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Abstract. We prove the local Hölder continuity of bounded generalized solutions of the Dirichlet problem associated to the equation

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) - c_0 |u|^{p-2} u = f(x, u, \nabla u),$$

assuming that the principal part of the equation satisfies the following degenerate ellipticity condition

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x, u, \eta)\eta_i \geq \nu(x)|\eta|^p,$$

and the lower-order term f has a natural growth with respect to ∇u.

Keywords: elliptic equations; weight function; regularity of solutions

Classification: 35J15, 35J70, 35B65

1. Introduction

Consider the equation

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) - c_0 |u|^{p-2} u = f(x, u, \nabla u) \qquad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^m$, $m \geq 2$, c_0 is a positive constant, ∇u is the gradient of unknown function u and f is a nonlinear function which has the growth of rate p, $1 < p < m$, with respect to the gradient ∇u. We shall suppose that the following degenerate ellipticity condition is satisfied:

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x, u, \eta)\eta_i \geq \nu(x)|\eta|^p, \tag{1.2}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_m) \in \mathbb{R}^m$, $|\eta| = (\eta_1^2 + \cdots + \eta_m^2)^{1/2}$, and $\nu\colon \Omega \to (0, \infty)$, $\lambda\colon [0, \infty) \to [1, \infty)$ are functions with properties to be specified later on.

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In the present article, we prove the local Hölder continuity in Ω of every bounded generalized solution of equation (1.1) under the condition (1.2). We want to emphasize that the study of the quasilinear equations where the lower-order term has natural p-growth deserves special attention because to obtain the existence of the solutions, if $1 < p \leq m$, it is not possible to directly apply the standard theory of the pseudomonotone operators; moreover, the solution in general is unbounded. In this regard, we refer, for instance, to [1], [8], [20].

Existence and L^∞-estimates of bounded solutions for quasilinear elliptic equations with natural growth of lower-order terms, in nondegenerate case, were established, for instance, in [2], [3], [7], [29], [30], [31], and the Hölder continuity on compact subsets of Ω of solutions was proved in [19, Chapter IX, Section 2], [32]. Similar results for elliptic equations and variational inequalities without the natural growth were obtained in [23], [25], [26] for the nondegenerate case, and also in [15]–[18], [22], [27], [28] for the degenerate case. We also mention the articles [10], [33] where the linear case is studied with weights belonging to Muckenhoupt's class.

Assuming the degenerate ellipticity condition (1.2), Drábek and Nicolosi in [9] obtained the existence of bounded generalized solutions of equation (1.1) establishing more general results than those obtained from Boccardo, Murat and Puel in [2], [3]. On the related topic and in degenerate-case, we also refer to [4]–[6] and [12], [13]. The results obtained in [9] are the starting point for this research.

The present paper is organized as follows. In Section 2 we formulate the hypotheses, we state our problem and the main results. Section 3 consists of preliminary lemmas which are sufficient in the proof of our main results. In Section 4 we prove local Hölder continuity of solutions of Dirichlet problem associated to equation (1.1). For the proof, we use an analogue of Moser's method (see [21]) proposed in [25] and modified in [32] for equations with natural growth terms. In Section 5 we give examples where all our assumptions are satisfied.

2. Hypotheses and formulation of the main results

We shall suppose that $\mathbb{R}^m$, $m \geq 2$, is the m-dimensional euclidean space with elements $x = (x_1, x_2, \dots, x_m)$. Let Ω be an open bounded nonempty subset of $\mathbb{R}^m$.

Let p be a real number such that $1 < p < m$.

Hypothesis 2.1. Let $\nu\colon \Omega \to (0,\infty)$ be a measurable function such that

$$\nu(x) \in L^1_{\mathrm{loc}}(\Omega), \qquad \Big(\frac{1}{\nu(x)}\Big)^{1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega).$$

We denote $W^{1,p}(\nu,\Omega)$ as the set of all functions $u \in L^p(\Omega)$ having for every $i = 1,\dots,m$ the weak derivative $\partial u/\partial x_i$ with the property $\nu|\partial u/\partial x_i|^p \in L^1(\Omega)$.

The space $W^{1,p}(\nu,\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p} = \left[\int_\Omega (|u|^p + \nu|\nabla u|^p)\,\mathrm{d}x\right]^{1/p}.$$

The space $\mathring{W}^{1,p}(\nu,\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$. Put $W = \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$.

Hypothesis 2.2. We assume that

$$\frac{1}{\nu(x)} \in L^s(\Omega)$$

with $s > \max(1/(p-1), m/p)$.

We set $\tilde{p} = mp/(m - p + m/s)$. Then, we have $W^{1,p}(\nu,\Omega) \subset L^{\tilde{p}}(\Omega)$ and there exists $\hat{c} > 0$ depending only on m, p, s, and Ω such that for every $u \in \mathring{W}^{1,p}(\nu,\Omega)$

$$\left(\int_\Omega |u|^{\tilde{p}}\,\mathrm{d}x\right)^{1/\tilde{p}} \le \hat{c}\left(\int_{\mathrm{supp}\,u}\left(\frac{1}{\nu}\right)^s \mathrm{d}x\right)^{1/(ps)} \left(\int_\Omega \sum_{i=1}^m \nu\left|\frac{\partial u}{\partial x_i}\right|^p \mathrm{d}x\right)^{1/p}.$$

In this connection see, for instance, [13] and [22].

Hypothesis 2.3. The functions $f(x,u,\eta)$, $a_i(x,u,\eta)$, $i = 1,2,\dots,m$, are Carathéodory functions in $\Omega \times \mathbb{R} \times \mathbb{R}^m$, i.e., measurable with respect to x for every $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$ and continuous with respect to (u,η) for almost every $x \in \Omega$.

Hypothesis 2.4. There exist a number σ and a function $f^*(x)$ such that

$$\max\left(0, \frac{2-p}{2}\right) < \sigma < 1, \qquad f^* \in L^1(\Omega)$$

and

$$|f(x,u,\eta)| \le \lambda(|u|)[f^*(x) + |u|^{p-1+\sigma} + (\nu^{1/p}(x)|\eta|)^{p-1+\sigma} + \nu(x)|\eta|^p] \tag{2.1}$$

holds for almost every $x \in \Omega$ and for all real numbers $u, \eta_1, \eta_2, \dots, \eta_m$.

Hypothesis 2.5. There exist a nonnegative number $c_1 < c_0$ and a function $f_0(x) \in L^\infty(\Omega)$ such that for almost all $x \in \Omega$ and for all real numbers u, η_1, $\eta_2, \dots, \eta_m$ the inequality

$$u\,f(x,u,\eta) + c_1|u|^p + \lambda(|u|)\nu(x)|\eta|^p + f_0(x) \ge 0 \tag{2.2}$$

holds.

Hypothesis 2.6. There exists a function $a^* \in L^{p/(p-1)}(\Omega)$ such that for almost every $x \in \Omega$ and for any real numbers $u, \eta_1, \eta_2, \dots, \eta_m$ the inequality

$$\text{(2.3)} \qquad \frac{|a_i(x,u,\eta)|}{\nu^{1/p}(x)} \le \lambda(|u|)[a^*(x) + |u|^{p-1} + \nu^{(p-1)/p}(x)|\eta|^{p-1}]$$

holds.

Hypothesis 2.7. The condition (1.2) is satisfied for almost every $x \in \Omega$ and for all real numbers $u, \eta_1, \eta_2, \dots, \eta_m$; the function $\lambda = \lambda(z)$ is monotone and non-decreasing.

Hypothesis 2.8. For almost all $x \in \Omega$ and for every real u, $\eta_1, \eta_2, \dots, \eta_m$, $\tau_1, \tau_2, \dots, \tau_m$ the inequality

$$\sum_{i=1}^{m} [a_i(x,u,\eta) - a_i(x,u,\tau)](\eta_i - \tau_i) \ge 0$$

holds while the inequality holds if and only if $\eta \ne \tau$.

Assumptions 2.1–2.4, 2.6 and 2.7 provide the correctness of the following definition.

Definition. A generalized solution of equation (1.1) is a function $u \in W$ such that for every function $w \in W$,

$$\text{(2.4)} \qquad \int_\Omega \left\{ \sum_{i=1}^{m} a_i(x,u,\nabla u) \frac{\partial w}{\partial x_i} + c_0|u|^{p-2}uw + \; f(x,u,\nabla u)w \right\} \mathrm{d}x = 0.$$

Note that if in addition to Assumptions 2.1–2.4, 2.6 and 2.7 they hold Assumptions 2.5 and 2.8, then there exists a generalized solution of equation (1.1). This follows from Theorem 2.1 of [9].

We will need the following further hypotheses on weight function.

Hypothesis 2.9. There exists a real number $\overline{t} > ms/(ps - m)$ such that

$$\nu(x) \in L^{\overline{t}}(\Omega).$$

For every $y \in \mathbb{R}^m$ and $R > 0$ we denote

$$B_R(y) = \{x \in \mathbb{R}^m \colon |x - y| < R\};$$

when not important, or clear from the context, we shall omit denoting the center as follows: $B_R = B_R(y)$.

Hypothesis 2.10. There exists $c > 0$ such that for every $y \in \Omega$ and $R > 0$ with $\overline{B_R(y)} \subset \Omega$ the following inequality holds

$$\left\{ R^{-m} \int_{B_R(y)} \left(\frac{1}{\nu}\right)^s \mathrm{d}x \right\}^{1/s} \left\{ R^{-m} \int_{B_R(y)} \nu^{\overline{t}} \, \mathrm{d}x \right\}^{1/\overline{t}} \le c.$$

As for Hypotheses 2.9 and 2.10 see, for example, [5]. Such kind of assumptions on weight function, introduced in [22], [27], [28], have been used in recent articles, see, for example, [16] and [18].

The main result of the present paper is a theorem on the local Hölder continuity of any generalized solution $u \in W$ of equation (1.1). More precisely, we prove the following

Theorem 2.11. *Assume that Hypotheses* 2.1–2.4, 2.6, 2.7, 2.9 *and* 2.10 *are satisfied with the functions* $|a^*|^{p/(p-1)}$, f^* *belonging to* $L^\tau(\Omega)$ *with* $\tau > ms/(ps-m)$.

Let $u \in W$ *be a generalized solution of equation* (1.1) *and* $M = \|u\|_{L^\infty(\Omega)}$.

Then there exist positive constants C *and* σ' *such that for every open set* Ω', $\overline{\Omega'} \subset \Omega$ *and every* $x', x'' \in \Omega'$

$$|u(x') - u(x'')| \le C[\operatorname{dist}(\Omega', \partial\Omega)]^{-\sigma'} |x' - x''|^{\sigma'},$$

where $\sigma' = \sigma'(\mathbf{data}) < 1$, $C = C(\mathbf{data})$ *and* $\mathbf{data} = (m, p, \tau, \bar{t}, s, M, \lambda(z),$ $\|f^*\|_{L^\tau(\Omega)}, \||a^*|^{p/(p-1)}\|_{L^\tau(\Omega)}, \sigma, \operatorname{meas}\Omega)$.

3. Auxiliary results

Lemma 3.1. *Let* $f \in W^{1,q}(B_R)$, $q \ge 1$. *Suppose there exist a measurable subset* $G \subset B_R$ *and positive constants* C' *and* C'' *such that*

$$\operatorname{meas} G \ge C' R^m, \qquad \max_G |f| \le C''.$$

Then

$$\int_{B_R} |f|^q \,\mathrm{d}x \le C R^q \Big(\sum_{i=1}^m \int_{B_R} \Big|\frac{\partial f}{\partial x_i}\Big|^q \,\mathrm{d}x + R^{m-q} \Big),$$

where C *is a positive constant depending only on* m, q, C', C''.

The proof of this lemma is given in [24, Chapter 1, Section 2, Lemma 4].

The following lemma is due to John and Nirenberg (see [14], and see also [11, Theorem 7.21]).

Lemma 3.2. *Let* $f \in W^{1,1}(\mathcal{O})$ *where* $\mathcal{O}$ *is a convex domain in* $\mathbb{R}^m$. *Suppose there exists a positive constant* K *such that*

$$\sum_{i=1}^m \int_{\mathcal{O}\cap B_\varrho} \Big|\frac{\partial f}{\partial x_i}\Big| \,\mathrm{d}x \le K \varrho^{m-1} \qquad \text{for all balls } B_\varrho.$$

Then there exist positive constants σ_0 *and* C *depending only on* m *such that*

$$\int_{\mathcal{O}} \exp\Big(\frac{\sigma}{K} |f - (f)_{\mathcal{O}}|\Big) \,\mathrm{d}x \le C(\operatorname{diam}\mathcal{O})^m,$$

where $\sigma = \sigma_0(\operatorname{meas}\mathcal{O})(\operatorname{diam}\mathcal{O})^{-m}$, $(f)_{\mathcal{O}} = \frac{1}{\operatorname{meas}\mathcal{O}} \int_{\mathcal{O}} f \,\mathrm{d}x$.

The following result is discussed in [11, Lemma 8.23] (see also [19, Lemma 4.8]).

Lemma 3.3. *Let ω be a non-decreasing function on an interval $(0, R_0]$ satisfying for all $R \le R_0$ the inequality*

$$\omega(\vartheta R) \le \theta\omega(R) + \varphi(R),$$

where φ is also non-decreasing function and $0 < \vartheta, \theta < 1$. Then for any $\delta \in (0,1)$ and $R \le R_0$ we have

$$\omega(R) \le C\Big(\Big(\frac{R}{R_0}\Big)^{\epsilon}\omega(R_0) + \varphi(R^{\delta}R_0^{1-\delta})\Big),$$

where $C = C(\vartheta, \theta)$ and $\epsilon = \epsilon(\vartheta, \theta, \delta)$ are positive constants.

4. Proof of Theorem 2.11

Suppose that Hypotheses 2.1–2.4, 2.6, 2.7, 2.9 and 2.10 hold with the functions $|a^*|^{p/(p-1)}$, $f^* \in L^{\tau}(\Omega)$, $\tau > ms/(ps-m)$. Let $u \in W$ be a generalized solution of equation (1.1). We set $M = \|u\|_\infty$, thus

$$|u| \le M < \infty \qquad \text{on } \Omega. \tag{4.1}$$

By d_i, $i = 1, 2, \dots$, we denote positive constants depending only on **data**.

Furthermore, let Ω' be an arbitrary open subset of Ω such that $\overline{\Omega'} \subset \Omega$ and $d = \operatorname{dist}(\Omega', \partial\Omega)$. We fix $x_0 \in \Omega'$. For every $R \in \big(0, \min\{1, d/4\}\big)$, we set

$$\omega_1(R) = \min_{B_R(x_0)} u, \qquad \omega_2(R) = \max_{B_R(x_0)} u, \qquad \omega(R) = \omega_2(R) - \omega_1(R).$$

Here the symbols min and max of course stands for *essential infimum* and *supremum.*

We fix a positive number r such that

$$r = 1 - \frac{m}{pt_\star} - \frac{m}{ps}, \tag{4.2}$$

where $t_\star = \min(\tau,\ \bar{t})$.

For every $R \in (0, \min\{1, d/4\})$, we shall establish the inequality

$$\omega(R) \le \alpha\omega(2R) + R^r \tag{4.3}$$

with a constant $\alpha \in (0,1)$ depending only on **data**. This inequality and Lemma 3.3 imply the validity of Theorem 2.11.

To prove (4.3), we fix R such that $0 < R < \min\{1, d/4\}$. If $\omega(2R) < R^r$, then inequality (4.3) is evident. Therefore, we shall suppose that

$$\omega(2R) \ge R^r. \tag{4.4}$$

We shall also assume that

$$\text{meas}\, G(R) \geq \frac{1}{2}\,\text{meas}\, B_{3R/2}(x_0), \tag{4.5}$$

where $G(R) = \{x \in B_{3R/2}(x_0) \colon u(x) \leq (\omega_1(2R) + \omega_2(2R))/2\}$.

Let $F_1 \colon \Omega \to \mathbb{R}$ be the function such that

$$F_1 = \begin{cases} \frac{2e\omega(2R)}{\omega_2(2R)-u+R^r} & \text{in } B_{2R}(x_0), \\ e & \text{in } \Omega \setminus B_{2R}(x_0). \end{cases}$$

Due to (4.4) we have $F_1 \geq e$ in Ω.

Now, we need some integral estimates of solution u.

Lemma 4.1. *Let $B_\varrho \subset \Omega$ and let $\zeta \in C_0^\infty(\Omega)$ be a function such that*

$$\zeta = 0 \qquad \text{in } \Omega \setminus B_\varrho \ \text{ and } \ 0 \leq \zeta \leq 1. \tag{4.6}$$

Then there exist positive constants d_1, d_2 such that

$$\int_{B_\varrho} \nu |\nabla u|^p\, \zeta^p \,\mathrm{d}x \leq d_1 \varrho^{m(\tau-1)/\tau} + d_2 \max_{B_\varrho} |\nabla \zeta|^p \Big(\int_{B_\varrho} \nu^{\bar t} \,\mathrm{d}x \Big)^{1/\bar t} \varrho^{m(\bar t-1)/\bar t}. \tag{4.7}$$

Proof: For every $x \in \Omega$ we set $v_1(x) = e^{\lambda_1 u(x)} \zeta^p(x)$ where

$$\lambda_1 = 3\lambda^2(M). \tag{4.8}$$

Simple calculations show that $v_1 \in \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$ and the following assertion holds:

(a) for every $i = 1, 2, \ldots, m$,

$$\frac{\partial v_1}{\partial x_i} = \lambda_1 e^{\lambda_1 u} \zeta^p \frac{\partial u}{\partial x_i} + p e^{\lambda_1 u} \zeta^{p-1} \frac{\partial \zeta}{\partial x_i} \qquad \text{a.e. in } \Omega.$$

Since $v_1 \in \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$, by virtue of (2.4), we have

$$\int_\Omega \Big\{ \sum_{i=1}^m a_i(x,u,\nabla u) \frac{\partial v_1}{\partial x_i} + c_0 |u|^{p-2} u v_1 + \ f(x,u,\nabla u) v_1 \Big\} \,\mathrm{d}x = 0.$$

From this equality, using (1.2), (2.1), (4.8) and assertion (a), we deduce that

$$\lambda(M) \int_{B_\varrho} \nu |\nabla u|^p e^{\lambda_1 u} \zeta^p \,\mathrm{d}x \leq I_\varrho + e^{\lambda_1 M} \int_{B_\varrho} g^\star(x) \,\mathrm{d}x, \tag{4.9}$$

where

$$I_\varrho = p\sum_{i=1}^{m}\int_{B_\varrho}|a_i(x,u,\nabla u)|\,\Big|\frac{\partial\zeta}{\partial x_i}\Big|e^{\lambda_1 u}\zeta^{p-1}\,\mathrm{d}x,$$
$$g^\star(x) = c_0M^{p-1} + 2\lambda(M)[|f^*(x)| + M^{p-1+\sigma} + 1].$$

Let us obtain suitable estimate for the first addend in the right-hand side of (4.9).
Estimate of I_ϱ. Using the Young's inequality with the exponents $p/(p-1)$ and p, (2.3), (4.1) and (4.6), we obtain

$$\tag{4.10} \begin{aligned} I_\varrho \le{} & \frac{\lambda(M)}{2}\int_{B_\varrho}\nu|\nabla u|^p e^{\lambda_1 u}\zeta^p\,\mathrm{d}x \\ & + d_3e^{\lambda_1 M}\int_{B_\varrho} g_1(x)\,\mathrm{d}x + d_4e^{\lambda_1 M}\int_{B_\varrho}\nu|\nabla\zeta|^p\,\mathrm{d}x, \end{aligned}$$

where

$$g_1(x) = 4^{1/(p-1)}(\lambda(M))^{p/(p-1)}[\,|a^*(x)|^{p/(p-1)} + M^p].$$

From (4.9), (4.10) it follows that

$$\tag{4.11} \frac{\lambda(M)}{2}\int_{B_\varrho}\nu|\nabla u|^p e^{\lambda_1 u}\zeta^p\,\mathrm{d}x \le d_5\int_{B_\varrho}(g^\star + g_1)\,\mathrm{d}x + d_6\int_{B_\varrho}\nu|\nabla\zeta|^p\,\mathrm{d}x.$$

By Hölder's inequality and the inequality $\tau > ms/(ps-m)$ we have

$$\int_{B_\varrho}(g^\star + g_1)\,\mathrm{d}x \le \|g^\star + g_1\|_\tau |B_\varrho|^{(\tau-1)/\tau} \le d_7\varrho^{m(\tau-1)/\tau}.$$

Moreover

$$\int_{B_\varrho}\nu|\nabla\zeta|^p\,\mathrm{d}x \le d_8\max_{B_\varrho}|\nabla\zeta|^p\varrho^{m(\bar t-1)/\bar t}\Big(\int_{B_\varrho}\nu^{\bar t}\,\mathrm{d}x\Big)^{1/\bar t}.$$

The last two inequalities and (4.11) imply inequality (4.7).
The lemma is proved. □

Lemma 4.2. *Let $B_\varrho \subset B_{2R}(x_0)$ and let $\zeta \in C_0^\infty(\Omega)$ be a function such that condition* (4.6) *is satisfied. Then there exist positive constants d_9, d_{10}, d_{11} such that*

$$\tag{4.12} \begin{aligned} \int_{B_\varrho}\frac{\nu|\nabla u|^p\,\zeta^p\,\mathrm{d}x}{(\omega_2(2R) - u + R^r)^p} \le{} & d_9\varrho^{m-p+m/s} + d_{10}\max_{B_\varrho}|\nabla\zeta|^p\Big(\int_{B_\varrho}\nu^{\bar t}\,\mathrm{d}x\Big)^{1/\bar t} \\ & \times\varrho^{m(\bar t-1)/\bar t} + d_{11}\varrho^{m(\tau-1)/\tau}. \end{aligned}$$

Proof: For every $x \in B_{2R}(x_0)$, we set $U(x) = \omega_2(2R) - u(x) + R^r$,

$$v_2(x) = \begin{cases} \zeta^p(x)[U(x)]^{1-p} & \text{if } x \in B_{2R}(x_0), \\ 0 & \text{if } x \in \Omega \setminus B_{2R}(x_0). \end{cases}$$

Simple calculations show that

$$v_2 \in \mathring{W}^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$$

and the following assertion holds:

(b) for every $i = 1, 2, \dots, m$

$$\frac{\partial v_2}{\partial x_i} = pU^{1-p}\zeta^{p-1}\frac{\partial \zeta}{\partial x_i} + (p-1)U^{-p}\zeta^p\frac{\partial u}{\partial x_i} \qquad \text{a.e. in } B_{2R}.$$

Putting the function v_2 into (2.4) instead of w and using (1.2), (2.1), (4.1) and assertion (b), we obtain

$$\frac{1}{\lambda(M)} \int_{B_\varrho} \nu|\nabla u|^p\, U^{-p}\zeta^p \,\mathrm{d}x \le I_{2,\varrho} + I_{3,\varrho} + \frac{(2M+1)}{p-1}\int_{B_\varrho} g^\star U^{-p}\,\mathrm{d}x, \tag{4.13}$$

where

$$I_{2,\varrho} = p\sum_{i=1}^m \int_{B_\varrho} |a_i(x,u,\nabla u)| \left|\frac{\partial \zeta}{\partial x_i}\right| U^{1-p}\zeta^{p-1}\,\mathrm{d}x,$$

$$I_{3,\varrho} = \frac{2\lambda(M)}{p-1}\int_{B_\varrho} \nu|\nabla u|^p\, U^{1-p}\zeta^p\,\mathrm{d}x.$$

Let us obtain suitable estimates for $I_{2,\varrho}$, $I_{3,\varrho}$.

Estimate of $I_{2,\varrho}$. Using the Young's inequality with the exponents $p/(p-1)$ and p, (2.3), (4.1) and (4.6), we obtain

$$I_{2,\varrho} \le \frac{1}{4\lambda(M)}\int_{B_\varrho} \nu|\nabla u|^p\, U^{-p}\zeta^p\,\mathrm{d}x + d_{12}\int_{B_\varrho} g_1 U^{-p}\,\mathrm{d}x + d_{13}\max_{B_\varrho}|\nabla\zeta|^p \left(\int_{B_\varrho}\nu^{\bar t}\,\mathrm{d}x\right)^{1/\bar t}\varrho^{m(\bar t-1)/\bar t}. \tag{4.14}$$

Estimate of $I_{3,\varrho}$. We use (4.1), the Young's inequality, (4.6) and (4.7) to obtain

$$I_{3,\varrho} \le \frac{1}{4\lambda(M)}\int_{B_\varrho} \nu|\nabla u|^p\, U^{-p}\zeta^p\,\mathrm{d}x + d_{14}\varrho^{m(\tau-1)/\tau} + d_{15}\max_{B_\varrho}|\nabla\zeta|^p \left(\int_{B_\varrho}\nu^{\bar t}\,\mathrm{d}x\right)^{1/\bar t}\varrho^{m(\bar t-1)/\bar t}. \tag{4.15}$$

Collecting (4.13), (4.14) and (4.15), we get

$$\frac{1}{2\lambda(M)}\int_{B_\varrho}\nu|\nabla u|^p\,U^{-p}\zeta^p\,\mathrm{d}x \le d_{14}\varrho^{m(\tau-1)/\tau} + d_{16}\max_{B_\varrho}|\nabla\zeta|^p\Big(\int_{B_\varrho}\nu^{\bar t}\,\mathrm{d}x\Big)^{1/\bar t}\varrho^{m(\bar t-1)/\bar t} + d_{17}\int_{B_\varrho} gU^{-p}\,\mathrm{d}x \tag{4.16}$$

where $g = g^\star + g_1$.

By Hölder's inequality, $U \ge R^r$ and $\varrho/2 < R < 1$, we have

$$\int_{B_\varrho} gU^{-p}\,\mathrm{d}x \le \|g\|_\tau\Big(\int_{B_\varrho} U^{-p\tau/(\tau-1)}\,\mathrm{d}x\Big)^{(\tau-1)/\tau} \le d_{18}\Big(\frac{2}{\varrho}\Big)^{rp}\varrho^{m(\tau-1)/\tau}.$$

From last inequality, taking into account relation (4.2), we get

$$\int_{B_\varrho} gU^{-p}\,\mathrm{d}x \le d_{19}\varrho^{m-p+m/s}. \tag{4.17}$$

Inequalities (4.16) and (4.17) imply inequality (4.12). The lemma is proved. □

Define in Ω the function $v_0 = \ln F_1$.

Let us prove that v_0 satisfies some integral inequalities.

Lemma 4.3. *Let $r_0 = sp/(s+1)$. Then, there exist positive constants $d_{20}, d_{21}, d_{22}, d_{23}$ such that*

$$\int_{B_{3R/2}(x_0)} v_0^{r_0}\,\mathrm{d}x \le d_{20}R^m + R^{r_0}\Big\{\Big[d_{21}\Big\|\frac{1}{\nu}\Big\|_{L^s(\Omega)} + d_{22}\Big]R^{m-p+m/s} + d_{23}\Big\|\frac{1}{\nu}\Big\|_{L^s(\Omega)}R^{m(\tau-1)/\tau}\Big\}^{s/(s+1)}. \tag{4.18}$$

Proof: We choose a function $\zeta_1 \in C_0^\infty(\Omega)$ such that

$$0 \le \zeta_1 \le 1 \quad \text{in } \Omega, \qquad \zeta_1 = 1 \quad \text{in } B_{3R/2}(x_0), \qquad \zeta_1 = 0 \quad \text{in } \Omega\setminus B_{7R/4}(x_0),$$
$$\Big|\frac{\partial\zeta_1}{\partial x_i}\Big| \le K_1R^{-1} \qquad \text{for } i = 1,2,\dots,m,$$

where K_1 is an absolute constant, not depending on R. By definition of v_0 it results $1 \le v_0 \le 1+\ln 4$ on $G(R)$; moreover from (4.5) it follows that $\operatorname{meas} G(R) \ge d_{24}R^m$.

Hence from Lemma 3.1 we find that:

$$\int_{B_{3R/2}(x_0)} v_0^{r_0}\,\mathrm{d}x \le d_{25}R^m + d_{25}R\int_{B_{3R/2}(x_0)}\Big\{\sum_{i=1}^m\Big|\frac{\partial u}{\partial x_i}\Big|\Big\}U^{-1}v_0^{r_0-1}\,\mathrm{d}x. \tag{4.19}$$

By means of Young inequality we get

$$
\begin{aligned}
(4.20)\quad & d_{25} R \int_{B_{3R/2}(x_0)} \Big\{\sum_{i=1}^{m} \Big|\frac{\partial u}{\partial x_i}\Big|\Big\} U^{-1} v_0^{r_0-1}\,\mathrm{d}x \\
&\le \frac{1}{r_0}(d_{25}R)^{r_0} \int_{B_{3R/2}(x_0)} \Big\{\sum_{i=1}^{m} \Big|\frac{\partial u}{\partial x_i}\Big|\Big\}^{r_0} U^{-r_0}\,\mathrm{d}x + \frac{r_0-1}{r_0} \int_{B_{3R/2}(x_0)} v_0^{r_0}\,\mathrm{d}x.
\end{aligned}
$$

From (4.19) and (4.20) we have:

$$
(4.21)\qquad \int_{B_{3R/2}(x_0)} v_0^{r_0}\,\mathrm{d}x \le r_0 d_{26} R^m + d_{27} R^{r_0} \int_{B_{3R/2}(x_0)} \sum_{i=1}^{m} \Big|\frac{\partial u}{\partial x_i}\Big|^{r_0} U^{-r_0}\,\mathrm{d}x.
$$

Using Hölder inequality with p/r_0 and $(1-r_0/p)^{-1} = (s+1)$ and the definition of the function ζ_1 we obtain

$$
\begin{aligned}
\int_{B_{3R/2}(x_0)} \sum_{i=1}^{m} \Big|\frac{\partial u}{\partial x_i}\Big|^{r_0} U^{-r_0}\,\mathrm{d}x &\le d_{28}\Big(\int_{B_{7R/4}(x_0)} \Big(\frac{1}{\nu}\Big)^{s}\mathrm{d}x\Big)^{1/(s+1)} \\
&\quad\times \Big(\int_{B_{7R/4}(x_0)} \nu|\nabla u|^p\,\zeta_1^p U^{-p}\,\mathrm{d}x\Big)^{s/(s+1)}.
\end{aligned}
$$

Finally, we acquire inequality (4.18) from (4.21) estimating the last integral of previous inequality by Lemma 4.2 and taking into account that from the Hypothesis 2.10 we have

$$
\Big\{\int_{B_{7R/4}(x_0)} \Big(\frac{1}{\nu}\Big)^{s}\mathrm{d}x\Big\}^{1/s} \Big\{\int_{B_{7R/4}(x_0)} \nu^{\bar t}\,\mathrm{d}x\Big\}^{1/\bar t} \le d_{29} R^{m(1/s+1/\bar t)}.
$$

The lemma is proved. □

Lemma 4.4. *For every $\kappa \ge 1$ there is a positive constant $c = c(\mathbf{data}, \kappa)$ such that* $\lim_{\kappa\to\infty} c(\mathbf{data}, \kappa) = \infty$ *and*

$$
(4.22)\qquad \int_{B_{3R/2}(x_0)} v_0^{\kappa}\,\mathrm{d}x \le c R^m.
$$

Proof: At first, we estimate from above the integral average

$$
(v_0)_{B_{3R/2}(x_0)} = \frac{1}{\operatorname{meas} B_{3R/2}(x_0)} \int_{B_{3R/2}(x_0)} v_0\,\mathrm{d}x
$$

by a constant depending only on **data**.

Using Hölder's inequality and Lemma 4.3 we get

$$(4.23)\qquad (v_0)_{B_{3R/2}(x_0)} \le d_{30} R^{-m/r_0} \Big(\int_{B_{3R/2}(x_0)} v_0^{r_0}\,\mathrm{d}x \Big)^{1/r_0} \le d_{30} R^{-m/r_0} [d_{31} R^{m/r_0} + d_{32} R (d_{33} R^{m-p+m/s} + d_{34} R^{m(\tau-1)/\tau})^{1/p}] \le d_{35}.$$

Next, let $B_{2\varrho} \subset B_{2R}(x_0)$, and let $\zeta_2 \in C_0^\infty(\Omega)$ be a function such that

$$0 \le \zeta_2 \le 1 \quad \text{in } \Omega, \qquad \zeta_2 = 1 \quad \text{in } B_\varrho, \qquad \zeta_2 = 0 \quad \text{in } \Omega \setminus B_{2\varrho},$$
$$\Big| \frac{\partial \zeta_2}{\partial x_i} \Big| \le K_2 \varrho^{-1} \qquad \text{for } i = 1, 2, \ldots, m,$$

where K_2 is an absolute constant, not depending on ϱ. Using Hölder's inequality, Lemma 4.2, Hypothesis 2.10, the properties of the function ζ_2 and that $\tau > ms/(ps-m)$, $s > 1/(p-1)$, we derive that

$$\sum_{i=1}^m \int_{B_\varrho} \Big| \frac{\partial v_0}{\partial x_i} \Big| \,\mathrm{d}x \le d_{36} \Big(\int_{B_\varrho} \nu^{-1/(p-1)}\,\mathrm{d}x \Big)^{(p-1)/p} \times \Big(\int_{B_{2\varrho}} \nu |\nabla u|^p \,\zeta_2^p U^{-p}\,\mathrm{d}x \Big)^{1/p} \le d_{37} \varrho^{m-1}.$$

Hence, by Lemma 3.2, we have

$$(4.24)\qquad \int_{B_{3R/2}(x_0)} \exp(d_{38} |\, v_0 - (v_0)_{B_{3R/2}}|)\,\mathrm{d}x \le d_{39} R^m.$$

Now let $\kappa \ge 1$. Then inequalities (4.23) and (4.24) imply (4.22).
The lemma is proved. □

Lemma 4.5. *There is a positive constant $c_3 = c_3(\mathbf{data})$ such that*

$$(4.25)\qquad \|v_0\|_{L^\infty(B_R(x_0))} \le c_3.$$

Proof: We proceed the proof in four steps.
Step 1. We fix a function $\psi_0 \in C_0^\infty(\mathbb{R})$ such that

$$0 \le \psi_0 \le 1 \quad \text{on } \mathbb{R}, \qquad \psi = 1 \quad \text{in } [-1,1], \qquad \psi = 0 \quad \text{in } \mathbb{R} \setminus (-3/2, 3/2).$$

For any $x \in \Omega$ we set $\psi(x) = \psi_0(|x - x_0|/R)$,

$$\tilde{v}(x) = \begin{cases} [v_0(x)]^k \psi^t(x) [U(x)]^{1-p} & \text{if } x \in B_{2R(x_0)}, \\ 0 & \text{if } x \in \Omega \setminus B_{2R(x_0)}, \end{cases}$$

where $U = \omega_2(2R) - u + R^r$,

$$k \geq \overline{k} := \max\{p, 2(6M+1)\lambda^2(M)\}, \tag{4.26}$$
$$t > p. \tag{4.27}$$

Simple calculations show that

$$\tilde{v} \in \mathring{W}^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$$

and the following assertion holds:

(c) for every $i = 1, 2, \ldots, m$,

$$\Big|\frac{\partial \tilde{v}}{\partial x_i} - z\frac{\partial u}{\partial x_i}\Big| \leq \frac{d_{40} t v_0^k \psi^{t-1}}{R\,U^{p-1}} \qquad \text{a.e. in } \Omega,$$

where $z = [(p-1)v_0^k + kv_0^{k-1}]\psi^t U^{-p}$ and $d_{40} > 0$ depends only on $\max_{\mathbb{R}} |\psi_0'|$.

Putting the function $\tilde{v}$ into (2.4) instead of w and using (1.2), (2.1), and assertion (c), we obtain

$$\begin{aligned} \frac{(p-1)}{\lambda(M)} \int_{B_{2R}(x_0)} & \nu|\nabla u|^p\, U^{-p} v_0^k \psi^t \,\mathrm{d}x \\ & + \frac{k}{\lambda(M)} \int_{B_{2R}(x_0)} \nu|\nabla u|^p\, U^{-p} v_0^{k-1} \psi^t \,\mathrm{d}x \\ & \leq 2\lambda(M) \int_{B_{2R}(x_0)} \nu|\nabla u|^p\, U^{1-p} v_0^k \psi^t \,\mathrm{d}x \\ & + \int_{B_{2R}(x_0)} g^\star v_0^k \psi^t U^{1-p} \,\mathrm{d}x + \mathcal{I}, \end{aligned} \tag{4.28}$$

where $g^\star(x)$ was defined in Lemma 4.1 and

$$\mathcal{I} = \frac{d_{40} t}{R} \sum_{i=1}^m \int_{B_{2R}(x_0)} |a_i(x, u, \nabla u)| U^{1-p} v_0^k \psi^{t-1} \,\mathrm{d}x. \tag{4.29}$$

Step 2. We show that the first term in the right-hand side of inequality (4.28) is absorbed by the second term in its left-hand side. For this we need the inequality

$$U v_0 \leq 6M + 1 \qquad \text{a.e. in } B_{2R}(x_0). \tag{4.30}$$

To prove it, we consider the function

$$\chi(s) = (s + R^r) \ln \frac{2\omega(2R)}{s + R^r}, \qquad s \in [0, \omega(2R)].$$

According to (4.4) and to the elementary inequality $\ln b < b,\ b > 0$, we obtain that for every $s \in [0, \omega(2R)]$

$$0 \le \chi(s) \le 2\omega(2R) \le 4M.$$

Now inequality (4.30) follows from the relations $R \le 1$ and

$$Uv_0 = \omega_2(2R) - u + R^r + \chi(\omega_2(2R) - u) \qquad \text{a.e. in } B_{2R}(x_0).$$

Using (4.30), the first term on the right-hand side of inequality (4.28) is estimated in the following way

$$(4.31) \qquad 2\lambda(M) \int_{B_{2R}(x_0)} \nu |\nabla u|^p\, U^{1-p} v_0^k \psi^t \, \mathrm{d}x \\ \le 2(6M+1)\lambda(M) \int_{B_{2R}(x_0)} \nu |\nabla u|^p\, U^{-p} v_0^{k-1} \psi^t \, \mathrm{d}x.$$

Now (4.26), (4.28) and (4.31) imply the inequality

$$(4.32) \qquad \frac{(p-1)}{\lambda(M)} \int_{B_{2R}(x_0)} \nu |\nabla u|^p\, U^{-p} v_0^k \psi^t \, \mathrm{d}x \\ \le (2M+1) \int_{B_{2R}(x_0)} g^\star v_0^k \psi^t U^{-p} \, \mathrm{d}x + \mathcal{I}\,.$$

Step 3. Let us estimate from above the quantity $\mathcal{I}$, which is defined by (4.29). We use (2.3) and Young's inequality

$$|yz| \le \varepsilon |y|^{p/(p-1)} + \varepsilon^{1-p} |z|^p,$$

where

$$y = |a_i(x, u, \nabla u)| U^{1-p} \psi^{(p-1)t/p} (\nu(x))^{-1/p}, \qquad i = 1, 2, \ldots, m, \\ z = t\, \psi^{(t-p)/p} \nu(x)^{1/p} / R,$$

and ε is an appropriate positive number, to obtain

$$(4.33) \qquad \mathcal{I} \le \frac{(p-1)}{2\lambda(M)} \int_{B_{2R}(x_0)} \nu |\nabla u|^p\, U^{-p} v_0^k \psi^t \, \mathrm{d}x \\ + d_{41} \int_{B_{2R}(x_0)} g_1 v_0^k \psi^t U^{-p} \, \mathrm{d}x + \frac{d_{42} t^p}{R^p} \int_{B_{2R}(x_0)} \nu\, v_0^k \psi^{t-p} \, \mathrm{d}x,$$

where $g_1(x)$ was defined in Lemma 4.1.

From (4.32) and (4.33), for every $k \geq \overline{k}$ and $t > p$, it follows that

$$\int_{B_{2R}(x_0)} \nu |\nabla u|^p \, U^{-p} v_0^k \psi^t \, \mathrm{d}x \leq d_{43} t^p \int_{B_{2R}(x_0)} \psi_1 v_0^k \psi^{t-p} \, \mathrm{d}x, \tag{4.34}$$

where $\psi_1(x) = R^{-rp}[g_1(x) + g^\star(x)] + \nu(x) R^{-p}$.

It results

$$\begin{aligned}\Big(\int_{B_{2R}(x_0)} \psi_1^{t_\star} \, \mathrm{d}x\Big)^{1/t_\star} &\leq R^{-p} \Big(\int_{B_{2R}(x_0)} \nu^{t_\star} \, \mathrm{d}x\Big)^{1/t_\star} \\ &\quad + R^{-rp} \|g_1 + g^\star\|_{L^{t_\star}(\Omega)}.\end{aligned} \tag{4.35}$$

Now, we fix arbitrary $k \geq \overline{k}\tilde{p}/p$ and $t > \tilde{p}$ and let

$$z_1 = v_0^{k/\tilde{p}} \psi^{t/\tilde{p}}.$$

We have $z_1 \in \mathring{W}^{1,p}(\nu, \Omega)$ and for every $i = 1, 2, \ldots, m$,

$$\begin{aligned}\int_\Omega \nu \Big|\frac{\partial z_1}{\partial x_i}\Big|^p \mathrm{d}x &\leq d_{44} k^p \int_{B_{2R}(x_0)} \nu \Big|\frac{\partial u}{\partial x_i}\Big|^p v_0^{kp/\tilde{p}} U^{-p} \psi^{tp/\tilde{p}} \, \mathrm{d}x \\ &\quad + d_{45} t^p \int_{B_{2R}(x_0)} \psi_1 v_0^{kp/\tilde{p}} \psi^{tp/\tilde{p}-p} \, \mathrm{d}x.\end{aligned}$$

From last inequality and (4.34) we obtain

$$\int_\Omega \nu \Big|\frac{\partial z_1}{\partial x_i}\Big|^p \mathrm{d}x \leq d_{46} k^p t^p \int_{B_{2R}(x_0)} \psi_1 v_0^{kp/\tilde{p}} \psi^{(t/\tilde{p}-1)p} \, \mathrm{d}x.$$

Estimating integral to second term of last inequality by Hölder's inequality with the exponents $t_\star$ and $t_\star/(t_\star - 1)$, we obtain that for every $k \geq \overline{k}\tilde{p}/p$ and $t > \tilde{p}$ the following inequality holds:

$$\begin{aligned}\int_\Omega \nu \Big|\frac{\partial z_1}{\partial x_i}\Big|^p \mathrm{d}x &\leq d_{46} k^p t^p \Big(\int_{B_{2R}(x_0)} \psi_1^{t_\star} \, \mathrm{d}x\Big)^{1/t_\star} \\ &\times \Big(\int_{B_{2R}(x_0)} v_0^{kpt_\star/(\tilde{p}(t_\star-1))} \psi^{tpt_\star/(\tilde{p}(t_\star-1)) - pt_\star/(t_\star-1)} \, \mathrm{d}x\Big)^{(t_\star-1)/t_\star}.\end{aligned} \tag{4.36}$$

Step 4. We set

$$H(k,t) = \int_{B_{2R}(x_0)} v_0^k \psi^t \, \mathrm{d}x, \qquad k \in \mathbb{R}, \ t > 0,$$

$$\theta = \frac{\tilde{p}(t_\star - 1)}{pt_\star}, \qquad \widetilde{m} = \frac{pt_\star}{t_\star - 1}.$$

Due to Hypothesis 2.2. we have

$$H(k,t) \le \hat{c}^{\tilde{p}} \Big[\int_{B_{2R}(x_0)} \Big(\frac{1}{\nu}\Big)^s \,\mathrm{d}x \Big]^{\tilde{p}/(ps)} \Big[\int_\Omega \sum_{i=1}^m \nu \Big| \frac{\partial z_1}{\partial x_i} \Big|^p \,\mathrm{d}x \Big]^{\tilde{p}/p}. \tag{4.37}$$

From (4.36) and (4.37) it follows that

$$H(k,t) \le d_{47}(k+t)^{2\tilde{p}} \Big\{ \Big[\int_{B_{2R}(x_0)} \Big(\frac{1}{\nu}\Big)^s \,\mathrm{d}x \Big]^{1/s} \Big[\int_{B_{2R}(x_0)} \psi_1^{t_\star} \,\mathrm{d}x \Big]^{1/t_\star} \Big\}^{\tilde{p}/p} \times \Big[H\Big(\frac{k}{\theta}, \frac{t}{\theta} - \widetilde{m}\Big)\Big]^\theta. \tag{4.38}$$

Using (4.35), (4.2) and Hypothesis 2.10, we obtain

$$\Big[\int_{B_{2R}(x_0)} \Big(\frac{1}{\nu}\Big)^s \,\mathrm{d}x \Big]^{1/s} \Big[\int_{B_{2R}(x_0)} \psi_1^{t_\star} \,\mathrm{d}x \Big]^{1/t_\star} \le d_{48} R^{-p+m/s+m/t_\star}. \tag{4.39}$$

Note that due to the definition of $\tilde{p}$ and θ we have

$$\Big(p - \frac{m}{s} - \frac{m}{t_\star}\Big)\frac{\tilde{p}}{p} = m(\theta - 1). \tag{4.40}$$

From (4.38), (4.39) and (4.40) we get

$$H(k,t) \le d_{49}(k+t)^{2\tilde{p}} R^{-m(\theta-1)} \Big[H\Big(\frac{k}{\theta}, \frac{t}{\theta} - \widetilde{m}\Big)\Big]^\theta \tag{4.41}$$

for every $k \ge \overline{k}\tilde{p}/p$ and $t > \tilde{p}$.

We choose a number $i_0 \in \mathbb{N}$ such that $\theta^{i_0} > \overline{k}\tilde{p}/p$ and set

$$k_i = \theta^{i_0+i}, \quad t_i = \frac{\widetilde{m}\theta}{\theta - 1}(\theta^{i_0+i} - 1), \qquad i = 0,1,2,\ldots$$

Then (4.41) and the inequality $\theta > 1$ imply that for every $i = 1,2,\ldots$,

$$[H(k_i,t_i)]^{1/k_i} \le [d_{50} R^{-m} H(k_0,t_0)]^{1/\theta^{i_0}}.$$

By Lemma 4.4 we have

$$H(k_0,t_0) \le \int_{B_{3R/2}(x_0)} v_0^{\theta^{i_0}} \,\mathrm{d}x \le d_{51} R^m.$$

From the last two inequalities it follows that

$$\|v_0\|_{L^\infty(B_R(x_0))} = \lim_{i\to\infty} \Big(\int_{B_R(x_0)} v_0^{k_i} \,\mathrm{d}x \Big)^{1/k_i} \le \limsup_{i\to\infty} [H(k_i,t_i)]^{1/k_i} \le c_3.$$

The lemma is proved. □

Inequality (4.25) implies (4.3). Recall that we proved (4.3) under assumption (4.5). If (4.5) is not true, we take instead of F_1 the function $F_2 = 2e\omega(2R) \times (u - \omega_1(2R) + R^r)^{-1}$ in $B_{2R}(x_0)$, and $F_2 = e$ in $\Omega \setminus B_{2R}(x_0)$, and arguing as above, we establish (4.3) again. Hence according to Lemma 3.3, the assertions of Theorem 2.11 are true.

Remark. The technique used to prove the local Hölder continuity for bounded generalized solutions of the Dirichlet problem associated with equation (1.1), assuming the degenerate ellipticity condition (1.2), can be repeated for bounded generalized solutions of equation (1.1) with the following boundary Neumann condition:

$$\sum_{i=1}^{m} a_i(x, u, \nabla u) \cos(\vec{n}, x_i) + c_2|u|^{p-2}u + F(x, u) = 0, \qquad c_2 > 0,\ x \in \partial\Omega,$$

where $\partial\Omega$ is locally Lipschitz boundary and $\vec{n} = \vec{n}(x)$ is the outwardly directed (relative to Ω) unit vector normal to $\partial\Omega$ at every point $x \in \partial\Omega$ (see [6] for the Existence theorem in such case). So, we can prove that every function $u \in W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$ satisfying

$$\int_\Omega \left\{ \sum_{i=1}^{m} a_i(x, u, \nabla u) \frac{\partial w}{\partial x_i} + c_0|u|^{p-2}uw + \ f(x, u, \nabla u)w \right\} \mathrm{d}x + \int_{\partial\Omega} \{c_2|u|^{p-2}uw + F(x, u)w\} \,\mathrm{d}s = 0$$

for any $w \in W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$, it is locally Hölder continuous in Ω.

5. Examples

Example 5.1. Let $\Omega \subset \mathbb{R}^m$ be an open bounded set. Suppose for simplicity that $0 \in \partial\Omega$ and, additionally, we assume that

$$p > \frac{m}{2}, \qquad m \geq 4.$$

Let $0 < \gamma < (m/2)(p - m/2)(3m/2 - p)^{-1}$, and let $\nu\colon \Omega \to (0, \infty)$ be defined by

$$\nu(x) = |x|^\gamma.$$

Let s be such that

$$\frac{m}{p - m/2} < s < 1 + \frac{m}{2\gamma}.$$

It results $m/p < s < m/\gamma$, then the function ν satisfies Hypotheses 2.1 and 2.2. Moreover, it is easy to verify that

$$|x|^{2\gamma} \in A_{1+1/s-1} \quad \text{(Muckenhoupt's class)}$$

then, Hypotheses 2.9 and 2.10 hold with $\overline{t} = 2$.

Consider the following boundary value problem

$$-\operatorname{div}\Big(\frac{|x|^{\gamma}}{1+|u|}|\nabla u|^{p-2}\nabla u\Big) + u\{|u|^{p-2} + e^{|u|}|x|^{\gamma}|\nabla u|^{p}\} = g(x) \qquad \text{in } \Omega, \tag{5.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{5.2}$$

where $g(x) \in L^{\infty}(\Omega)$.

In this case we have:

$$a_i(x,u,\nabla u) = \frac{|x|^{\gamma}}{1+|u|}|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}, \qquad i = 1,2,\ldots,m,$$

$$f(x,u,\nabla u) = ue^{|u|}|x|^{\gamma}|\nabla u|^{p} - \frac{1}{2}u|u|^{p-2} - g(x), \qquad c_0 = \frac{3}{2}.$$

If we put $\lambda(|u|) = (1+|u|)e^{|u|}$, it is possible to verify all the Hypotheses 2.3–2.8. To verify (2.1), for example, it will be sufficient to note that the function $|u|^{p-2}/e^{-|u|}$ is bounded from above by $((p-2)/e)^{p-2}$ in $]-\infty,\infty[$.

Hence, boundary value problem (5.1), (5.2) has at least one weak solution in the sense (2.4), i.e., there exists at least one $u \in W$ such that

$$\int_{\Omega} \frac{|x|^{\gamma}}{1+|u|}|\nabla u|^{p-2}\nabla u \nabla w \,\mathrm{d}x + \int_{\Omega} u\{|u|^{p-2} + e^{|u|}|x|^{\gamma}|\nabla u|^{p}\}w \,\mathrm{d}x = \int_{\Omega} g(x)w \,\mathrm{d}x$$

holds for every $w \in W$.

Moreover, from Theorem 2.11, u is locally Hölder continuous in Ω.

Example 5.2. Let $\Omega \subset \mathbb{R}^m$ be an open bounded set and let $g \in L^{\infty}(\Omega)$. Put $\nu(x) = 1$ in Ω and consider boundary value problem

$$-\operatorname{div}\Big(\frac{1}{1+|u|^{p}}|\nabla u|^{p-2}\nabla u\Big) + e^{u} - |u|^{p} + |\nabla u|^{p} = g(x) \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

In this case we have:

$$a_i(x,u,\nabla u) = \frac{1}{1+|u|^{p}}|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}, \qquad i = 1,2,\ldots,m,$$

$$f(x,u,\nabla u) = e^{u} - |u|^{p} - u|u|^{p-2} + |\nabla u|^{p} - g(x), \qquad c_0 = 1,$$

and $\lambda(|u|) = e^{|u|^{p}}$. All Hypotheses 2.1–2.8 are satisfied. It may be worth noting that the function $u(e^{u} - |u|^{p} - u|u|^{p-2})$ has minimum (negative) in $\mathbb{R}$. Hence every functions $u \in W$ satisfying

$$\int_\Omega \frac{1}{1+|u|^p} |\nabla u|^{p-2} \nabla u \nabla w \, \mathrm{d}x + \int_\Omega \{e^u - |u|^p + |\nabla u|^p\} w \, \mathrm{d}x = \int_\Omega g(x) w \, \mathrm{d}x$$

for every $w \in W$ is locally Hölder continuous in Ω.

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