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AN ENTIRE FUNCTION SHARING A POLYNOMIAL WITH ITS LINEAR DIFFERENTIAL POLYNOMIAL

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Cordially dedicated to my teacher Professor Indrajit Lahiri

Abstract. We study the uniqueness of entire functions which share a polynomial with their linear differential polynomials.

Keywords: entire function; differential polynomial; derivative; sharing

MSC 2010: 30D35

1. Introduction, definitions and results

Let f be a noncostant meromorphic function in the open complex plane $\mathbb{C}$ and $a = a(z)$ be a polynomial. We denote by $E(a;f)$ the set of zeros of $f-a$, counted with multiplicities, and $\overline{E}(a;f)$ the set of all distinct zeros of $f-a$. Let $N(r,a;f)$ be the counting function of zeros of $f-a$ in $\{z\colon\ |z|\leqslant r\}$. If $A\subset\mathbb{C}$, then the counting function $N_A(r,a;f)$ of zeros of $f-a$ in $\{z\colon\ |z|\leqslant r\}\cap A$ is defined as

$$N_A(r,a;f)=\int_0^r\frac{n_A(t,a;f)-n_A(0,a;f)}{t}\,\mathrm{d}t+n_A(0,a;f)\log r,$$

where $n_A(t,a;f)$ is the number of zeros of $f-a$, counted with multiplicities, in $\{z\colon\ |z|\leqslant r\}\cap A$. For standard definitions and notations we refer the reader to [1] and [6].

There are some results related to value sharing and polynomial sharing. In the beginning, Jank, Mues and Volkmann [2] considered the situation that an entire

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function shares a nonzero value with its derivatives and they proved the following theorem.

Theorem A ([2]). *Let f be a nonconstant entire function and a be a nonzero finite value. If $\overline{E}(a;f)=\overline{E}(a;f^{(1)})\subset\overline{E}(a;f^{(2)})$, then $f\equiv f^{(1)}$.*

The following example shows that in Theorem A the second derivative cannot be replaced by any higher order derivatives.

Example 1.1 ([7]). Let k ($\geqslant 3$) be an integer and ω ($\neq 1$) be a $(k-1)$th root of unity. We put $f=\mathrm{e}^{\omega z}+\omega-1$. Then f, $f^{(1)}$ and $f^{(k)}$ share the value ω CM, but $f\not\equiv f^{(1)}$.

On the basis of this example, Zhong [7] improved Theorem A by considering higher order derivatives in the following way.

Theorem B ([7]). *Let f be a nonconstant entire function and a be a nonzero finite number. If $E(a;f)=E(a;f^{(1)})$ and $\overline{E}(a;f)\subset\overline{E}(a;f^{(n)})\cap\overline{E}(a;f^{(n+1)})$ for n ($\geqslant 1$), then $f\equiv f^{(n)}$.*

In 1999 Li [5] considered linear differential polynomials and proved the following result.

Theorem C ([5]). *Let f be a nonconstant entire function and $L=a_1f^{(1)}+a_2f^{(2)}+\ldots+a_nf^{(n)}$, where $a_1,a_2,\ldots,a_n$ ($\neq 0$) are constants, and a ($\neq 0$) be a finite number. If $\overline{E}(a;f)=\overline{E}(a;f^{(1)})\subset\overline{E}(a;L)\cap\overline{E}(a;L^{(1)})$, then $f\equiv f^{(1)}\equiv L$.*

Lahiri and Kaish [3] improved Theorem B by considering a shared polynomial. They proved the following theorem.

Theorem D ([3]). *Let f be a nonconstant entire function and $a=a(z)$ ($\not\equiv 0$) be a polynomial with $\deg(a)\neq\deg(f)$. Suppose that $A=\overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$ and $B=\overline{E}(a;f^{(1)})\backslash\{\overline{E}(a;f^{(n)})\cap\overline{E}(a;f^{(n+1)})\}$, where Δ denotes the symmetric difference of sets and n ($\geqslant 1$) is an integer. If*

(1) $N_A(r,a;f)+N_A(r,a;f^{(1)})=O\{\log T(r,f)\}$,
(2) $N_B(r,a;f^{(1)})=S(r,f)$, *and*
(3) *each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity,*

then $f=\lambda\mathrm{e}^z$, where λ ($\neq 0$) is a constant.

In Theorem D, Lahiri and Kaish considered an entire function which shares a polynomial with its derivatives. In our paper we improve Theorem D by considering an entire function which shares a polynomial with its linear differential polynomials.

The main result of the paper is the following theorem.

Theorem 1.1. *Let f be a nonconstant entire function and $L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)}$, where $a_2, a_3, \ldots, a_n$ $(\neq 0)$ are constants, and n $(\geqslant 2)$ be an integer. Also let $a(z)$ $(\neq 0)$ be a polynomial with $\deg(a) \neq \deg(f)$. Suppose that $A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$. If*

(1) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,
(2) $N_B(r, a; f^{(1)}) = S(r, f)$, *and*
(3) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

then $f = L = \lambda \mathrm{e}^z$, where λ $(\neq 0)$ is a constant.

In the theorem we assume that the degree of a transcendental entire function is infinity.

Putting $A = B = \Phi$, we get the following corollary.

Corollary 1.1. *Let f be a nonconstant entire function and $a = a(z)$ $(\not\equiv 0)$ be a polynomial with $\deg(a) \neq \deg(f)$. Also let $L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)}$, where $a_2, a_3, \ldots, a_n$ $(\neq 0)$ are constants and n $(\geqslant 2)$ is an integer. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$, then $f = L = \lambda \mathrm{e}^z$, where λ $(\neq 0)$ is a constant.*

In Theorem C, Li considered the linear differential polynomial as $L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)}$, where $a_1, a_2, \ldots, a_n$ $(\geqslant 0)$ are constants. Here we consider the linear differential polynomial L with the first coefficient $a_1 = 0$. That is, we consider $L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)}$. In Corollary 1.1 if we consider $a = a(z)$ as a nonzero finite constant, then we get a particular case of Theorem C when L will be considered with the first coefficient zero. Therefore Corollary 1.1 shows that our result is an improvement of a particular case of Theorem C when L is considered with the first coefficient $a_1 = 0$.

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 ([3]). *Let f be transcendental entire function of finite order and $a = a(z)$ $(\not\equiv 0)$ be a polynomial and $A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)})$. If*

(1) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,
(2) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

then $m(r, a; f) = m(r, (f - a)^{-1}) = S(r, f)$.

Lemma 2.2. *Let f be a transcendental entire function and $a(z)$ ($\not\equiv 0$) be a polynomial. Also let $L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)}$ and $b(z) = a_2 a^{(2)} + a_3 a^{(3)} + \ldots + a_n a^{(n)}$, where $a_2, a_3, \ldots, a_n$ ($\geqslant 0$) are constants and n ($\geqslant 2$) is an integer. Suppose $h = ((a - a^{(1)})(L - b) - (a - b)(f^{(1)} - a^{(1)}))(f - a)^{-1}$ and $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$. If*

(1) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$,
(2) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*
(3) *h is transcendental entire or meromorphic,*

then $m(r, a; f^{(1)}) = m(r, (f^{(1)} - a)^{-1}) = S(r, f)$.

P r o o f. Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, if z_0 is a common zero of $f - a$ and $f^{(1)} - a$ with multiplicity q ($\geqslant 2$), then z_0 is a zero of $a - a^{(1)}$ with multiplicity $q - 1$. So

$$N_{(2}(r, a; f) \leqslant 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f),$$

where $N_{(2}(r, a; f)$ is the counting function of multiple zeros of $f - a$.

Hence, by the hypothesis we see that

$$N(r, h) \leqslant N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2}(r, a; f) + S(r, f) = S(r, f).$$

Since $m(r, h) = S(r, f)$, we have $T(r, h) = S(r, f)$.

Now by a simple calculation we get

$$\begin{aligned} f &= a + \frac{1}{h}((a - a^{(1)})(L - b) - (a - b)(f^{(1)} - a^{(1)})) \\ &= a + \frac{1}{h}((a - a^{(1)})(L - a) - (a - b)(f^{(1)} - a)). \end{aligned}$$

Differentiating we obtain

$$\begin{aligned} f^{(1)} = a^{(1)} &+ \Big(\frac{1}{h}\Big)^{(1)}((a - a^{(1)})(L - a) - (a - b)(f^{(1)} - a)) \\ &+ \frac{1}{h}((a - a^{(1)})(L^{(1)} - a^{(1)}) + (a^{(1)} - a^{(2)})(L - a) \\ &- (a^{(1)} - b^{(1)})(f^{(1)} - a) - (a - b)(f^{(2)} - a^{(1)})). \end{aligned}$$

This implies

$$\begin{aligned} (f^{(1)} - a)\Big(1 + \Big(\frac{1}{h}\Big)^{(1)}(a - b) + \frac{1}{h}(a^{(1)} - b^{(1)})\Big) \\ = a^{(1)} - a + \Big(\Big(\frac{1}{h}\Big)^{(1)}(a - a^{(1)}) + \frac{1}{h}(a^{(1)} - a^{(2)})\Big)(L - a) \\ + \frac{1}{h}(a - a^{(1)})(L^{(1)} - a^{(1)}) - \frac{a - b}{h}(f^{(2)} - a^{(1)}) \end{aligned}$$

$$= \Big(\frac{a-a^{(1)}}{h}\Big)^{(1)}(L-c) + \frac{a-a^{(1)}}{h}(L^{(1)}-c^{(1)})$$
$$- \frac{a-b}{h}(f^{(2)}-a^{(1)}) + a^{(1)} - a + \Big(\frac{(c-a)(a-a^{(1)})}{h}\Big)^{(1)},$$

where $c(z) = a_2 a^{(1)} + a_3 a^{(2)} + \ldots + a_n a^{(n-1)}$.

Therefore

$$\Big(1 + \Big(\frac{a-b}{h}\Big)^{(1)}\Big)(f^{(1)} - a)$$
$$= a^{(1)} - a + \Big(\frac{(c-a)(a-a^{(1)})}{h}\Big)^{(1)} + \Big(\frac{a-a^{(1)}}{h}\Big)^{(1)}(L-c)$$
$$+ \frac{a-a^{(1)}}{h}(L^{(1)}-c^{(1)}) - \frac{a-b}{h}(f^{(2)}-a^{(1)}).$$

This implies

$$\frac{1}{f^{(1)}-a} = \frac{\mu}{\nu} - \frac{1}{\nu}\Big(\frac{a-a^{(1)}}{h}\Big)^{(1)}\frac{L-c}{f^{(1)}-a} - \frac{a-a^{(1)}}{h\nu}\frac{L^{(1)}-c^{(1)}}{f^{(1)}-a} + \frac{a-b}{h\nu}\frac{f^{(2)}-a^{(1)}}{f^{(1)}-a}, \tag{2.1}$$

where $\mu = 1 + ((a-b)h^{-1})^{(1)}$ and $\nu = a^{(1)} - a + ((c-a)(a-a^{(1)})h^{-1})^{(1)}$.

We now verify that $\mu \not\equiv 0$ and $\nu \not\equiv 0$. If $\mu \equiv 0$, then $1 + ((a-b)h^{-1})^{(1)} \equiv 0$. Integrating we get $h = (a-b)(c_1 - z)^{-1}$, where c_1 is a constant. This is a contradiction as h is transcendental. Therefore $\mu \not\equiv 0$.

If $\nu \equiv 0$, then $((c-a)(a-a^{(1)})h^{-1})^{(1)} \equiv a - a^{(1)}$. Integrating we get $(c-a) \times (a-a^{(1)})h^{-1} = P(z)$, i.e. $h = (c-a)(a-a^{(1)})/P(z)$, where $P(z)$ is a polynomial. This is a contradiction because h is transcendental. Therefore $\nu \not\equiv 0$.

Again $T(r,\mu) + T(r,\nu) = S(r,f)$. Therefore from (2.1) we get $m(r,a;f^{(1)}) = m(r,(f^{(1)}-a)^{-1}) = S(r,f)$. This proves the lemma. □

Lemma 2.3 ([4], page 58)**.** *Each solution of the differential equation*

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0,$$

where a_0 $(\not\equiv 0), a_1, \ldots, a_n$ $(\not\equiv 0)$ are polynomials, is an entire function of finite order.

Lemma 2.4 ([4], page 47)**.** *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 be three distinct meromorphic functions satisfying $T(r,a_\nu) = S(r,f)$ for $\nu = 1,2,3$. Then*

$$T(r,f) \leqslant \overline{N}(r,0;f-a_1) + \overline{N}(r,0;f-a_2) + \overline{N}(r,0;f-a_3) + S(r,f).$$

Lemma 2.5 ([6], page 92). *Let* $f_1, f_2, \ldots, f_n$ *be meromorphic functions which are nonconstant except possibly for* f_n, *where* $n \geqslant 3$. *If* $f_n \not\equiv 0$ *and* $\sum_{j=1}^{n} f_j \equiv 1$ *and* $\sum_{j=1}^{n} N(r,0;f_j) + (n-1)\sum_{j=1}^{n} N(r,\infty;f_j) < \{\mu + o(1)\}T(r,f_k)$ *for* $k = 1,2,\ldots,n-1$, *then* $f_n \equiv 1$.

3. Proof of the theorem

First, we verify that f cannot be a polynomial. We suppose that f is a polynomial. Then $T(r,f) = O(\log r)$ and $N_A(r,a;f) + N_A(r,a;f^{(1)}) = O(\log T(r,f)) = S(r,f)$ imply $A = \Phi$. Also $N_B(r,a;f^{(1)}) = S(r,f)$ implies $B = \Phi$. Therefore $E(a;f) = E(a;f^{(1)})$ and $\overline{E}(a;f^{(1)}) \subset \overline{E}(a,L) \cap \overline{E}(a;L^{(1)})$.

Let $\deg(f) = m$ and $\deg(a) = p$. If $m \geqslant p+1$, then $\deg(f-a) = m$, $\deg(f^{(1)}-a) \leqslant m-1$. Since each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, it contradicts the fact that $E(a;f) = E(a;f^{(1)})$.

Next let $m \leqslant p-1$. Then $\deg(f-a) = p$, $\deg(f^{(1)}-a) = p$. Again $E(a;f) = E(a;f^{(1)})$, we can write $f^{(1)} - a \equiv (f-a)k$, where k $(\geqslant 0)$ is a constant.

If $k \neq 1$, then $kf - f^{(1)} \equiv (k-1)a$, which is impossible as $\deg((k-1)a) = p > m = \deg(kf - f^{(1)})$.

If $k = 1$, then $f = f^{(1)}$, which is again a contradiction. Therefore f is a transcendental entire function.

Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, a common zero of $f-a$ and $f^{(1)}-a$ of multiplicity q $(\geqslant 2)$ is a zero of $a - a^{(1)}$ with multiplicity $q-1$ $(\geqslant 1)$. Therefore $N_{(2}(r,a;f^{(1)}|f=a) \leqslant 2N(r,0;a-a^{(1)}) = S(r,f)$, where $N_{(2}(r,a;f^{(1)}|f=a)$ denotes the counting function (counted with multiplicities) of those multiple zeros of $f^{(1)}-a$, which are also zeros of $f-a$.

Now

$$
\begin{aligned}
N_{(2}(r,a;f^{(1)}) &\leqslant N_A(r,a;f^{(1)}) + N_B(r,a;f^{(1)}) \\
&\quad + N_{(2}(r,a;f^{(1)}|f=a) + S(r,f) = S(r,f).
\end{aligned} \tag{3.1}
$$

First we suppose that $L^{(1)} \not\equiv f^{(1)}$. Then using (3.1) we get by the hypothesis

$$
\begin{aligned}
N(r,a;f^{(1)}) &\leqslant N_B(r,a;f^{(1)}) + N\Big(r, \frac{a-b^{(1)}}{a-a^{(1)}}; \frac{L^{(1)}-b^{(1)}}{f^{(1)}-a^{(1)}}\Big) + S(r,f) \\
&\leqslant T\Big(r, \frac{L^{(1)}-b^{(1)}}{f^{(1)}-a^{(1)}}\Big) + S(r,f) = N\Big(r, \frac{L^{(1)}-b^{(1)}}{f^{(1)}-a^{(1)}}\Big) + S(r,f) \\
&\leqslant N(r,a^{(1)};f^{(1)}) + S(r,f),
\end{aligned} \tag{3.2}
$$

where $b(z) = a_2a^{(2)}(z) + a_3a^{(3)}(z) + \ldots + a_na^{(n)}(z)$.

Again

$$\begin{aligned}m(r,a;f) &\leqslant m\Big(r,\frac{f^{(1)}-a^{(1)}}{f-a};\frac{1}{f^{(1)}-a^{(1)}}\Big)\\ &\leqslant m(r,a^{(1)};f^{(1)})+S(r,f)\\ &= T(r,f^{(1)})-N(r,a^{(1)};f^{(1)})+S(r,f)\\ &= m(r,f^{(1)})-N(r,a^{(1)};f^{(1)})+S(r,f)\\ &\leqslant m(r,f)-N(r,a^{(1)};f^{(1)})+S(r,f)\\ &= T(r,f)-N(r,a^{(1)};f^{(1)})+S(r,f),\end{aligned}$$

i.e. $N(r,a^{(1)};f^{(1)})\leqslant N(r,a;f)+S(r,f)$.

Therefore from (3.2) we get

$$N(r,a;f^{(1)})\leqslant N(r,a;f)+S(r,f). \tag{3.3}$$

Again

$$N(r,a;f)\leqslant N_A(r,a;f)+N(r,a;f^{(1)}|f=a)\leqslant N(r,a;f^{(1)})+S(r,f). \tag{3.4}$$

Therefore from (3.3) and (3.4) we get

$$N(r,a;f^{(1)})=N(r,a;f)+S(r,f). \tag{3.5}$$

Let $h=((a-a^{(1)})(L-b)-(a-b)(f^{(1)}-a^{(1)}))(f-a)^{-1}$ be transcendental. Then

$$\begin{aligned}T(r,f)=m(r,f) &\leqslant m\Big(r,\frac{1}{h}((a-a^{(1)})L-(a-b)f^{(1)})\Big)+S(r,f)\\ &\leqslant m(r,f^{(1)})+m\Big(r,(a-a^{(1)})\frac{L}{f^{(1)}}-(a-b)\Big)+S(r,f)\\ &\leqslant m(r,f^{(1)})+S(r,f)=T(r,f^{(1)})+S(r,f)\\ &= m(r,f^{(1)})+S(r,f)\leqslant m(r,f)+S(r,f)\\ &= T(r,f)+S(r,f).\end{aligned}$$

Therefore

$$T(r,f^{(1)})=T(r,f)+S(r,f). \tag{3.6}$$

Again by Lemma 2.2 we get $m(r,a;f^{(1)})=S(r,f)$. Then from (3.5) and (3.6) we get $m(r,a;f)=S(r,f)$. Therefore

$$m(r,a;f)+m(r,a;f^{(1)})=S(r,f). \tag{3.7}$$

Next we suppose that h is rational. Then by Lemma 2.3 we see that f is of finite order and by Lemma 2.1 we get $m(r,a;f)=S(r,f)$. Since

$$T(r,f^{(1)})=m(r,f^{(1)})\leqslant m(r,f)+S(r,f)=T(r,f)+S(r,f)$$

and from (3.5) we get $m(r,a;f^{(1)})\leqslant m(r,a;f)+S(r,f)=S(r,f)$. Hence in this case also we obtain (3.7).

Let $\xi=(f^{(1)}-a)(f-a)^{-1}$ and $\eta=(L-a)(f^{(1)}-a)^{-1}$. Then by (3.7) we get $m(r,\xi)+m(r,\eta)=S(r,f)$. Also $N(r,\xi)\leqslant N_A(r,a;f)+N_B(r,a;f^{(1)})+N_{(2}(r,a;f)+S(r,f)=S(r,f)$ because $N_{(2}(r,a;f)\leqslant N_A(r,a;f)+2N(r,0;a-a^{(1)})+S(r,f)=S(r,f)$.

Using (3.2) we get

$$N(r,\eta)\leqslant N_A(r,a;f^{(1)})+N_B(r,a;f^{(1)})+N_{(2}(r,a;f^{(1)})+S(r,f)=S(r,f).$$

Therefore

$$T(r,\xi)+T(r,\eta)=S(r,f). \tag{3.8}$$

Let z_1 be a simple zero of $f-a$ such that $z_1\notin A\cup B$ and $a(z_1)-a^{(1)}(z_1)\neq 0$. Then by Taylor's expansion in some neighbourhood of z_1 we get

$$\begin{aligned} f(z)-a(z)&=(a(z_1)-a^{(1)}(z_1))(z-z_1)+O(z-z_1)^2,\\ f^{(1)}(z)-a(z)&=(f^{(2)}(z_1)-a^{(1)}(z_1))(z-z_1)+O(z-z_1)^2, \end{aligned}$$

and

$$L(z)-a(z)=(a(z_1)-a^{(1)}(z_1))(z-z_1)+O(z-z_1)^2.$$

Therefore in some neighbourhood of z_1 we get

$$\xi(z)=\frac{f^{(2)}(z_1)-a^{(1)}(z_1)}{a(z_1)-a^{(1)}(z_1)}+O(z-z_1), \tag{3.9}$$

and

$$\eta(z)=\frac{a(z_1)-a^{(1)}(z_1)}{f^{(2)}(z_1)-a^{(1)}(z_1)}+O(z-z_1). \tag{3.10}$$

We put $\chi=\eta-\xi^{-1}$. Then from (3.8) we get $T(r,\chi)\leqslant T(r,\eta)+T(r,\xi)+S(r,f)=S(r,f)$.

Also in some neighbourhood of z_1 we have by (3.9) and (3.10),

$$\begin{aligned}\chi(z) &= \eta(z) - \frac{1}{\xi(z)}\\ &= \frac{a(z_1)-a^{(1)}(z_1)}{f^{(2)}(z_1)-a^{(1)}(z_1)} + O(z-z_1) - \Big(\frac{f^{(2)}(z_1)-a^{(1)}(z_1)}{a(z_1)-a^{(1)}(z_1)} + O(z-z_1)\Big)^{-1}\\ &= \frac{a(z_1)-a^{(1)}(z_1)}{f^{(2)}(z_1)-a^{(1)}(z_1)} + O(z-z_1) - \Big(\frac{a(z_1)-a^{(1)}(z_1)}{f^{(2)}(z_1)-a^{(1)}(z_1)} + O(z-z_1)\Big)\\ &= O(z-z_1).\end{aligned}$$

If $\chi \not\equiv 0$, then

$$\begin{aligned}N(r,a;f) &\leqslant N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f) + N(r,0;a-a^{(1)}) + N(r,0;\chi)\\ &= S(r,f),\end{aligned}$$

and so by (3.7) we get $T(r,f) = S(r,f)$, a contradiction.

Therefore $\chi \equiv 0$ and so

$$L \equiv f. \tag{3.11}$$

Differentiating (3.11) we get $L^{(1)} \equiv f^{(1)}$, which contradicts our hypothesis that $L^{(1)} \not\equiv f^{(1)}$. Therefore, indeed we have $L^{(1)} \equiv f^{(1)}$.

Next we suppose that $L^{(1)} \not\equiv L$. Then by the hypothesis and (3.1) we get

$$\begin{aligned}N(r,a;f^{(1)}) &\leqslant N_B(r,a;f^{(1)}) + N\Big(r,\frac{a-b^{(1)}}{a-b};\frac{L^{(1)}-b^{(1)}}{L-b}\Big) + S(r,f)\\ &\leqslant T\Big(r,\frac{L^{(1)}-b^{(1)}}{L-b}\Big) + S(r,f) = N\Big(r,\frac{L^{(1)}-b^{(1)}}{L-b}\Big) + S(r,f)\\ &= \overline{N}(r,b;L) + S(r,f).\end{aligned} \tag{3.12}$$

Again

$$\begin{aligned}m(r,a;f) &= m\Big(r,\frac{L-b}{f-a}\frac{1}{L-b}\Big) \leqslant m(r,b;L) + S(r,f)\\ &= T(r,L) - N(r,b;L) + S(r,f) = m(r,L) - N(r,b;L) + S(r,f)\\ &\leqslant m\Big(r,\frac{L}{f}\Big) + m(r,f) - N(r,b;L) + S(r,f)\\ &= m(r,f) - N(r,b;L) + S(r,f) = T(r,f) - N(r,b;L) + S(r,f)\end{aligned}$$

and so $N(r,b;L) \leqslant N(r,a;f) + S(r,f)$. Now by (3.12) we get $N(r,a;f^{(1)}) \leqslant N(r,a;f) + S(r,f)$.

Also

$$N(r,a;f) \leqslant N_A(r,a;f) + N(r,a;f^{(1)}|f=a) \leqslant N(r,a;f^{(1)}) + S(r,f).$$

Therefore $N(r,a;f^{(1)}) = N(r,a;f) + S(r,f)$, which is (3.5).

Now using Lemma 2.1, Lemma 2.2, Lemma 2.3 and (3.5) we similarly obtain (3.7). Using ξ and η and proceeding likewise we get (3.11), which implies $L \equiv f$ or $a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)} - f \equiv 0$. Solving this we get

$$f = p_1 \mathrm{e}^{\alpha_1 z} + p_2 \mathrm{e}^{\alpha_2 z} + \ldots + p_t \mathrm{e}^{\alpha_t z}, \tag{3.13}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the roots of $a_2\zeta^2 + a_3\zeta^3 + \ldots + a_n\zeta^n - 1 = 0$ and $p_1, p_2, \ldots, p_t$ are constants or polynomials, not all identically zero and t ($\leqslant n$) is an integer.

Differentiating (3.13) we get

$$f^{(1)} = \sum_{i=1}^{t} (p_i^{(1)} + p_i \alpha_i) \mathrm{e}^{\alpha_i z}. \tag{3.14}$$

Now from (3.13), (3.14) and $\xi = (f^{(1)} - a)(f-a)^{-1}$ we get

$$\sum_{i=1}^{t} (\xi p_i - p_i^{(1)} - p_i \alpha_i) \mathrm{e}^{\alpha_i z} \equiv a(\xi - 1). \tag{3.15}$$

We suppose that $\xi \not\equiv 1$. Then from (3.15) we get

$$\sum_{i=1}^{t} \frac{\xi p_i - p_i^{(1)} - p_i \alpha_i}{a(\xi-1)} \mathrm{e}^{\alpha_i z} \equiv 1. \tag{3.16}$$

Here $T(r,f) = O(T(r,\mathrm{e}^{\alpha_i z}))$ for $i = 1, 2, \ldots, t$.

First we suppose that the left-hand side of (3.16) contains only one term, say,

$$\frac{\xi p_k - p_k^{(1)} - p_k \alpha_k}{a(\xi-1)} \mathrm{e}^{\alpha_k z} \equiv 1.$$

Then $T(r,\mathrm{e}^{\alpha_k z}) = S(r,f) = S(r,\mathrm{e}^{\alpha_k z})$, a contradiction.

Next we suppose that the left-hand side of (3.16) contains only two terms, say,

$$\frac{\xi p_k - p_k^{(1)} - p_k \alpha_k}{a(\xi-1)} \mathrm{e}^{\alpha_k z} + \frac{\xi p_l - p_l^{(1)} - p_l \alpha_l}{a(\xi-1)} \mathrm{e}^{\alpha_l z} \equiv 1.$$

So by Lemma 2.4 we get from above

$$\begin{aligned}T(r, \mathrm{e}^{\alpha_k z}) &\leqslant \overline{N}(r, 0; \mathrm{e}^{\alpha_k z}) + \overline{N}(r, \infty; \mathrm{e}^{\alpha_k z}) \\ &\quad + \overline{N}\Big(r, \frac{a(\xi-1)}{\xi p_k - p_k^{(1)} - p_k \alpha_k}; \mathrm{e}^{\alpha_k z}\Big) + S(r, \mathrm{e}^{\alpha_k z}) \\ &= \overline{N}(r, 0; \mathrm{e}^{\alpha_l z}) + S(r, \mathrm{e}^{\alpha_k z}) = S(r, \mathrm{e}^{\alpha_k z}),\end{aligned}$$

a contradiction.

Finally we suppose that the left-hand side of (3.16) contains more than two terms, then by Lemma 2.5 we get

$$\frac{\xi p_i - p_i^{(1)} - p_i \alpha_i}{a(\xi-1)} \mathrm{e}^{\alpha_i z} \equiv 1 \tag{3.17}$$

for one value of $i \in \{1, 2, \ldots, t\}$.

From (3.17) we see that $T(r, \mathrm{e}^{\alpha_i z}) = S(r, f) = S(r, \mathrm{e}^{\alpha_i z})$, a contradiction. Therefore $\xi \equiv 1$ and so $f^{(1)} \equiv f$. Hence, from $L \equiv f$ we get $L \equiv L^{(1)}$, a contradiction to the supposition. Therefore, indeed we have $L \equiv L^{(1)}$.

Now $L \equiv L^{(1)} \equiv f^{(1)}$ implies $L = L^{(1)} = f^{(1)} = \lambda \mathrm{e}^z$, where λ ($\geqslant 0$) is a constant. Therefore $f = \lambda \mathrm{e}^z + K$, where K is a constant.

By Lemma 2.4 we get

$$\begin{aligned}T(r, \lambda \mathrm{e}^z) &\leqslant \overline{N}(r, 0; \lambda \mathrm{e}^z) + \overline{N}(r, \infty; \lambda \mathrm{e}^z) + \overline{N}(r, a-K; \lambda \mathrm{e}^z) + S(r, \lambda \mathrm{e}^z) \\ &= \overline{N}(r, a; f) + S(r, \lambda \mathrm{e}^z).\end{aligned} \tag{3.18}$$

If $\overline{N}(r, a; f) = S(r, f)$, then from (3.18) we get $T(r, \lambda \mathrm{e}^z) = S(r, \lambda \mathrm{e}^z)$, which is a contradiction. Therefore $\overline{N}(r, a; f) \neq S(r, f)$.

Again

$$\overline{N}(r, a; f) \leqslant N_A(r, a; f) + N(r, a; f | f^{(1)} = a). \tag{3.19}$$

Since $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$, from (3.19) we must have $\overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \neq \Phi$, otherwise $\overline{N}(r, a; f) = S(r, f)$.

Let $z_3 \in \overline{E}(a; f) \cap \overline{E}(a; f^{(1)})$. Then $f(z_3) = f^{(1)}(z_3)$ and then $f(z) = f^{(1)}(z) + K$ implies $K = 0$. Therefore $f = L = \lambda \mathrm{e}^z$. This proves the theorem. □

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