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AN ENTIRE FUNCTION SHARING A POLYNOMIAL WITH ITS LINEAR DIFFERENTIAL POLYNOMIAL

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Cordially dedicated to my teacher Professor Indrajit Lahiri

Abstract. We study the uniqueness of entire functions which share a polynomial with their linear differential polynomials.

Keywords: entire function; differential polynomial; derivative; sharing

MSC 2010: 30D35

1. Introduction, definitions and results

Let \( f \) be a nonconstant meromorphic function in the open complex plane \( \mathbb{C} \) and \( a = a(z) \) be a polynomial. We denote by \( E(a; f) \) the set of zeros of \( f - a \), counted with multiplicities, and \( \overline{E}(a; f) \) the set of all distinct zeros of \( f - a \). Let \( N(r, a; f) \) be the counting function of zeros of \( f - a \) in \( \{z: |z| \leq r\} \). If \( A \subset \mathbb{C} \), then the counting function \( N_A(r, a; f) \) of zeros of \( f - a \) in \( \{z: |z| \leq r\} \cap A \) is defined as

\[
N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} \, dt + n_A(0, a; f) \log r,
\]

where \( n_A(t, a; f) \) is the number of zeros of \( f - a \), counted with multiplicities, in \( \{z: |z| \leq r\} \cap A \). For standard definitions and notations we refer the reader to [1] and [6].

There are some results related to value sharing and polynomial sharing. In the beginning, Jank, Mues and Volkmann [2] considered the situation that an entire

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function shares a nonzero value with its derivatives and they proved the following theorem.

**Theorem A** ([2]). Let $f$ be a nonconstant entire function and $a$ be a nonzero finite value. If $E(a; f) = E(a; f^{(1)}) \subset E(a; f^{(2)})$, then $f \equiv f^{(1)}$.

The following example shows that in Theorem A the second derivative cannot be replaced by any higher order derivatives.

**Example 1.1** ([7]). Let $k (\geq 3)$ be an integer and $\omega (\neq 1)$ be a $(k - 1)$th root of unity. We put $f = e^{\omega z} + \omega - 1$. Then $f$, $f^{(1)}$ and $f^{(k)}$ share the value $\omega \text{CM}$, but $f \not\equiv f^{(1)}$.

On the basis of this example, Zhong [7] improved Theorem A by considering higher order derivatives in the following way.

**Theorem B** ([7]). Let $f$ be a nonconstant entire function and $a$ be a nonzero finite number. If $E(a; f) = E(a; f^{(1)})$ and $E(a; f) \subset E(a; f^{(n)}) \cap E(a; f^{(n+1)})$ for $n (\geq 1)$, then $f \equiv f^{(n)}$.

In 1999 Li [5] considered linear differential polynomials and proved the following result.

**Theorem C** ([5]). Let $f$ be a nonconstant entire function and $L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)}$, where $a_1, a_2, \ldots, a_n (\neq 0)$ are constants, and $a (\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.

Lahiri and Kaish [3] improved Theorem B by considering a shared polynomial. They proved the following theorem.

**Theorem D** ([3]). Let $f$ be a nonconstant entire function and $a = a(z) (\neq 0)$ be a polynomial with $\deg(a) \neq \deg(f)$. Suppose that $A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap E(a; f^{(n+1)})\}$, where $\Delta$ denotes the symmetric difference of sets and $n (\geq 1)$ is an integer. If

1. $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,
2. $N_B(r, a; f^{(1)}) = S(r, f)$, and
3. each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

then $f = \lambda e^z$, where $\lambda (\neq 0)$ is a constant.

In Theorem D, Lahiri and Kaish considered an entire function which shares a polynomial with its derivatives. In our paper we improve Theorem D by considering an entire function which shares a polynomial with its linear differential polynomials.
The main result of the paper is the following theorem.

**Theorem 1.1.** Let \( f \) be a nonconstant entire function and \( L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)} \), where \( a_2, a_3, \ldots, a_n \ (\neq 0) \) are constants, and \( n \ (\geq 2) \) be an integer. Also let \( a(z) \ (\neq 0) \) be a polynomial with \( \deg(a) \neq \deg(f) \). Suppose that \( A = \mathcal{E}(a; f) \Delta \mathcal{E}(a; f^{(1)}) \) and \( B = \mathcal{E}(a; f^{(1)}) \setminus \{ \mathcal{E}(a; L) \cap \mathcal{E}(a; L^{(1)}) \} \). If

1. \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\} \),
2. \( N_B(r, a; f^{(1)}) = S(r, f) \), and
3. each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity,

then \( f = L = \lambda e^z \), where \( \lambda \ (\neq 0) \) is a constant.

In the theorem we assume that the degree of a transcendental entire function is infinity.

Putting \( A = B = \Phi \), we get the following corollary.

**Corollary 1.1.** Let \( f \) be a nonconstant entire function and \( a = a(z) \ (\neq 0) \) be a polynomial with \( \deg(a) \neq \deg(f) \). Also let \( L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)} \), where \( a_2, a_3, \ldots, a_n \ (\neq 0) \) are constants, and \( n \ (\geq 2) \) be an integer. If \( E(a; f) = E(a; f^{(1)}) \) and \( \mathcal{E}(a; f^{(1)}) \subset \{ \mathcal{E}(a; L) \cap \mathcal{E}(a; L^{(1)}) \} \), then \( f = L = \lambda e^z \), where \( \lambda \ (\neq 0) \) is a constant.

In Theorem C, Li considered the linear differential polynomial as \( L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_n f^{(n)} \), where \( a_1, a_2, \ldots, a_n \ (\geq 0) \) are constants. Here we consider the linear differential polynomial \( L \) with the first coefficient \( a_1 = 0 \). That is, we consider \( L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)} \). In Corollary 1.1 if we consider \( a = a(z) \) as a nonzero finite constant, then we get a particular case of Theorem C when \( L \) will be considered with the first coefficient zero. Therefore Corollary 1.1 shows that our result is an improvement of a particular case of Theorem C when \( L \) is considered with the first coefficient \( a_1 = 0 \).

2. **Lemmas**

In this section we present some necessary lemmas.

**Lemma 2.1** ([3]). Let \( f \) be transcendental entire function of finite order and \( a = a(z) \ (\neq 0) \) be a polynomial and \( A = \mathcal{E}(a; f) \Delta \mathcal{E}(a; f^{(1)}) \). If

1. \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\} \),
2. each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity,

then \( m(r, a; f) = m(r, (f - a)^{-1}) = S(r, f) \).
Lemma 2.2. Let $f$ be a transcendental entire function and $a(z) (\neq 0)$ be a polynomial. Also let $L = a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)}$ and $b(z) = a_2 a^{(2)} + a_3 a^{(3)} + \ldots + a_n a^{(n)}$, where $a_2, a_3, \ldots, a_n (\geq 0)$ are constants and $n (\geq 2)$ is an integer. Suppose $h = ((a - a^{(1)})(L - b) - (a - b)(f^{(1)} - a^{(1)})) (f - a)^{-1}$ and $A = E(a; f) \setminus E(a; f^{(1)})$, $B = E(a; f^{(1)}) \setminus \{E(a; L) \cap E(a; L^{(1)})\}$. If

1. $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f),$
2. each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,
3. $h$ is transcendental entire or meromorphic,

then $m(r, a; f^{(1)}) = m(r, (f^{(1)} - a)^{-1}) = S(r, f)$.

Proof. Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, if $z_0$ is a common zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q (\geq 2)$, then $z_0$ is a zero of $a - a^{(1)}$ with multiplicity $q - 1$. So

$$N(2, r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f),$$

where $N(2, r, a; f)$ is the counting function of multiple zeros of $f - a$.

Hence, by the hypothesis we see that

$$N(r, h) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(2, r, a; f) + S(r, f) = S(r, f).$$

Since $m(r, h) = S(r, f)$, we have $T(r, h) = S(r, f)$.

Now by a simple calculation we get

$$f = a + \frac{1}{h}((a - a^{(1)})(L - b) - (a - b)(f^{(1)} - a^{(1)}))$$

$$= a + \frac{1}{h}((a - a^{(1)})(L - a) - (a - b)(f^{(1)} - a)).$$

Differentiating we obtain

$$f^{(1)} = a^{(1)} + \left(\frac{1}{h}\right)^{(1)}((a - a^{(1)})(L - a) - (a - b)(f^{(1)} - a))$$

$$+ \frac{1}{h}((a - a^{(1)})(L^{(1)} - a^{(1)}) + (a^{(1)} - a^{(2)})(L - a)$$

$$- (a^{(1)} - b^{(1)})(f^{(1)} - a) - (a - b)(f^{(2)} - a^{(1)})).$$

This implies

$$(f^{(1)} - a) \left(1 + \left(\frac{1}{h}\right)^{(1)}(a - b) + \frac{1}{h}(a^{(1)} - b^{(1)})\right)$$

$$= a^{(1)} - a + \left(\left(\frac{1}{h}\right)^{(1)}(a - a^{(1)}) + \frac{1}{h}(a^{(1)} - a^{(2)})\right)(L - a)$$

$$+ \frac{1}{h}(a - a^{(1)})(L^{(1)} - a^{(1)}) - \frac{a - b}{h}(f^{(2)} - a^{(1)}).$$
\[
= \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(L-c) + \frac{a-a^{(1)}}{h}(L^{(1)}-c^{(1)})
\]
\[- \frac{a-b}{h}(f^{(2)}-a^{(1)}) + a^{(1)} - a + \left(\frac{(c-a)(a-a^{(1)})}{h}\right)^{(1)},
\]
where \(c(z) = a_2a^{(1)} + a_3a^{(2)} + \ldots + a_na^{(n-1)}\).

Therefore
\[
\left(1 + \left(\frac{a-b}{h}\right)^{(1)}\right)(f^{(1)}-a)
= a^{(1)} - a + \left(\frac{(c-a)(a-a^{(1)})}{h}\right)^{(1)} + \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(L-c)
\]
\[+ \frac{a-a^{(1)}}{h}(L^{(1)}-c^{(1)}) - \frac{a-b}{h}(f^{(2)}-a^{(1)}).
\]

This implies
\[
\frac{1}{f^{(1)}-a} = \frac{\mu}{\nu} - \frac{1}{\nu} \left(\frac{a-a^{(1)}}{h}\right)^{(1)} \frac{L-c}{f^{(1)}-a} - \frac{a-a^{(1)}}{h\nu} \frac{L^{(1)}-c^{(1)}}{f^{(1)}-a}
\]
\[+ \frac{a-b}{h\nu} \frac{f^{(2)}-a^{(1)}}{f^{(1)}-a},
\]
where \(\mu = 1 + ((a-b)h^{-1})^{(1)}\) and \(\nu = a^{(1)} - a + ((c-a)(a-a^{(1)})h^{-1})^{(1)}\).

We now verify that \(\mu \neq 0\) and \(\nu \neq 0\). If \(\mu \equiv 0\), then \(1 + ((a-b)h^{-1})^{(1)} \equiv 0\). Integrating we get \(h = (a-b)(c_1-z)^{-1}\), where \(c_1\) is a constant. This is a contradiction as \(h\) is transcendental. Therefore \(\mu \neq 0\).

If \(\nu \equiv 0\), then \((c-a)(a-a^{(1)})h^{-1})^{(1)} \equiv a-a^{(1)}\). Integrating we get \((c-a) \times (a-a^{(1)})h^{-1} = P(z)\), i.e. \(h = (c-a)(a-a^{(1)})/P(z)\), where \(P(z)\) is a polynomial. This is a contradiction because \(h\) is transcendental. Therefore \(\nu \neq 0\).

Again \(T(r, \mu) + T(r, \nu) = S(r, f)\). Therefore from (2.1) we get \(m(r, a; f^{(1)}) = m(r, (f^{(1)}-a)^{-1}) = S(r, f)\). This proves the lemma. \(\square\)

**Lemma 2.3** ([4], page 58). Each solution of the differential equation
\[
a_nf^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_0f = 0,
\]
where \(a_0 \neq 0\), \(a_1, \ldots, a_n \neq 0\) are polynomials, is an entire function of finite order.

**Lemma 2.4** ([4], page 47). Let \(f\) be a nonconstant meromorphic function and \(a_1, a_2, a_3\) be three distinct meromorphic functions satisfying \(T(r, a_\nu) = S(r, f)\) for \(\nu = 1, 2, 3\). Then
\[\]
\[T(r, f) \leq \overline{N}(r, 0; f-a_1) + \overline{N}(r, 0; f-a_2) + \overline{N}(r, 0; f-a_3) + S(r, f)\].
Lemma 2.5 ([6], page 92). Let \( f_1, f_2, \ldots, f_n \) be meromorphic functions which are nonconstant except possibly for \( f_n \), where \( n \geq 3 \). If \( f_n \neq 0 \) and \( \sum_{j=1}^{n} f_j \equiv 1 \) and
\[ \sum_{j=1}^{n} N(r, 0; f_j) + (n - 1) \sum_{j=1}^{n} N(r, \infty; f_j) < \{ \mu + o(1) \} T(r, f_k) \text{ for } k = 1, 2, \ldots, n - 1, \]
then \( f_n \equiv 1 \).

3. Proof of the theorem

First, we verify that \( f \) cannot be a polynomial. We suppose that \( f \) is a polynomial. Then \( T(r, f) = O(\log r) \) and \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = O(\log T(r, f)) = S(r, f) \) imply \( A = \Phi \). Also \( N_B(r, a; f^{(1)}) = S(r, f) \) implies \( B = \Phi \). Therefore \( E(a; f) = E(a; f^{(1)}) \) and \( E(a; f^{(1)}) \subset E(a, L) \cap E(a; L^{(1)}) \).

Let \( \deg(f) = m \) and \( \deg(a) = p \). If \( m \geq p + 1 \), then \( \deg(f - a) = m, \deg(f^{(1)} - a) \leq m - 1 \). Since each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity, it contradicts the fact that \( E(a; f) = E(a; f^{(1)}) \).

Next let \( m \leq p - 1 \). Then \( \deg(f - a) = p, \deg(f^{(1)} - a) = p \). Again \( E(a; f) = E(a; f^{(1)}) \), we can write \( f^{(1)} - a \equiv (f - a)k \), where \( k \geq 0 \) is a constant.

If \( k \neq 1 \), then \( kf - f^{(1)} \equiv (k - 1)a \), which is impossible as \( \deg((k - 1)a) = p > m = \deg(kf - f^{(1)}) \).

If \( k = 1 \), then \( f = f^{(1)} \), which is again a contradiction. Therefore \( f \) is a transcendental entire function.

Since \( a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a) \), a common zero of \( f - a \) and \( f^{(1)} - a \) of multiplicity \( q \geq 2 \) is a zero of \( a - a^{(1)} \) with multiplicity \( q - 1 \geq 1 \). Therefore \( N_2(r, a; f^{(1)} | f = a) \leq 2N(r, 0; a-a^{(1)}) = S(r, f) \), where \( N_2(r, a; f^{(1)} | f = a) \) denotes the counting function (counted with multiplicities) of those multiple zeros of \( f^{(1)} - a \), which are also zeros of \( f - a \).

Now
\[ (3.1) \quad N_2(r, a; f^{(1)}) \leq N_A(r, a; f^{(1)}) + N_B(r, a; f^{(1)}) \]
\[ + N_2(r, a; f^{(1)} | f = a) + S(r, f) = S(r, f). \]

First we suppose that \( L^{(1)} \neq f^{(1)} \). Then using (3.1) we get by the hypothesis
\[ (3.2) \quad N(r, a; f^{(1)}) \leq N_B(r, a; f^{(1)}) + N\left(r, \frac{a - b^{(1)}}{a - a^{(1)}}; \frac{L^{(1)} - b^{(1)}}{f^{(1)} - a^{(1)}} \right) + S(r, f) \]
\[ \leq T\left(r, \frac{L^{(1)} - b^{(1)}}{f^{(1)} - a^{(1)}} \right) + S(r, f) = N\left(r, \frac{L^{(1)} - b^{(1)}}{f^{(1)} - a^{(1)}} \right) + S(r, f) \]
\[ \leq N(r, a^{(1)}; f^{(1)}) + S(r, f), \]
where \( b(z) = a_2a^{(2)}(z) + a_3a^{(3)}(z) + \ldots + a_n a^{(n)}(z) \).
Again

$$m(r, a; f) \leq m\left(r, \frac{f^{(1)} - a^{(1)}}{f - a}; \frac{1}{f^{(1)} - a^{(1)}}\right)$$

$$\leq m(r, a^{(1)}; f^{(1)}) + S(r, f)$$

$$= T(r, f^{(1)}) - N(r, a^{(1)}; f^{(1)}) + S(r, f)$$

$$= m(r, f^{(1)}) - N(r, a^{(1)}; f^{(1)}) + S(r, f)$$

$$\leq m(r, f) - N(r, a^{(1)}; f^{(1)}) + S(r, f)$$

$$= T(r, f) - N(r, a^{(1)}; f^{(1)}) + S(r, f),$$

i.e. $N(r, a^{(1)}; f^{(1)}) \leq N(r, a; f) + S(r, f)$.

Therefore from (3.2) we get

(3.3) \hspace{1cm} N(r, a; f^{(1)}) \leq N(r, a; f) + S(r, f).

Again

(3.4) \hspace{1cm} N(r, a; f) \leq N_A(r, a; f) + N(r, a; f^{(1)}|f = a) \leq N(r, a; f^{(1)}) + S(r, f).

Therefore from (3.3) and (3.4) we get

(3.5) \hspace{1cm} N(r, a; f^{(1)}) = N(r, a; f) + S(r, f).

Let $h = ((a - a^{(1)})(L - b) - (a - b)(f^{(1)} - a^{(1)}))(f - a)^{-1}$ be transcendental. Then

$$T(r, f) = m(r, f) \leq m\left(r, \frac{1}{h}((a - a^{(1)})(L - (a - b)f^{(1)}))\right) + S(r, f)$$

$$\leq m(r, f^{(1)}) + m\left(r, (a - a^{(1)})\frac{L}{f^{(1)}} - (a - b)\right) + S(r, f)$$

$$\leq m(r, f^{(1)}) + S(r, f) = T(r, f^{(1)}) + S(r, f)$$

$$= m(r, f^{(1)}) + S(r, f) \leq m(r, f) + S(r, f)$$

$$= T(r, f) + S(r, f).$$

Therefore

(3.6) \hspace{1cm} T(r, f^{(1)}) = T(r, f) + S(r, f).

Again by Lemma 2.2 we get $m(r, a; f^{(1)}) = S(r, f)$. Then from (3.5) and (3.6) we get $m(r, a; f) = S(r, f)$. Therefore

(3.7) \hspace{1cm} m(r, a; f) + m(r, a; f^{(1)}) = S(r, f).
Next we suppose that \( h \) is rational. Then by Lemma 2.3 we see that \( f \) is of finite order and by Lemma 2.1 we get \( m(r, a; f) = S(r, f) \). Since

\[
T(r, f^{(1)}) = m(r, f^{(1)}) \leq m(r, f) + S(r, f) = T(r, f) + S(r, f)
\]

and from (3.5) we get \( m(r, a; f^{(1)}) \leq m(r, a; f) + S(r, f) = S(r, f) \). Hence in this case also we obtain (3.7).

Let \( \xi = (f^{(1)} - a)(f - a)^{-1} \) and \( \eta = (L - a)(f^{(1)} - a)^{-1} \). Then by (3.7) we get \( m(r, \xi) + m(r, \eta) = S(r, f) \). Also \( N(r, \xi) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_2(r, a; f) + S(r, f) = S(r, f) \) because \( N_2(r, a; f) \leq N_A(r, a; f) + 2N(r, 0; a - a^{(1)}) + S(r, f) = S(r, f) \).

Using (3.2) we get

\[
N(r, \eta) \leq N_A(r, a; f^{(1)}) + N_B(r, a; f^{(1)}) + N_2(r, a; f^{(1)}) + S(r, f) = S(r, f).
\]

Therefore

\[
(3.8) \quad T(r, \xi) + T(r, \eta) = S(r, f).
\]

Let \( z_1 \) be a simple zero of \( f - a \) such that \( z_1 \notin A \cup B \) and \( a(z_1) - a^{(1)}(z_1) \neq 0 \). Then by Taylor’s expansion in some neighbourhood of \( z_1 \) we get

\[
\begin{align*}
f(z) - a(z) &= (a(z_1) - a^{(1)}(z_1))(z - z_1) + O(z - z_1)^2, \\
f^{(1)}(z) - a(z) &= (f^{(2)}(z_1) - a^{(1)}(z_1))(z - z_1) + O(z - z_1)^2,
\end{align*}
\]

and

\[
L(z) - a(z) = (a(z_1) - a^{(1)}(z_1))(z - z_1) + O(z - z_1)^2.
\]

Therefore in some neighbourhood of \( z_1 \) we get

\[
(3.9) \quad \xi(z) = \frac{f^{(2)}(z_1) - a^{(1)}(z_1)}{a(z_1) - a^{(1)}(z_1)} + O(z - z_1),
\]

and

\[
(3.10) \quad \eta(z) = \frac{a(z_1) - a^{(1)}(z_1)}{f^{(2)}(z_1) - a^{(1)}(z_1)} + O(z - z_1).
\]

We put \( \chi = \eta - \xi^{-1} \). Then from (3.8) we get \( T(r, \chi) \leq T(r, \eta) + T(r, \xi) + S(r, f) = S(r, f) \).
Also in some neighbourhood of $z_1$ we have by (3.9) and (3.10),

$$
\chi(z) = \eta(z) - \frac{1}{\xi(z)}
= \frac{a(z_1) - a^{(1)}(z_1)}{f^{(2)}(z_1) - a^{(1)}(z_1)} + O(z - z_1) - \left( \frac{f^{(2)}(z_1) - a^{(1)}(z_1)}{a(z_1) - a^{(1)}(z_1)} + O(z-z_1) \right)^{-1}
= \frac{a(z_1) - a^{(1)}(z_1)}{f^{(2)}(z_1) - a^{(1)}(z_1)} + O(z - z_1) - \left( \frac{a(z_1) - a^{(1)}(z_1)}{f^{(2)}(z_1) - a^{(1)}(z_1)} + O(z-z_1) \right)
= O(z - z_1).
$$

If $\chi \not\equiv 0$, then

$$
N(r, a; f) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(2r, a; f) + N(r, 0; a - a^{(1)}) + N(r, 0; \chi)
= S(r, f),
$$
and so by (3.7) we get $T(r, f) = S(r, f)$, a contradiction.
Therefore $\chi \equiv 0$ and so

$$
(3.11) \quad L \equiv f.
$$

Differentiating (3.11) we get $L^{(1)} \equiv f^{(1)}$, which contradicts our hypothesis that $L^{(1)} \not\equiv f^{(1)}$. Therefore, indeed we have $L^{(1)} \equiv f^{(1)}$.

Next we suppose that $L^{(1)} \not\equiv L$. Then by the hypothesis and (3.1) we get

$$
(3.12) \quad N(r, a; f^{(1)}) \leq N_B(r, a; f^{(1)}) + N \left( r, \frac{a - b^{(1)}}{a-b} ; \frac{L^{(1)} - b^{(1)}}{L-b} \right) + S(r, f)
\leq T \left( r, \frac{L^{(1)} - b^{(1)}}{L-b} \right) + S(r, f) = N \left( r, \frac{L^{(1)} - b^{(1)}}{L-b} \right) + S(r, f)
= \overline{N}(r, b; L) + S(r, f).
$$

Again

$$
m(r, a; f) = m \left( r, \frac{L-b}{f-a} \right) \leq m(r, b; L) + S(r, f)
= T(r, L) - N(r, b; L) + S(r, f) = m(r, L) - N(r, b; L) + S(r, f)
\leq m \left( r, \frac{L}{f} \right) + m(r, f) - N(r, b; L) + S(r, f)
= m(r, f) - N(r, b; L) + S(r, f) = T(r, f) - N(r, b; L) + S(r, f)
$$
and so $N(r, b; L) \leq N(r, a; f) + S(r, f)$. Now by (3.12) we get $N(r, a; f^{(1)}) \leq N(r, a; f) + S(r, f)$. Now by (3.12) we get $N(r, a; f^{(1)}) \leq N(r, a; f) + S(r, f)$. Now by (3.12) we get $N(r, a; f^{(1)}) \leq N(r, a; f) + S(r, f)$.
Also

\[ N(r, a; f) \leq N_A(r, a; f) + N(r, a; \{f \mid f = a\}) \leq N(r, a; f^{(1)}) + S(r, f). \]

Therefore \( N(r, a; f^{(1)}) = N(r, a; f) + S(r, f) \), which is (3.5).

Now using Lemma 2.1, Lemma 2.2, Lemma 2.3 and (3.5) we similarly obtain (3.7).

Using \( \xi \) and \( \eta \) and proceeding likewise we get (3.11), which implies \( L \equiv f \) or \( a_2 f^{(2)} + a_3 f^{(3)} + \ldots + a_n f^{(n)} - f \equiv 0 \). Solving this we get

\[ f = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + \ldots + p_t e^{\alpha_t z}, \tag{3.13} \]

where \( \alpha_1, \alpha_2, \ldots, \alpha_t \) are the roots of \( a_2 \zeta^2 + a_3 \zeta^3 + \ldots + a_n \zeta^n - 1 = 0 \) and \( p_1, p_2, \ldots, p_t \)

are constants or polynomials, not all identically zero and \( t \ (\leq n) \) is an integer.

Differentiating (3.13) we get

\[ f^{(1)} = \sum_{i=1}^{t} (p_i^{(1)} + p_i \alpha_i) e^{\alpha_i z}. \tag{3.14} \]

Now from (3.13), (3.14) and \( \xi = (f^{(1)} - a)(f - a)^{-1} \) we get

\[ \sum_{i=1}^{t} (\xi p_i - p_i^{(1)} - p_i \alpha_i) e^{\alpha_i z} \equiv a(\xi - 1). \tag{3.15} \]

We suppose that \( \xi \neq 1 \). Then from (3.15) we get

\[ \sum_{i=1}^{t} \frac{\xi p_i - p_i^{(1)} - p_i \alpha_i}{a(\xi - 1)} e^{\alpha_i z} \equiv 1. \tag{3.16} \]

Here \( T(r, f) = O(T(r, e^{\alpha_k z})) \) for \( i = 1, 2, \ldots, t \).

First we suppose that the left-hand side of (3.16) contains only one term, say,

\[ \frac{\xi p_k - p_k^{(1)} - p_k \alpha_k}{a(\xi - 1)} e^{\alpha_k z} \equiv 1. \]

Then \( T(r, e^{\alpha_k z}) = S(r, f) = S(r, e^{\alpha_k z}), \) a contradiction.

Next we suppose that the left-hand side of (3.16) contains only two terms, say,

\[ \frac{\xi p_k - p_k^{(1)} - p_k \alpha_k}{a(\xi - 1)} e^{\alpha_k z} + \frac{\xi p_l - p_l^{(1)} - p_l \alpha_l}{a(\xi - 1)} e^{\alpha_l z} \equiv 1. \]
So by Lemma 2.4 we get from above

\[
T(r, e^{\alpha z}) \leq \overline{N}(r, 0; e^{\alpha z}) + \overline{N}(r, \infty; e^{\alpha z}) + \overline{N}\left(r, \frac{a(\xi - 1)}{\xi p_k - p_k^{(1)} - p_k\alpha_k}; e^{\alpha z}\right) + S(r, e^{\alpha z}),
\]

\[
= \overline{N}(r, 0; e^{\alpha z}) + S(r, e^{\alpha z}) = S(r, e^{\alpha z}),
\]

a contradiction.

Finally we suppose that the left-hand side of (3.16) contains more than two terms, then by Lemma 2.5 we get

\[
(3.17) \xi p_i - p_i^{(1)} - p_i\alpha_i \equiv e^{\alpha z}
\]

for one value of \(i \in \{1, 2, \ldots, t\}\).

From (3.17) we see that \(T(r, e^{\alpha z}) = S(r, f) = S(r, e^{\alpha z})\), a contradiction. Therefore \(\xi \equiv 1\) and so \(f^{(1)} \equiv f\). Hence, from \(L \equiv f\) we get \(L \equiv L^{(1)}\), a contradiction to the supposition. Therefore, indeed we have \(L \equiv L^{(1)}\).

Now \(L \equiv L^{(1)} \equiv f^{(1)}\) implies \(L = L^{(1)} = f^{(1)} = \lambda e^z\), where \(\lambda (\geq 0)\) is a constant. Therefore \(f = \lambda e^z + K\), where \(K\) is a constant.

By Lemma 2.4 we get

\[
(3.18) \quad T(r, \lambda e^z) \leq \overline{N}(r, 0; \lambda e^z) + \overline{N}(r, \infty; \lambda e^z) + \overline{N}(r, a - K; \lambda e^z) + S(r, \lambda e^z)
\]

\[
= \overline{N}(r, a; f) + S(r, \lambda e^z).
\]

If \(\overline{N}(r, a; f) = S(r, f)\), then from (3.18) we get \(T(r, \lambda e^z) = S(r, \lambda e^z)\), which is a contradiction. Therefore \(\overline{N}(r, a; f) \neq S(r, f)\).

Again

\[
(3.19) \quad \overline{N}(r, a; f) \leq N_A(r, a; f) + N(r, a; f|f^{(1)} = a).
\]

Since \(N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}\), from (3.19) we must have \(\overline{E}(a; f) \cap \overline{E}(a; f^{(1)}) \neq \Phi\), otherwise \(\overline{N}(r, a; f) = S(r, f)\).

Let \(z_3 \in \overline{E}(a; f) \cap \overline{E}(a; f^{(1)})\). Then \(f(z_3) = f^{(1)}(z_3)\) and then \(f(z) = f^{(1)}(z) + K\) implies \(K = 0\). Therefore \(f = L = \lambda e^z\). This proves the theorem.

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References


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