terms in the summation (denoted \(32\))

\[\sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1_v ^{\otimes u} \otimes \delta_u \otimes 1_v ^{\otimes n-i-u} \right) \circ \hat{H}|_{(sV) ^{\otimes n}}\]

\[\otimes \hat{H}|_{(sV) ^{\otimes n}}\]
and use (34) a (35) to rewrite terms in the definition of $\hat{q}_n$:

\[
\sum_{i=2}^{n-1} \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1_{V}^u \otimes \delta_i \otimes 1_{V}^{n-i-u} \right) \circ \hat{H}_{(sV)^n} = \sum_{C(n),k \neq n} \delta_k \left( [\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{k-i} \right).
\]

Certainly,

\[
\delta_n \circ \hat{H}_{(sV)^n} = \sum_{C(n),k=n} \delta_k \left( [\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{k-i} \right),
\]

which together with (20) completes the proof. □

**Remark 4.7.** Adopting slight changes in the proofs, our claims can be reformulated as follows:

**Lemma 4.4.** On the assumption (23) holds for all $n \geq 2$

\[
\left( \sum_{B(n)} \hat{h} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \right) \circ \hat{g}^{\otimes r_k} = \hat{g}^{\otimes n} \\
+ \sum_{i=2}^{n-1} \sum_{C(n-i+1)} [\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{k-i} \\
\circ \left( \sum_{u=0}^{n-i} 1_{V}^u \otimes \delta_i \otimes 1_{V}^{n-i-u} \right) \circ \hat{g}^{\otimes n},
\]

(36)

where we write $\hat{h} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k}$ instead of $\delta_k(\hat{h} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k})$ and $[\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{k-i}$ instead of $\delta_k([\hat{\psi}\hat{\varphi}]_{r_1} \otimes \cdots \otimes [\hat{\psi}\hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{k-i})$;

**Lemma 4.5.** On the assumption (23) holds for all $n \geq 2$

\[
\hat{f} \circ \hat{q}_n + \sum_{B(n)} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{f} \circ \hat{q}_{r_k} - \hat{f}^{\otimes n} = \hat{f} \circ \delta_n \circ \hat{H}_{(sV)^n} \\
+ \sum_{i=2}^{n-1} \left( \hat{f} \circ \hat{q}_{n-i+1} + \sum_{B(n-i+1)} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{f} \circ \hat{q}_{r_k} \right) \\
\circ \left( \sum_{u=0}^{n-i} 1_{V}^u \otimes \delta_i \otimes 1_{V}^{n-i-u} \right) \circ \hat{H}_{(sV)^n},
\]

(37)

where we write $\hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{f} \circ \hat{q}_{r_k}$ instead of $\hat{h} \circ \hat{p}_k(g \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \hat{g} \hat{f} \circ \hat{q}_{r_k})$;
Lemma 4.6. On the assumption \([23]\), we have for all \(n \geq 2\)

\[
\sum_{C(n)} [[\hat{\psi} \hat{\varphi}]]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i} = \hat{H} \big|_{(sV)^{\otimes n}}
\]

\[+
\sum_{i=2}^{n-1} \sum_{C(n-i+1)} [[\hat{\psi} \hat{\varphi}]]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i}
\]

\[
\circ \left( \sum_{u=0}^{n-i} 1_{V}^{\otimes n} \otimes \delta_{i} \otimes 1_{V}^{\otimes n-i-u} \right) \circ \hat{H} \big|_{(sV)^{\otimes n}} ,
\]

where we write \( [[\hat{\psi} \hat{\varphi}]]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i} \) instead of \( \delta_{k}([[\hat{\psi} \hat{\varphi}]]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i}) \).

Theorem 4.8. On the assumption \([23]\), the formulas produced by the homological perturbation lemma \([22]\) fulfill

1. \( \delta_{\nu} \big|_{(sV)^{\otimes n}} = \hat{f} \circ \hat{p}_{n} \circ \hat{g}^{\otimes n} + \sum_{A(n)} 1_{W}^{\otimes i-1} \otimes \hat{f} \circ \hat{p}_{i} \circ \hat{g}^{\otimes \ell} \otimes 1_{W}^{\otimes n-k} , \)

2. \( \hat{\psi} \big|_{(sV)^{\otimes n}} = \hat{h} \circ \hat{q}_{n} \circ \hat{g}^{\otimes n} + \sum_{B(n)} \hat{h} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{h} \circ \hat{p}_{r_k} \circ \hat{g}^{\otimes r_k} , \)

3. \( \hat{\varphi} \big|_{(sV)^{\otimes n}} = \hat{f} \circ \hat{q}_{n} + \sum_{B(n)} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{f} \circ \hat{q}_{r_k} , \)

4. \( \hat{H} \big|_{(sV)^{\otimes n}} = \hat{h} \circ \hat{q}_{n} + \sum_{C(n)} [[\hat{\psi} \hat{\varphi}]]_{r_1} \otimes \cdots \otimes [\hat{\psi} \hat{\varphi}]_{r_{i-1}} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_{V}^{\otimes k-i} \)

for all \(n \geq 2\). In particular, \( \delta_{W} + \delta_{\nu} \) is a codifferential, \( \hat{\psi}, \hat{\varphi} \) are morphisms and \( \hat{H} \) is a homotopy between \( \hat{\psi} \hat{\varphi} \) and \( 1 \). When expressed in terms of \( A_{\infty} \) algebras, the relevant objects fulfill \([9]\).

Proof. We already noticed

\[
(\delta_{\mu} \hat{H})((sV)^{\otimes n}) \subseteq sV \oplus \cdots \oplus (sV)^{\otimes n-1} ,
\]

\[
(\delta_{\mu} \hat{H})^{n-1}((sV)^{\otimes n}) \subseteq sV , \quad (\delta_{\mu} \hat{H})^{n}((sV)^{\otimes n}) = 0 .
\]

\([\hat{\psi}] \) & \([\hat{\varphi}]\): We prove by the induction hypothesis \([\hat{\varphi}]\). For \(n = 2\), we get by \([22]\)

\[
\hat{\varphi} \big|_{(sV)^{\otimes 2}} = \hat{F} \big|_{(sV)^{\otimes 2}} + \hat{F} \delta_{\mu} \hat{H} \big|_{(sV)^{\otimes 2}} = \hat{f} \otimes \hat{f} + \hat{f} \circ \left( \delta_{2}(\hat{g} \hat{f} \otimes \hat{h}) + \delta_{2}(\hat{h} \otimes 1_{V}) \right)
\]

\[
= \hat{f} \circ \hat{q}_{1} \otimes \hat{f} \circ \hat{q}_{1} + \hat{f} \circ \hat{q}_{2} .
\]
Let us assume \([3]\) holds for all natural number greater than 1 and less than \(n\). Because \(\delta_\mu \hat{H}\) decreases the homogeneity,

\[
\hat{\phi}|_{(sV)^\otimes n} = \hat{F}|_{(sV)^\otimes n} + \hat{F} \delta_\mu \hat{H}|_{(sV)^\otimes n} + \hat{F} \left( \sum_{m=1}^{n-2} (\delta_\mu \hat{H})^m \right) \delta_\mu \hat{H}|_{(sV)^\otimes n}
\]

\[
= \hat{F}|_{(sV)^\otimes n} + \hat{F} \circ \left( \sum_{u=0}^{n-i-1} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}
\]

\[
+ \hat{F} \left( \sum_{m=1}^{n-2} (\delta_\mu \hat{H})^m \right) \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}.
\]

The mapping \(\left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}\) is of homogeneity \(n - i + 1\), so \([22]\) allows us to rewrite the last result as

\[
\hat{F}|_{(sV)^\otimes n} + \hat{f} \circ \delta_n \circ \hat{H}|_{(sV)^\otimes n}
\]

\[
+ \sum_{i=2}^{n-1} \hat{\phi}|_{(sV)^\otimes n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{H}|_{(sV)^\otimes n}
\]

and the combination of induction hypothesis \(\hat{\phi}|_{(sV)^\otimes n-i+1}\) and Lemma \([4.6\) or \([37]\), gives the required form

\[
\hat{F}|_{(sV)^\otimes n} + \hat{f} \circ \hat{q}_n + \sum_{B(n)} \hat{f} \circ \hat{q}_{r_1} \otimes \cdots \otimes \hat{f} \circ \hat{q}_{r_k} - \hat{f}^\otimes n.
\]

Let us remark that \([22]\) gives for all \(n \geq 2\)

\[
\delta_v|_{(sV)^\otimes n} = \hat{f} \circ \delta_n \circ \hat{g}^\otimes n + \sum_{i=2}^{n-1} \hat{\phi}|_{(sV)^\otimes n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{g}^\otimes n.
\]

Choosing \(2 \leq j \leq n - 1\), Lemma \([4.2]\) implies

\[
\hat{\phi}|_{(sV)^\otimes n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{g}^\otimes n
\]

\[
= \hat{f} \circ \hat{q}_{n-i+1} \circ \left( \sum_{u=0}^{n-i} 1^\otimes u \otimes \delta_i \otimes 1^\otimes n-i-u \right) \circ \hat{g}^\otimes n
\]

\[
+ \sum_{u=1}^{n-i+1} (\hat{f} \hat{g})^\otimes u-1 \otimes \hat{f} \circ \delta_i \circ \hat{g}^\otimes i \otimes (\hat{f} \hat{g})^{\otimes n-i+1-u}
\]

\[
+ \sum_{A(n-i+1)} (\hat{f} \hat{g})^{\otimes i-1} \otimes \hat{f} \circ \hat{q}_{\ell} \circ \left( \sum_{u=0}^{\ell-1} 1^\otimes u \otimes \delta_i \otimes 1^\otimes \ell-1-u \right) \circ \hat{g}^\otimes \ell-1+i \otimes (\hat{f} \hat{g})^{\otimes n-i+1-k}.
\]
We take into account \((23)\), \(\hat{f}\hat{g} = 1_W\), and sum up over all terms of the form \(1_W^{\otimes*} \otimes \star \otimes 1_W^{\otimes*}\):

\[
\delta_n|_{(aV)^{\otimes n}} = \hat{f} \circ \delta_n \circ \hat{g}^\otimes + \sum_{i=2}^{n-1} \hat{f} \circ \hat{q}_{n-i+1} \\
\otimes \left( \sum_{u=0}^{n-i} 1_V^\otimes \otimes \delta_j \otimes 1_V^{\otimes n-i-u} \right) \otimes \hat{g}^\otimes + \sum_{A(n)} (\hat{f} \hat{g})^{\otimes i-1} \\
\otimes \left[ \hat{f} \circ \delta_\ell \circ \hat{g}^\otimes + \sum_{i=2}^{\ell-1} \hat{f} \circ \hat{q}_{\ell-i+1} \circ \left( \sum_{u=0}^{\ell-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^{\otimes \ell-i-u} \right) \circ \hat{g}^\otimes \right] \\
\otimes (\hat{f} \hat{g})^{\otimes n-k}.
\]

The application of Lemma \([4.4]\) concludes the proof. \([4\) & \([2\): Similarly to the previous part of the proof, we first concentrate on \([4\) and then derive \([2\). For \(n = 2\), it follows from \((22)\)

\[
\hat{H}|_{(aV)^{\otimes 2}} = \hat{H}|_{(aV)^{\otimes 2}} + \hat{H}_\delta \hat{H}|_{(aV)^{\otimes 2}} \\
= \hat{g} \hat{f} \otimes \hat{h} + \hat{h} \otimes 1_V \circ \left( \delta_2(\hat{g} \hat{f} \otimes \hat{h}) + \delta_2(\hat{h} \otimes 1_V) \right) \\
= [\hat{\psi}\hat{\varphi}] \otimes \hat{h} \circ \hat{q}_1 + \hat{h} \circ \hat{q}_1 \otimes 1_V \circ \hat{h} \circ \hat{q}_2.
\]

By the induction hypothesis, we assume \((4)\) holds for all natural numbers greater than 1 and less than \(n\). We can write

\[
\hat{H}|_{(aV)^{\otimes n}} = \hat{H}|_{(aV)^{\otimes n}} + \hat{h} \circ \delta_n \circ \hat{H}|_{(aV)^{\otimes n}} \\
+ \sum_{i=2}^{n-1} \hat{H}|_{(aV)^{\otimes n-i+1}} \circ \left( \sum_{u=0}^{n-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^{\otimes n-i-u} \right) \circ \hat{H}|_{(aV)^{\otimes n}}
\]

Thanks to the induction hypothesis we can expand \(\hat{H}|_{(aV)^{\otimes n-i+1}}\), and apply Lemma \([4.6]\) together with \((38)\):

\[
\hat{h} \circ \hat{q}_n + \sum_{r_i} \hat{\psi}\hat{\varphi}_{1} \otimes \cdots \otimes \hat{\psi}\hat{\varphi}_{r_i-1} \otimes \hat{h} \circ \hat{q}_{r_i} \otimes 1_V^{\otimes k-i},
\]

which completes the proof of the first assertion. Now we use again \((22)\) for \(n \geq 2\):

\[
\hat{\psi}|_{(aV)^{\otimes n}} = \hat{G}|_{(aV)^{\otimes n}} + \hat{h} \circ \delta_n \circ \hat{G}|_{(aV)^{\otimes n}} \\
+ \sum_{i=2}^{n-1} \hat{H}|_{(aV)^{\otimes n-i+1}} \circ \left( \sum_{u=0}^{n-i} 1_V^\otimes \otimes \delta_i \otimes 1_V^{\otimes n-i-u} \right) \circ \hat{G}|_{(aV)^{\otimes n}}.
\]
A consequence of Lemma 4.2, $2 \leq j \leq n - 1$ arbitrary, is

$$\hat{H}|_{(sV)^{\otimes n-j+1}} \circ \left( \sum_{u=0}^{n-j} 1_V^u \otimes \delta_j \otimes 1_V^{n-j-u} \right) \circ \hat{G}|_{(sV)^{\otimes n}} = \hat{H} \circ \hat{q}_{n-j+1} \circ \left( \sum_{u=0}^{n-j} 1_V^u \otimes \delta_j \otimes 1_V^{n-j-u} \right) \circ \hat{G}|_{(sV)^{\otimes n}} + \sum_{C(n-j+1)} \hat{H} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{H} \circ \hat{p}_{r_{i-1}} \circ \hat{g}^{\otimes r_{i-1}} \otimes \hat{H} \circ \hat{q}_i \circ \left( \sum_{u=0}^{r_i-1} 1_V^u \otimes \delta_j \otimes 1_V^{r_i-1-u} \right) \circ \hat{G} \otimes (\hat{H} \circ \hat{p}_1 \circ \hat{g})^{\otimes k-i},$$

because $\hat{H} \circ \hat{q}_m \circ g^{\otimes m} = 0$ for all $m \geq 1$. In other words, if $\delta_*$ in the last summation would not fit into $\hat{H} \circ \hat{q}_*$ the corresponding term will be trivial. The summation then leads to

$$\hat{\psi}|_{(sV)^{\otimes n}} = \hat{G}|_{(sV)^{\otimes n}} + \hat{H} \circ \delta_n \circ \hat{G}|_{(sV)^{\otimes n}} + \sum_{i=2}^{n-1} \hat{H} \circ \hat{q}|_{(sV)^{\otimes n-i+1}} \circ \left( \sum_{u=0}^{n-i} 1_V^u \otimes \delta_i \otimes 1_V^{n-i-u} \right) \circ \hat{G}|_{(sV)^{\otimes n}} + \sum_{C(n)} \hat{H} \circ \hat{p}_{r_1} \circ \hat{g}^{\otimes r_1} \otimes \cdots \otimes \hat{H} \circ \hat{p}_{r_{i-1}} \circ \hat{g}^{\otimes r_{i-1}} \otimes \left[ \hat{H} \circ \delta_i \circ \hat{G} + \sum_{i=2}^{\ell-1} \hat{H} \circ \hat{q}_{\ell-i+1} \circ \left( \sum_{u=0}^{i-1} 1_V^u \otimes \delta_i \otimes 1_V^{i-u} \right) \circ \hat{G} \right] \otimes (\hat{H} \circ \hat{p}_1 \circ \hat{g})^{\otimes k-i}.$$

In order to finish the proof, we remind the equality $\hat{H} \circ \hat{p}_1 = 1_V$ and use Lemma 4.4.

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