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Geometric properties of Wright function


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Abstract. In the present paper, we investigate certain geometric properties and inequalities for the Wright function and mention a few important consequences of our main results. A nonlinear differential equation involving the Wright function is also investigated.

Keywords: analytic function; univalent function; starlike function; strongly starlike function; convex function; close-to-convex function; Wright function; Bessel function; subordination of functions

MSC 2010: 30C45, 33C10

1. Introduction

The entire function (of $z$)

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C},$$

called the Wright function, has appeared for the first time in connection with the partitions of natural numbers, see [28]. Later on, it has been used in the asymptotic theory of partitions, Mikusinski operational calculus, integral transforms and in fractional differential equations (see [10], [13]). The Wright function can be represented in terms of familiar hypergeometric functions (see [10], page 389) and in terms of the Bessel functions $J_\nu$ (see [23], page 204).

Also, the Wright function generalizes various functions like array function, Whittaker function, (Wright-type) entire auxiliary functions, etc. The reader is referred to [10], [12] for details and many interesting results on the Wright function.

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Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ having the form

\begin{equation}
(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.
\end{equation}

By $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}$. For two analytic functions $f$ and $F$ in $\mathbb{D}$, we say that $f$ is subordinated to $F$, and express this symbolically by $f(z) \prec F(z)$, if $f(z) = F(w(z))$ in $\mathbb{D}$, for some analytic function $w$ in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. In particular, if $f \in \mathcal{S}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

A function $f \in \mathcal{A}$ is called starlike, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. The class of starlike functions in $\mathcal{A}$ is denoted by $\mathcal{S}^*$. Analytically, a function $f \in \mathcal{A}$ is called starlike if and only if it satisfies $\Re\{zf'(z)/f(z)\} > 0$, $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ which maps $\mathbb{D}$ onto a convex domain is called a convex function and the class of such functions is denoted by $K$. A function $f \in \mathcal{A}$ is called convex if and only if it satisfies $1 + \Re\{zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$. Let $\tilde{\mathcal{S}}^*(\alpha), 0 < \alpha \leq 1$ be the class of strongly starlike functions of order $\alpha$ in $\mathbb{D}$, which is defined by

\begin{equation}
(1.3) \quad \tilde{\mathcal{S}}^*(\alpha) = \{ f \in \mathcal{A} : \arg\left(\frac{zf'(z)}{f(z)}\right) \leq \frac{\alpha \pi}{2}, \ z \in \mathbb{D} \}.
\end{equation}

Note that $\tilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. Further, a function $f \in \mathcal{A}$ is called close-to-convex in $\mathbb{D}$ if the complement of $f(\mathbb{D})$ can be written as the union of non-intersecting half-lines. A function $f \in \mathcal{A}$ is close-to-convex with respect to a starlike function $g$, denoted by $\mathcal{C}_g$, if it satisfies $\Re\{zf'(z)/g(z)\} > 0$, $z \in \mathbb{D}$. For more details about these classes one can refer to [7], [9].

In this paper, we consider the following normalized form of the Wright function:

\begin{equation}
(1.4) \quad \mathbb{W}_{\lambda,\mu}(z) = z \Gamma(\mu)W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu)z^{n+1}}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \ \mu > 0, \ z \in \mathbb{D}.
\end{equation}

The normalized Wright function $\mathbb{W}_{\lambda,\mu}$ was studied recently by the present author in [23] (see also [17]). Note that

\begin{equation}
(1.5) \quad \mathbb{W}_{1,\nu+1}(-z) = J_\nu(z) = \Gamma(\nu + 1)z^{1-\nu/2}J_\nu(2\sqrt{z}).
\end{equation}

Here, $J_\nu(z)$ denotes the normalized Bessel function, investigated recently for the geometric properties in [2], [22], [25]. The function $J_\nu(z)$ is the well known Bessel function, defined by

\begin{equation}
(1.6) \quad J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n+\nu}}{n!\Gamma(n + \nu + 1)}.
\end{equation}
The special functions play an important role in function theory, especially the hypergeometric function, which appeared in De-Branges’ solution of the famous Bieberbach conjecture (see [6]). Several researchers studied classes of analytic functions involving special functions \( F \subset A \), to find different conditions such that the members of \( F \) to have certain geometric properties such as univalency, starlikeness or convexity in \( \mathbb{D} \). In this context many results are available in the literature regarding the hypergeometric functions (see [14], [24], [21], [20]), normalized Bessel functions (see [2], [4], [22], [25]), generalized Bessel functions (see [3], [16]), generalized Struve functions (see [30], [31]), Lommel functions (see [29]), Wright functions (see [23]) and Mittag-Leffler function (see [1]). In this paper, our main aim is to examine the geometric properties and inequalities of the Wright function \( \mathbb{W}_{\lambda,\mu} \). We also investigate an initial value problem involving the Wright function.

2. Close-to-convexity and starlikeness of \( \mathbb{W}_{\lambda,\mu} \)

In this section we obtain certain sufficient conditions for close-to-convexity and starlikeness of \( \mathbb{W}_{\lambda,\mu} \) in \( \mathbb{D} \). To prove our results, we shall need the following known results.

**Lemma 2.1** (Fejér [8]). Let \( f \in A \) be of the form (1.2) with \( a_n \geq 0 \). If the sequences \( \{na_n\} \) and \( \{na_n-(n+1)a_{n+1}\} \) are non-increasing, then \( f \) is starlike in \( \mathbb{D} \).

**Lemma 2.2** (Ozaki [18]). Let \( f \in A \) be of the form (1.2). If

\[
1 \geq 2a_2 \geq \ldots \geq na_n \geq (n+1)a_{n+1} \ldots \geq 0
\]

or

\[
1 \leq 2a_2 \leq \ldots \leq na_n \leq (n+1)a_{n+1} \ldots \leq 2,
\]

then \( f \) is close-to-convex with respect to \( g(z) = z/(1-z) \).

**Lemma 2.3** (Halenbeck and Ruscheweyh [11]). Let \( G(z) \) be convex and univalent in \( \mathbb{D} \) and \( F(z) \) be analytic in \( \mathbb{D} \) with \( G(0) = F(0) = 1 \). If \( F(z) \prec G(z) \) in \( \mathbb{D} \), then

\[
(n+1)z^{-n-1}\int_0^zt^nF(t)\,dt \prec (n+1)z^{-n-1}\int_0^zt^nG(t)\,dt, \quad n \in \mathbb{N} \cup \{0\}.
\]

Our first result is given below by Theorem 2.1:

**Theorem 2.1.** Let \( \lambda \geq 1 \) and \( \mu \geq 1 \). If \( \Gamma(\lambda+\mu) \geq 2\Gamma(\mu) \), then \( \mathbb{W}_{\lambda,\mu} \) is close-to-convex with respect to \( g(z) = z/(1-z) \).
Proof. By using Lemma 2.2, it is sufficient to show that

\[(2.1) \quad 1 \geq 2a_2 \geq \ldots \geq na_n \geq (n + 1)a_{n+1} \ldots \geq 0.\]

From (1.4), we have

\[
n a_n - (n + 1)a_{n+1} = na_n - \frac{(n + 1)\Gamma(\lambda(n - 1) + \mu)}{n\Gamma(\lambda n + \mu)} a_n
\]

\[
= \frac{a_n}{n\Gamma(\lambda n + \mu)} \left(n^2\Gamma(\lambda n + \mu) - (n + 1)\Gamma(\lambda(n - 1) + \mu)\right)
\]

\[
= \frac{a_n}{n\Gamma(\lambda n + \mu)} X(n),
\]

where

\[X(n) = n^2\Gamma(\lambda n + \mu) - (n + 1)\Gamma(\lambda(n - 1) + \mu).\]

Under the hypothesis, it is clear that

\[n^2\Gamma(\lambda n + \mu) = n^2\Gamma(\lambda(n - 1) + \lambda + \mu) \geq n^2\Gamma(\lambda(n - 1) + 1 + \mu)\]

\[= n^2(\lambda(n - 1) + \mu)\Gamma(\lambda(n - 1) + \mu)\]

\[\geq (n + 1)\Gamma(\lambda(n - 1) + \mu), \quad n \in \mathbb{N} \setminus \{1\}.\]

Also, \(X(1) \geq 0\) and \(\Gamma(\lambda n + \mu) \geq \Gamma(\lambda(n - 1) + \mu), n \geq 2.\) Hence \(X(n) \geq 0\) for all \(n \geq 1.\) This shows that the inequality (2.1) holds. This completes the proof. \(\square\)

Taking \(\lambda = 1, \mu = \nu + 1 (\nu > -1)\) and replacing \(z\) by \(-z\) in Theorem 2.1, we get the following result:

Corollary 2.1. If \(\nu \geq 1,\) then \(J_\nu\) is close-to-convex in \(\mathbb{D}\) with respect to \(g(z) = z/(1 - z)\).

Example 2.1. Taking \(\lambda = 1\) in Theorem 2.1, we obtain that the function \(W_{1,\mu}\) is close-to-convex for \(\mu \geq 2.\) Also, we obtain that the function \(W_{2,\mu}\) is close-to-convex for \(\mu \geq 1.\) In particular, functions \(W_{1,2}\) and \(W_{2,1}\) are close to convex and their image domains under \(\mathbb{D}\) are given below in Figures (a) and (b).

Theorem 2.2. Let \(\lambda \geq 1\) and \(\mu \geq 1.\) If \(\Gamma(\lambda + \mu) \geq 4\Gamma(\mu),\) then \(W_{\lambda,\mu}\) is starlike in \(\mathbb{D}\).

Proof. In view of Lemma 2.1, it is sufficient to prove that \(\{na_n\}\) and \(\{na_n - \)}
\((n+1)a_{n+1}\) are non-increasing sequences for all \(n \geq 1\). Clearly, the sequence \(\{na_n\}\) is non-increasing by Theorem 2.1. Therefore, it suffices to show that

\[
(2.2) \quad na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \geq 0 \quad \forall n \geq 1.
\]

Under the hypothesis, we have

\[
n^2 \Gamma(\lambda n + \mu) \geq n^2 \Gamma(\lambda(n-1) + \mu + 1) = n^2(\lambda(n-1) + \mu)\Gamma(\lambda(n-1) + \mu)
\]
\[
\geq 2(n+1)\Gamma(\lambda(n-1) + \mu), \quad n \in \mathbb{N}.
\]

Hence

\[
n a_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} = \frac{n\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)(n-1)!} - \frac{2(n+1)\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} + \frac{(n+2)\Gamma(\mu)}{\Gamma(\lambda(n+1) + \mu)(n+1)!}
\]
\[
= \frac{\Gamma(\mu)}{(n-1)!} \left( \frac{n}{\Gamma(\lambda(n-1) + \mu)} - \frac{2(n+1)}{\Gamma(\lambda n + \mu)n} + \frac{(n+2)}{\Gamma(\lambda(n+1) + \mu)(n+1)n} \right) \geq 0.
\]

This shows that the inequality (2.2) holds, hence \(W_{\lambda,\mu}(z)\) is starlike in \(\mathbb{D}\). \(\square\)

Taking \(\lambda = 1\), \(\mu = \nu + 1\), \(\nu > -1\) and replacing \(z\) by \(-z\) in Theorem 2.2, we get the following result:

**Corollary 2.2.** If \(\nu \geq 3\), then \(J_{\nu}\) is starlike in \(\mathbb{D}\).
Example 2.2. Taking $\lambda = 1$ in Theorem 2.2, we obtain that the function $W_{1,\mu}$ is starlike for $\mu \geq 4$. Also, we obtain that the function $W_{2,\mu}$ is starlike for $\mu \geq \frac{1}{2}(-1 + \sqrt{17})$. Further, we observe that, as $\lambda$ increases, $\mu$ decreases to preserve the starlikeness of the function $W_{\lambda,\mu}$.

Theorem 2.3. If $\lambda \geq 1$ and $\mu \geq 1 + \sqrt{3}$, then $W_{\lambda,\mu} \in \tilde{S}^*(\alpha)$. Here $\alpha$ is given by

$$\alpha = \frac{2}{\pi} \arcsin \left( \eta \sqrt{1 - \frac{1}{4} \eta^2 + \frac{1}{2} \eta \sqrt{1 - \eta^2}} \right),$$

where $\eta = 2(\mu + 1)/\mu^2$.

Proof. Under the hypothesis, the inequality $\Gamma(\mu + n) \leq \Gamma(\lambda n + \mu)$, $n \in \mathbb{N}$ holds and is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \leq \frac{\Gamma(\mu)}{\Gamma(n + \mu)} = \frac{1}{(\mu)_n}, \quad n \in \mathbb{N},$$

where $(x)_n$ is the well known Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\ldots(x+n-1), & n \in \mathbb{N}. \end{cases}$$

For $n \in \mathbb{N}$, we have

$$(x)_n = x(x+1)_{n-1}, \quad x^n \leq (x)_n.$$ 

Using (2.5) in (2.4), we have

$$|W'_{\lambda,\mu}(z) - 1| \leq \sum_{n=1}^{\infty} \frac{(n+1)\Gamma(\mu)}{n! \Gamma(\lambda n + \mu)} |z|^n < \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{n+1}{n!} \frac{1}{(\mu)_{n-1}}$$

$$= \frac{1}{\mu} \left( \sum_{n=0}^{\infty} \frac{n}{n!} (\mu)_n + \sum_{n=0}^{\infty} \frac{1}{n!} (\mu)_n \right) < \frac{1}{\mu} \left( \sum_{n=0}^{\infty} \frac{1}{(\mu)_n} + \sum_{n=0}^{\infty} \frac{1}{(\mu)_n} \right)$$

$$< 2 \mu \sum_{n=0}^{\infty} \frac{1}{(\mu + 1)^n} = \frac{2(\mu + 1)}{\mu^2} = \eta.$$ 

Note that under the hypothesis $0 < \eta \leq 1$. From (2.6), we conclude that $W'_{\lambda,\mu}(z) \prec 1 + \eta z$, $z \in \mathbb{D}$, which implies that

$$|\arg(W'_{\lambda,\mu}(z))| < \arcsin \eta, \quad z \in \mathbb{D}.$$
Using Lemma 2.3, for \( F(z) = W_{\lambda,\mu}(z) \), \( G(z) = 1 + \eta z \) and \( n = 0 \), we obtain 
\( W_{\lambda,\mu}(z)/z < 1 + \frac{1}{2} \eta z \), \( z \in \mathbb{D} \), and consequently

\[
(2.8) \quad \left| \arg \left( \frac{W_{\lambda,\mu}(z)}{z} \right) \right| < \arcsin \frac{\eta}{2}, \quad z \in \mathbb{D}.
\]

Now from (2.7) and (2.8), we conclude that

\[
\left| \arg \left( \frac{z W'_{\lambda,\mu}(z)}{W_{\lambda,\mu}(z)} \right) \right| = \left| \arg \left( \frac{z}{W_{\lambda,\mu}(z)} \right) + \arg(W'_{\lambda,\mu}(z)) \right|
\leq \left| \arg \left( \frac{z}{W_{\lambda,\mu}(z)} \right) \right| + |\arg(W'_{\lambda,\mu}(z))|
< \arcsin \frac{\eta}{2} + \arcsin \eta
= \arcsin \left( \eta \sqrt{1 - \frac{1}{4} \eta^2} + \frac{1}{2} \eta \sqrt{1 - \eta^2} \right),
\]

i.e., \( W_{\lambda,\mu}(z) \in \tilde{S}^*(\alpha) \) for \( \alpha \) given by (2.3). \( \Box \)

**Corollary 2.3.** Let \( \lambda \geq 1 \) and \( \mu \geq 1 + \sqrt{3} \). If \( 0 < \alpha \leq 1 \) and

\[
(2.9) \quad \eta = \frac{2(\mu + 1)}{\mu^2} = 2\sqrt{\frac{5 - 4\sqrt{1 - \nu^2}}{16\nu^2 + 9}},
\]

where \( \nu = \sin \left( \frac{1}{2} \alpha \pi \right) \), then \( W_{\lambda,\mu} \in \tilde{S}^*(\alpha) \).

**Proof.** If we substitute the value of \( \eta \) from (2.9) to (2.3), we obtain the required \( \alpha \). Hence the result. \( \Box \)

Taking \( \alpha = 1 \) in Corollary 2.3, we get

\[
\nu = 1 \Rightarrow \eta = \frac{2(\mu + 1)}{\mu^2} = \frac{2}{\sqrt{5}} \Rightarrow \frac{\mu + 1}{\mu^2} = \frac{1}{\sqrt{5}}.
\]

Hence, we get the following result:

**Corollary 2.4.** Let \( \lambda \geq 1 \) and \( \mu = \mu^* \), where \( \mu^* \) is the positive root of \( \mu^2 - \sqrt{5} \mu - \sqrt{5} = 0 \). Then \( W_{\lambda,\mu} \) is starlike in \( \mathbb{D} \).
3. A NONLINEAR DIFFERENTIAL EQUATION

In this section, we aim to study a nonlinear differential equation involving the Wright function. For this, we shall need the following lemmas:

**Lemma 3.1** (Miller and Mocanu [15]). Let $\Omega \subset \mathbb{C}$. Suppose that the function $\psi(z): \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq M(M - |a|)/(M + |a|)$, $\theta \in \mathbb{R}$ and $z \in \mathbb{D}$. Let $p(z)$ be an analytic function of the form

$$p(z) = a + a_1z + a_2z^2 + \ldots, \quad z \in \mathbb{D},$$

such that $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{D}$. Then $|p(z)| < M$, where $0 \leq |a| < M$.

**Lemma 3.2** (Tuneski [26]). If $f \in A$ and $|f''(z)| \leq 1$, $z \in \mathbb{D}$, then $f$ is starlike in $\mathbb{D}$.

**Theorem 3.1.** For all $\lambda > -1$ and $\mu > 0$, let $W_{\lambda,\mu}(z)$ satisfy the inequality

$$|zW_{\lambda,\mu}(z)| < \frac{M(M - |a|)}{(M + 1)(M + |a|)}, \quad 0 \leq |a| < M \leq 1; \quad z \in \mathbb{D}. \quad (3.2)$$

Let $\varphi$ be the (unique) solution of the initial value problem

$$\varphi^{(n+1)}(z) + W_{\lambda,\mu}(z)\varphi^{(n)}(z) = W_{\lambda,\mu}(z), \quad z \in \mathbb{D}$$

$$(n \in \mathbb{N} \cup \{0\}, \quad \varphi(0) = 0, \quad \varphi'(0) = 1, \quad \varphi^{(k)}(0) = 0, \quad k = 2, \ldots, n - 1, \quad \varphi^{(n)}(0) = a),$$

where $\varphi^{(n)}$ denotes the $n$th derivative with respect to $z$. Then the inequality $|\varphi^{(n)}(z)| < M$, $z \in \mathbb{D}$ holds.

**Proof.** Let the function $p(z)$ be defined by $p(z) = \varphi^{(n)}(z)$, $z \in \mathbb{D}$. Note that $p(z)$ has the form (3.1), and then it follows that

$$\frac{zp'(z)}{1 + p(z)} = \frac{z\varphi^{(n+1)}(z)}{1 + \varphi^{(n)}(z)} = zW_{\lambda,\mu}(z), \quad z \in \mathbb{D}; \quad \varphi^{(n)}(z) \neq -1.$$ 

We denote $\psi(r, s; z)$ and $\Omega$ by

$$\psi(r, s; z) := \frac{s}{1 + r}, \quad r \neq -1$$

and

$$\Omega := \{w \in \mathbb{C}: |w| < \frac{M(M - |a|)}{(M + 1)(M + |a|)}, \quad 0 \leq |a| < M \leq 1\},$$

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respectively. Then clearly

\[
\psi(p(z), zp'(z); z) = \frac{zp'(z)}{1 + p(z)} = \frac{z\varphi^{(n+1)}(z)}{1 + \varphi^{(n)}(z)} \in \Omega, \quad z \in \mathbb{D}.
\]

Further, for any \( \theta \in \mathbb{R} \), \( K \geq M(M - |a|)/(M + |a|) \) and \( z \in \mathbb{D} \), we have

\[
|\psi(Me^{i\theta}, Ke^{i\theta}; z)| = \left| \frac{Ke^{i\theta}}{1 + Me^{i\theta}} \right| \geq \frac{M(M - |a|)}{(M + 1)(M + |a|)},
\]

which gives that

\[
\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega.
\]

Therefore, in view of Lemma 3.1 it follows that

\[
|p(z)| = |\varphi^{(n)}(z)| < M, \quad z \in \mathbb{D}; \quad 0 \leq |a| < M \leq 1.
\]

This completes the proof. \( \square \)

By taking \( n = 2 \) in the above theorem we get the following result:

**Corollary 3.1.** For all \( \lambda > -1 \) and \( \mu > 0 \), let \( \mathbb{W}_{\lambda,\mu}(z) \) satisfy the inequality (3.2) in \( \mathbb{D} \). Let \( \varphi \) be the (unique) solution of the initial value problem given by

\[
\begin{aligned}
\varphi''''(z) + \mathbb{W}_{\lambda,\mu}(z)\varphi''(z) &= \mathbb{W}_{\lambda,\mu}(z), \quad z \in \mathbb{D} \\
(\varphi(0) = 0, \ \varphi'(0) = 1, \ \varphi''(0) = a).
\end{aligned}
\]

Then the inequality \( |\varphi''(z)| < M, \ 0 \leq |a| < M \leq 1 \) holds.

**Corollary 3.2.** If \( \mathbb{W}_{\lambda,\mu}(z) \) satisfies the inequality

\[
|z\mathbb{W}_{\lambda,\mu}(z)| < \frac{1 - |a|}{2(1 + |a|)}, \quad 0 \leq |a| < 1
\]

and the function \( \varphi(z) \) is the (unique) solution of the initial value problem given by (3.3), then \( \varphi \) is starlike in \( \mathbb{D} \).

**Proof.** The proof can be obtained easily by taking \( M = 1 \) in Corollary 3.1 and then using Lemma 3.2. \( \square \)
4. Inequalities

The following result by Fejér will be needed in this section.

**Lemma 4.1** (Fejér [8]). Let \( \{a_n\} \) be a sequence of nonnegative real numbers such that \( a_1 = 1 \). If the sequence \( \{a_n\} \) is convex decreasing, i.e., \( 0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n \) for all \( n \in \mathbb{N} \setminus \{1\} \), then

\[
\Re \left( \sum_{n=1}^\infty a_n z^{n-1} \right) > \frac{1}{2}, \quad z \in \mathbb{D}.
\]

The convex hull of \( K \), denoted by \( \overline{\overline{K}} \), is the set of all convex combinations of functions belonging to \( K \). We recall from [5] that the closure of the set \( \overline{\overline{K}} \) is

\[
\overline{\overline{K}} = \left\{ f \in A : \Re \left( \frac{f(z)}{z} \right) > \frac{1}{2}, \; z \in \mathbb{D} \right\}.
\]

It is well known (see [27]) that a sequence \( \{b_n\}_{n=1}^\infty \) of complex numbers is said to be a subordinating sequence for the class \( \mathcal{X} \subset A \), whenever we have

\[
\sum_{n=1}^\infty b_n a_n z^n \prec \sum_{n=1}^\infty a_n z^n, \quad z \in \mathbb{D}
\]

for all \( \sum_{n=1}^\infty a_n z^n \in \mathcal{X} \).

**Lemma 4.2** (Piejko and Sokół [19]). The function of the form (1.2) is in the set \( \overline{\overline{K}} \) if and only if \( a_2, a_3, \ldots \) is a subordinating factor sequence for the class \( K \).

**Theorem 4.1.** For each \( \lambda \geq 1 \) and \( \mu \geq 1 \), we have

\[
|W_{\lambda,\mu}(z)| \leq r_0 F_1(-; \mu; r),
\]

where \( _0F_1(-; \mu; r) \) is the well known hypergeometric function and \(|z| = r < 1\).

**Proof.** Using (2.4), we get

\[
|W_{\lambda,\mu}(z)| \leq |z| + \sum_{n=2}^\infty \frac{\Gamma(\mu)|z|^n}{\Gamma((n-1)\lambda + \mu)(n-1)!}
\]

\[
\leq |z| + \sum_{n=1}^\infty \frac{|z|^{n+1}}{(\mu)n!} = r_0 F_1(-; \mu; r).
\]

This proves the result. \( \square \)
By using (1.4) and Theorem 4.1, we get the following result:

**Corollary 4.1.** For each \( \lambda \geq 1 \) and \( \mu \geq 1 \), we have \(|W_{\lambda,\mu}(z)| \leq _0F_1(-;\mu;r)/\Gamma(\mu), |z| = r < 1\).

**Theorem 4.2.** Let \( \lambda \geq 1 \) and \( \mu \geq 1 \). If \( \Gamma(\lambda + \mu) \geq 2\Gamma(\mu) \), then

\[
\Re\left( \frac{W_{\lambda,\mu}(z)}{z} \right) > \frac{1}{2}, \quad z \in \mathbb{D}.\tag{4.4}
\]

**Proof.** Under the hypothesis, the inequality

\[
n!\Gamma(\lambda n + \mu) \geq (n - 1)!\Gamma(\lambda(n - 1) + \mu)
\]

holds, which is equivalent to

\[
\frac{1}{\Gamma(\lambda(n - 1) + \mu)(n - 1)!} \geq \frac{1}{\Gamma(\lambda n + \mu)n!}.\tag{4.5}
\]

Now we need to show that

\[
\{a_n\}_{n=1}^\infty = \left\{ \frac{\Gamma(\mu)}{\Gamma(\lambda(n - 1) + \mu)(n - 1)!} \right\}_{n=1}^\infty
\]

is a convex decreasing sequence. We observe that

\[
a_{n+2} - 2a_{n+1} + a_n = \frac{\Gamma(\mu)}{\Gamma(\lambda(n + 1) + \mu)(n + 1)!} - \frac{2\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} + \frac{\Gamma(\mu)}{\Gamma(\lambda(n - 1) + \mu)(n - 1)!} \geq 0,
\]

which shows that \( \{a_n\}_{n=1}^\infty \) is a convex decreasing sequence. Now applying Lemma 4.1, we get

\[
\Re\left\{ \sum_{n=1}^\infty a_n z^{n-1} \right\} > \frac{1}{2}, \quad z \in \mathbb{D}
\]

which is equivalent to (4.4). This proves the result. \(\square\)

Proceeding similarly as in Theorem 4.2, we get the following result:

**Theorem 4.3.** Let \( \lambda \geq 1 \) and \( \mu \geq 1 \). If \( \Gamma(\lambda + \mu) \geq 4\Gamma(\mu) \), then

\[
\Re\{W_{\lambda,\mu}(z)\} > \frac{1}{2}, \quad z \in \mathbb{D}.\tag{4.6}
\]
Corollary 4.2. Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 2\Gamma(\mu)$, then the sequence

$$
\left\{ \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} \right\}_{n=1}^{\infty}
$$

is a subordinating sequence for the class $\mathcal{K}$.

**Proof.** By using (4.1) and (4.4), we have $W_{\lambda,\mu}(z) \in \overline{co}\mathcal{K}$. \hfill $\Box$

Now applying Lemma 4.2, we get the desired result.

Corollary 4.3. Let $\lambda \geq 1$ and $\mu \geq 1$. If $\Gamma(\lambda + \mu) \geq 4\Gamma(\mu)$, then the sequence

$$
\left\{ \frac{(n+1)\Gamma(\mu)}{\Gamma(\lambda n + \mu)n!} \right\}_{n=1}^{\infty}
$$

is a subordinating sequence for the class $\mathcal{K}$.

**Proof.** By using (4.1) and (4.6), we have $zw'_{\lambda,\mu}(z) \in \overline{co}\mathcal{K}$. Now applying Lemma 4.2, we get the desired result. \hfill $\Box$

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