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ON THE DIOPHANTINE EQUATION $\sum_{j=1}^{k} jF_j^p = F_n^q$

Gökhan Soydan, László Németh, and László Szalay

Abstract. Let $F_n$ denote the $n^{th}$ term of the Fibonacci sequence. In this paper, we investigate the Diophantine equation $F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q$ in the positive integers $k$ and $n$, where $p$ and $q$ are given positive integers. A complete solution is given if the exponents are included in the set $\{1,2\}$. Based on the specific cases we could solve, and a computer search with $p, q, k \leq 100$ we conjecture that beside the trivial solutions only $F_8 = F_1 + 2F_2 + 3F_3 + 4F_4$, $F_4 = F_1 + 2F_2 + 3F_3$, and $F_4^3 = F_1^3 + 2F_2^3 + 3F_3^3$ satisfy the title equation.

1. Introduction

As usual, let $(F_n)_{n \geq 0}$ and $(L_n)_{n \geq 0}$ denote the sequences of Fibonacci and Lucas numbers, respectively, given by the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, and by the recurrence relations

\[(1) \quad F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \quad \text{for all} \quad n \geq 0,\]

respectively. Putting $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$ for the two roots of the common characteristic equation $x^2 - x - 1 = 0$ of the two sequences, the formulae

\[F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n\]

hold for all $n \geq 0$. These numbers are well-known for possessing amazing and wonderful properties (consult, for instance, [13] and [5] together with their very rich annotated bibliography for history and additional references). Observing

\[F_1 = F_2, \quad F_1 + 2F_2 = F_4, \quad F_1 + 2F_2 + 3F_3 = F_4^2, \quad F_1 + 2F_2 + 3F_3 + 4F_4 = F_8,\]

the question arises naturally: is there any rule for $F_1 + 2F_2 + 3F_3 + \cdots + kF_k$? We study this question more generally, according to the title equation.
Diophantine equations among the terms of Fibonacci numbers have a very extensive literature. Here we quote a few results that partially motivated us.

By the defining equality \([1]\) of the Fibonacci numbers and the identity \(F_n^2 + F_{n+1}^2 = F_{2n+1}\) (Lemma 1.8), we see that \(F_n^s + F_{n+1}^s\) \((n \geq 0)\) is a Fibonacci number for \(s \in \{1, 2\}\). For larger \(s\) Marques and Togbé \([8]\) proved in 2010 that if \(F_n^s + F_{n+1}^s\) is a Fibonacci number for all sufficiently large \(n\) then \(s = 1\) or \(2\). Next year Luca and Oyono \([6]\) completed the solution of the question by showing that apart from \(F_1^s + F_2^s = F_3\) there is no solution \(s \geq 3\) to the equation \(F_n^s + F_{n+1}^s = F_m\).

Let \(l, s_1, \ldots, s_t, a_1, \ldots, a_t\) be integers with \(l \geq 1\) and \(s_j \geq 1\). Suppose that there exists \(1 \leq t \leq l\) such that \(a_t \neq 0\) and \(s_t > s_j\), for all \(j \neq t\). Chaves, Marques and Togbé \([1]\), showed that if either \(s_t\) is even or \(a_t\) is not a positive power of 5, then the sum

\[
a_1 F_{n+1}^{s_1} + a_2 F_{n+2}^{s_2} + \cdots + a_t F_{n+t}^{s_t}
\]

does not belong to the Fibonacci sequence for all sufficiently large \(n\).

A balancing problem having similar flavor has been considered by Behera et al. \([3]\). They studied the equation

\[
F_1^p + F_2^p + \cdots + F_{k-1}^p = F_{k+1}^q + \cdots + F_{k+r}^q,
\]

and solved it for the cases \((p, q) = (2, 1), (3, 1), (3, 2)\), and for \(2 \leq p \leq q\) by showing the non-existence of any solution. Further the authors conjectured that only the quadruple \((k, r, p, q) = (4, 3, 8, 2)\) of positive integers satisfies \([2]\). The conjecture was completely justified by Alvarado et al. \([1]\). Note that if \((p, q) = (1, 1)\) we obtain the problem of sequence balancing numbers handled by Panda \([9]\).

Recalling the formulae \(F_1 + F_2 + \cdots + F_k = F_{k+2} - 1\) and \(F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}\), it is obvious that the problems

\[
F_1 + F_2 + \cdots + F_k = F_n^q, \quad \text{and} \quad F_1^2 + F_2^2 + \cdots + F_k^2 = F_n^q
\]

are rather simple. Indeed, the equations above lead to the lightsome ones

\[
F_{k+2} - 1 = F_n^q, \quad F_k F_{k+1} = F_n^q.
\]

However the equation \(F_1^p + F_2^p + \cdots + F_k^p = F_n^q\) might be taken an interest if \(p \geq 3\).

The last motivation of our examination was the Diophantine equation

\[
x^2 + 2(x+1)^2 + \cdots + n(x+n-1)^2 = y^2
\]

to determine the values of \(n\) for which it has finitely or infinitely many positive integer solutions \((x, y)\) (see Wulczyn \([14]\), and for details, see also \([2]\)). For variations of the equation \([3]\), we refer the reader to \([12]\).

In this paper, we investigate the Diophantine equation

\[
F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q
\]

in the positive integers \(k\) and \(n\), where \(p\) and \(q\) are fixed positive integers. We consider

\[
F_1^p = 1 = F_1^q = F_2^q, \quad \text{and} \quad F_1^p + 2F_2^p = 3 = F_4
\]

as trivial solutions to \((4)\). We have the following conjecture based upon the specific cases we could solve, and a computer search with \(p, q, k \leq 100\).
Conjecture 1. The non-trivial solutions to (4) are only
\[ F_4^2 = 9 = F_1 + 2F_2 + 3F_3, \]
\[ F_8 = 21 = F_1 + 2F_2 + 3F_3 + 4F_4, \]
\[ F_4^3 = 27 = F_1^3 + 2F_2^3 + 3F_3^3. \]

This work handles the particular cases \( p, q \in \{1, 2\} \) (hence the first two solutions above will be obtained), the precise results proved are described as follow.

Theorem 1. If
\[ F_1 + 2F_2 + \cdots + kF_k = F_n, \]
then \((k,n) = (1,1), (1,2), (2,4), (4,8), \) among them only the last one is non-trivial solution.

Theorem 2. The Diophantine equation
\[ F_1^2 + 2F_2^2 + \cdots + kF_k^2 = F_n^2 \]
possesses only the trivial solutions \((k,n) = (1,1), (1,2).\)

Theorem 3. If
\[ F_1 + 2F_2 + \cdots + kF_k = F_n^2, \]
then \((k,n) = (1,1), (1,2), (3,4), \) among them only the last one is non-trivial solution.

Theorem 4. The Diophantine equation
\[ F_1^2 + 2F_2^2 + \cdots + kF_k^2 = F_n \]
possesses only the trivial solutions \((k,n) = (1,1), (1,2), (2,4).\)

2. Lemmata

In this section, we present the lemmata that are needed in the proofs of the theorems. The first lemma is a collection of a few well-known results, we state them without proof, and in the proof of the theorems sometimes we do not refer to them.

Lemma 1. Let \( k \) and \( n \) be arbitrary integers.

(i) \[ \sum_{j=1}^{k} jF_j = kF_{k+2} - F_{k+3} + 2. \]

(ii) \[ \sum_{j=1}^{k} jF_j^2 = F_k(kF_{k+1} - F_k) + \tau, \] where \( \tau = 0 \) if \( k \) is even, and \( \tau = 1 \) otherwise.

(iii) For \( k \geq 0 \) we have \( F_{-k} = (-1)^{k+1}F_k, \) further \( L_{-k} = (-1)^kL_k \) (extension of the sequences for negative subscripts).

(iv) \( \gcd(F_k, F_n) = F_{\gcd(k,n)}. \)

(v) \( \gcd(F_k, L_n) = 1 \) or 2 or \( L_{\gcd(k,n)}. \)

(vi) \( F_k \mid F_n \) if and only if \( k \mid n. \)

(vii) \( F_{k+1}F_n - F_kF_{n+1} = (-1)^{n+1}F_{k-n} \) (d’ Ocagne’s identity).
(viii) $F_{k+n} = F_k F_{n+1} + F_{k-1} F_n$.
(ix) $F_{2k} = F_k L_k$.
(x) $F_{k+n}^2 - F_{k-n}^2 = F_{2k} L_{2n}$.

**Lemma 2.**

$$F_k - 1 = \begin{cases} F_{(k+2)/2} L_{(k-2)/2}, & \text{if } k \equiv 0 \pmod{4}, \\ F_{(k-1)/2} L_{(k+1)/2}, & \text{if } k \equiv 1 \pmod{4}, \\ F_{(k-2)/2} L_{(k+2)/2}, & \text{if } k \equiv 2 \pmod{4}, \\ F_{(k+1)/2} L_{(k-1)/2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

$$F_k + 1 = \begin{cases} F_{(k-2)/2} L_{(k+2)/2}, & \text{if } k \equiv 0 \pmod{4}, \\ F_{(k+1)/2} L_{(k-1)/2}, & \text{if } k \equiv 1 \pmod{4}, \\ F_{(k+2)/2} L_{(k-2)/2}, & \text{if } k \equiv 2 \pmod{4}, \\ F_{(k-1)/2} L_{(k+1)/2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** See, for instance, [7] and [11].

**Lemma 3.**

$$F_k^2 - 1 = \begin{cases} F_{k-1} F_{k+1}, & \text{if } k \equiv 1 \pmod{2}, \\ F_{k-2} F_{k+2}, & \text{if } k \equiv 0 \pmod{2}. \end{cases}$$

$$F_k^2 + 1 = \begin{cases} F_{k-1} F_{k+1}, & \text{if } k \equiv 0 \pmod{2}, \\ F_{k-2} F_{k+2}, & \text{if } k \equiv 1 \pmod{2}. \end{cases}$$

**Proof.** See Lemma 3 in [10].

**Lemma 4.** If $j \geq 4$ is even, then

$$2F_{j-1}^{\varphi(F_j)} - 1 \equiv F_{j-3} \pmod{F_j}.$$

**Proof.** Since $\gcd(F_{j-1}, F_j) = 1$, and $F_{j-1}^{\varphi(F_j)} \equiv 1 \pmod{F_j}$, it is sufficient to show that

$$2 \equiv F_{j-3} F_{j-1} \pmod{F_j}.$$ But

$$F_{j-1} F_{j-3} = (F_{j+1} - F_j) F_{j-3} \equiv F_{j+1} F_{j-3} = F_j F_{j-2} + (-1)^{j-2} F_3 \equiv 2 \pmod{F_j}$$

follows from the definition of the Fibonacci numbers, d’ Ocagne’s identity (Lemma [11]), and the parity of $j$.

**Lemma 5.** If $j \geq 3$ is odd, then

$$F_{j-1}^{\varphi(F_j)} - 1 \equiv F_{j-2} \pmod{F_j}.$$

**Proof.** Similarly to the proof of the previous lemma, the statement is equivalent to

$$1 \equiv F_{j-2} F_{j-1} \pmod{F_j}.$$ And it is easy to see that

$$F_{j-2} F_{j-1} = (F_j - F_{j-1}) F_{j-1} \equiv - F_{j-1}^2 = -(F_{j-2} F_j + (-1)^j F_{j-1}) \equiv 1 \pmod{F_j}.$$
Lemma 6. Let $k_0$ be a positive integer, and for $i \in \{0,1\}$ put
\[
\delta_i = \log_\alpha \left( \frac{1 + (-1)^{i-1} (|\beta|/\alpha)^{k_0}}{\sqrt{5}} \right),
\]
where $\log_\alpha$ is the logarithm in base $\alpha = (1 + \sqrt{5})/2$. Then for all integers $k \geq k_0$, the two inequalities
\[
\alpha^{k+\delta_0} \leq F_k \leq \alpha^{k+\delta_1}
\]
hold.

Proof. This is a part of Lemma 5 in [7].

In order to make the application of Lemma 6 more convenient, we shall suppose that $k_0 \geq 1$. Then we have

Corollary 5. If $k \geq 1$, then
\[
\alpha^{k-2} \leq F_k \leq \alpha^{k-1},
\]
and equality holds if and only if $k = 2$, and $k = 1$, respectively.

Now, we are ready to justify the theorems.

3. Proofs

Proof of Theorem [1]
Verifying the cases $k = 1, \ldots, 5$ by hand we found the solutions listed in Theorem [1]. Put $\kappa = k + 2$, and suppose that $\kappa \geq 8$. Consequently, $F_{\kappa-3} \geq 5$ and $F_\kappa \geq 21$. If equation [5] holds, then $n > \kappa$, and then by Lemma [1] we conclude

(9) \[ k = \frac{F_n + F_{\kappa+1} - 2}{F_\kappa} = \frac{F_n + F_{\kappa-1} - 2}{F_\kappa} + 1 \in \mathbb{N}. \]

In the sequel, we study the sequence $(F_u)_{u=0}^\infty$ modulo $F_\kappa$ if $\kappa$ is fixed. Note that we indicate a suitable value congruent to $F_u$ modulo $F_\kappa$, not always the smallest non-negative remainder. The period can be deduced from the range

\[
0, 1, 2, \ldots, F_{\kappa-2}, F_{\kappa-1}, 0, F_{\kappa-1}, F_{\kappa-1}, 2F_{\kappa-1}, \ldots, F_{\kappa-2}F_{\kappa-1}, F_{\kappa-1}F_{\kappa-1},
\]
of length $2\kappa$ if $\kappa$ is even, since then, by Lemma 1 we have $F_{\kappa-1}^2 \equiv 1$ (mod $F_\kappa$) and then
\[
F_{\kappa-2}F_{\kappa-1} = (F_\kappa - F_{\kappa-1})F_{\kappa-1} \equiv -1 \pmod{F_\kappa}.
\]
In case of odd \( \kappa \) we have \( F_{\kappa-1}^2 \equiv -1 \pmod{F_{\kappa}} \), therefore the length of the period is \( 4\kappa \) coming from

\[
0, 1, 1, 2, \ldots, F_{\kappa-2}, F_{\kappa-1}, 0, F_{\kappa-1}, F_{\kappa-1}, 2F_{\kappa-1}, \ldots, F_{\kappa-2}F_{\kappa-1}, F_{\kappa-1}F_{\kappa-1}, \]

\[
0, -1, -1, -2, \ldots, -F_{\kappa-2}, -F_{\kappa-1},
\]

\[
0, -F_{\kappa-1}, -F_{\kappa-1}, -2F_{\kappa-1}, \ldots, -F_{\kappa-2}F_{\kappa-1}, -F_{\kappa-1}F_{\kappa-1}.
\]

Based on the length of the period we distinguish two cases.

**Case I: \( \kappa \) is even.** Either \( F_n \equiv F_j \) or \( F_n \equiv F_j F_{\kappa-1} \) modulo \( F_\kappa \) holds for some \( j = 0, 1, \ldots, \kappa - 1 \). Hence

\[
F_n + F_{\kappa-1} - 2 \equiv \begin{cases} F_j + F_{\kappa-1} - 2, & \text{or} \\ F_j F_{\kappa-1} + F_{\kappa-1} - 2 \end{cases} \pmod{F_\kappa}.
\]

We will show that none of them is congruent to 0 modulo \( F_\kappa \). In the first branch

\[
F_j + F_{\kappa-1} - 2 \geq F_{\kappa-1} - 2 \geq 11,
\]

further if \( j \neq \kappa - 1 \), then

\[
F_j + F_{\kappa-1} - 2 \leq F_{\kappa-2} + F_{\kappa-1} - 2 \leq F_{\kappa} - 2.
\]

Thus \( F_j + F_{\kappa-1} - 2 \not\equiv 0 \pmod{F_\kappa} \), hence \( (9) \) does not hold. Assume now, that \( j = \kappa - 1 \). Then, together with the definition of the Fibonacci sequence we have

\[
F_j + F_{\kappa-1} - 2 = F_{\kappa-1} + (F_\kappa - F_{\kappa-2}) - 2 \equiv F_{\kappa-3} - 2 \pmod{F_\kappa}.
\]

But \( 3 \leq F_{\kappa-3} - 2 < F_\kappa \) contradicts to \( (9) \).

Choosing the second branch of \( (10) \), suppose that \( F_{\kappa-1}(F_j + 1) - 2 \) is congruent to 0 modulo \( F_\kappa \). Then

\[
F_{\kappa-1}^{\varphi(F_\kappa)}(F_j + 1) \equiv 2F_{\kappa-1}^{\varphi(F_\kappa) - 1} \pmod{F_\kappa}.
\]

Subsequently, by Lemma \( 4 \) it leads to

\[
F_j + 1 \equiv F_{\kappa-3} \pmod{F_\kappa}.
\]

Since \( j = 0, 1, \ldots, \kappa - 1 \), \( (\kappa \geq 8) \) it follows that \( F_j = F_{\kappa-3} - 1 \), a contradiction.

**Case II: \( \kappa \) is odd.** Now \( \kappa \geq 9 \), and either \( F_n \equiv \pm F_j \pmod{F_\kappa} \) or \( F_n \equiv \pm F_j F_{\kappa-1} \pmod{F_\kappa} \) holds for some \( j = 0, 1, \ldots, \kappa - 1 \). Hence

\[
F_n + F_{\kappa-1} - 2 \equiv \begin{cases} \pm F_j + F_{\kappa-1} - 2 \\
\pm F_j F_{\kappa-1} + F_{\kappa-1} - 2 \end{cases} \pmod{F_\kappa}.
\]

First, obviously, if \( j \neq \kappa - 1 \), then

\[
6 \leq F_{\kappa-3} - 2 \leq \pm F_j + F_{\kappa-1} - 2 \leq F_{\kappa} - 2,
\]
so dividing $±F_j + F_{κ−1} − 2$ by $F_κ$, the result is not an integer. If $j = κ − 1$, then the treatment of the “+” case coincides the treatment when $κ$ was even. The “−” case leads to $F_n + F_κ = 2 ≡ −2 (mod F_κ)$, a contradiction.

Assume now that $F_n + F_κ = 2 ≡ ±F_j F_κ + F_κ − 2 (mod F_n)$. Thus $F_κ(1 ± F_j) ≡ 2 (mod F_n)$. Multiplying both sides by $F_κ^{2−(F_n−1)}$, by Lemma 5 it gives

$$1 ± F_j ≡ 2F_κ (mod F_κ).$$

First let $F_j = 2F_κ−2 − 1$, which leads immediately a contradiction via $0 < 2F_κ−2 − 1 = F_κ−1 + F_κ−4 − 1 < F_κ$. If $F_κ − F_j + 1 = 2F_κ−2$, then $F_j = F_κ−3 + 1$ follows, a contradiction again. The proof of Theorem 1 is complete.

**Proof of Theorem 2**

For the range $k = 1, 2, . . . , 20$ we checked (6) by hand. From now we assume $k ≥ 21$. Based on Lemma 1 (1), we must distinguish two cases.

**Case I: k is even.** Consider the equation

$$F_k(kF_{k+1} − F_k) = F_n^2.$$  

Trivially, $n > k$. Put $ν = gcd(k, n)$.

If $ν = k$, then $F_k | F_n$ by Lemma 1 (1). Consequently,

$$
\left( \frac{F_n}{F_k} \right)^2 = \frac{kF_{k+1} − F_k}{F_k} = \frac{kF_{k+1}}{F_k} − 1
$$

is integer. But $F_k$ and $F_{k+1}$ are coprime, hence $F_k | k$, and it results $k ≤ 5$, a contradiction.

Examine the possibility $ν = k/2$. Put $κ = k/2$. Now $F_κ L_κ(kF_{κ+1} − F_κ L_κ) = F_n^2$ leads to

$$\frac{L_κ(kF_{κ+1} − F_κ L_κ)}{F_κ} = \left( \frac{F_n}{F_κ} \right)^2.$$ 

This is an equality of integers, which together with $gcd(F_κ, F_{κ+1})$ and $gcd(F_κ, L_κ) = 1, 2$ (see Lemma 1 (1)) shows that $2k/F_κ$ is integer. Thus $k ≤ 14$, a contradiction.

Finally, we have $3 ≤ ν ≤ k/3$. Since $gcd(F_k/F_ν, F_n/F_ν) = 1$, then from the equation

$$\frac{F_k}{F_ν} (kF_{k+1} − F_k) = \frac{F_n}{F_ν} F_n$$

we conclude

$$\frac{F_k}{F_ν} | F_n$$  and  $$\frac{F_n}{F_ν} | kF_{k+1} − F_k.$$ 

Subsequently, $F_k | F_ν F_n$ and $F_n | F_ν(kF_{k+1} − F_k)$. Thus $F_k | F_ν^2(kF_{k+1} − F_k)$, and then $F_k | kF_ν^2$ holds since $gcd(F_k, F_{k+1}) = 1$. Applying Corollary 5 we obtain

$$α^{k−2} ≤ F_k ≤ kF_ν^2 ≤ kα^{2ν−2} ≤ kα^{2/3k−2},$$

which implies $k < 19$. 

Case II: $k$ is odd. In this part, we follow the idea of the previous case. Recall that $k \geq 21$. Now
\[
F_k(kF_{k+1} - F_k) = F_{n-\varepsilon}F_{n+\varepsilon},
\]
where $\varepsilon = 1$ or $2$ depending on the parity of $n$ (see Lemma 3). Clearly, $\gcd(n - \varepsilon, n + \varepsilon) = 2$ or $4$. Thus $\gcd(F_{n-\varepsilon}, F_{n+\varepsilon}) = 1$ or $3$, respectively.

Put $\nu_1 = \gcd(k, n - \varepsilon)$ and $\nu_2 = \gcd(k, n + \varepsilon)$. Obviously, $\gcd(\nu_1, \nu_2)$ divides $\gcd(n - \varepsilon, n + \varepsilon)$. Hence $\nu = \gcd(\nu_1, \nu_2) = 1$ or $2$ or $4$, and then $F_\nu = \gcd(F_{\nu_1}, F_{\nu_2}) = 1$ or $3$. Thus $F_{\nu_1}F_{\nu_2} | F_\nu F_k$. The terms of both the left and right sides of
\[
\frac{F_k}{F_{\nu_1}}(kF_{k+1} - F_k) = F_{n-\varepsilon}F_{n+\varepsilon} \quad \text{and} \quad \frac{F_k}{F_{\nu_2}}(kF_{k+1} - F_k) = F_{n-\varepsilon}F_{n+\varepsilon}
\]
are integers, and
\[
\frac{F_{n-\varepsilon}}{F_{\nu_1}} | kF_{k+1} - F_k, \quad \frac{F_k}{F_{\nu_2}} | F_{n-\varepsilon}.
\]
Combining them, $F_k | F_{\nu_1}F_{\nu_2}(kF_{k+1} - F_k)$ follows, and then $F_k | kF_{\nu_1}F_{\nu_2}$. The remaining part of the proof consists of three cases.

Suppose first that $\nu_1 = k$, i.e. $F_k | F_{n-\varepsilon}$. Observe, that $n - \varepsilon$ and $n + \varepsilon$ are even, and $k \mid (n - \varepsilon)/2$. Thus $k + 1 \leq (n - \varepsilon)/2 + 1$, which does not exceed $(n + \varepsilon)/2$. Then
\[
kF_{k+1} > \frac{F_{n-\varepsilon}}{F_{\nu_1}}F_{n+\varepsilon} = \frac{F_{(n-\varepsilon)/2}}{F_{\nu_1}}L_{(n-\varepsilon)/2}F_{n+\varepsilon} \geq L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2}F_{(n+\varepsilon)/2}
\]
\[
\geq L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2}L_{k+1}.
\]
Simplifying by $F_{k+1}$ we conclude
\[
\frac{n - \varepsilon}{2} \geq k > L_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2},
\]
and we arrived at a contradiction since $21 \leq k < n$. Note that the same machinery works when $\nu_2 = k$, i.e. $F_k | F_{n+\varepsilon}$.

If none of the two conditions above holds, we can assume $\nu_1 \leq k/3$ and $\nu_2 \leq k/3$. Indeed, $k$ is odd, so the largest non-trivial divisor of $k$ is at most $k/3$. The application of Corollary 5 gives
\[
\alpha^{k-2} \leq F_k \leq kF_{\nu_1}F_{\nu_2} \leq k\alpha^{\nu_1-1}\alpha^{\nu_2-1} \leq k\alpha^{2k/3-2},
\]
and then $k < 19$.

The proof of the theorem is complete.

Proof of Theorem 3

The proof partially follows the proof of Theorem 1. The small cases of (7) can be verified by hand. Suppose $\kappa = k + 2 \geq 9$. Similarly to (9), we have
\[
k = \frac{F_2^2 + F_{\kappa+1} - 2}{F_\kappa} = \frac{F_2^2 + F_{\kappa-1} - 2}{F_\kappa} + 1 \in \mathbb{N}.
\]
Now we study the sequence $(F_2^2)_{u=0}^\infty$ modulo $F_\kappa$, and we again indicate the most suitable values by modulo $F_\kappa$, not always the smallest non-negative remainders. Lemma 1 implies
\[
F_2^2 \equiv F_j^2 \pmod{F_\kappa},
\]
Lemma 1 (11) implies
\[
F_2^2 \equiv F_{j+1}^2 \pmod{F_\kappa},
\]
where $j = 0, 1, \ldots, \kappa - 1$. Hence the period having length $2\kappa$ can be given by

$$0, 1, 1, 2^2, \ldots, F_{\kappa-2}^2, F_{\kappa-1}^2, 0, F_{\kappa-1}^2, F_{\kappa-2}^2, \ldots, 2^2, 1, 1.$$ 

Let us distinguish two cases according to the parity of $\kappa$.

**Case I:** $\kappa$ is even. Put $\kappa = 2\ell$. Again by Lemma 1(1), together with $\kappa - i = 2\ell - i = \ell + (\ell - i)$ and $i = \ell - (\ell - i)$ admits $F_{\kappa-i}^2 \equiv F_i^2 \pmod{F_\kappa}$. It reduces the possibilities to $j = 0, 1, \ldots, \ell$.

If $j \leq \ell - 1 = (\kappa - 2)/2$, then

$$0 < F_j^2 + F_{\kappa-1} - 2 \leq F_{(\kappa-2)/2}^2 + F_{\kappa-1} - 2 \leq F_{\kappa-2} + F_{\kappa-1} - 2 < F_\kappa$$

holds since $F_{(\kappa-2)/2}^2 \leq F_{(\kappa-2)/2}L_{(\kappa-2)/2} = F_{\kappa-2}$.

Suppose now that $j = \ell = \kappa/2$. Repeating the previous idea we find $F_{\ell/2}^2 + F_{\kappa-1} - 2 \leq F_\kappa + F_{\kappa-1} - 2 < 2F_\kappa$. Consequently, $F_{\ell/2}^2 + F_{\kappa-1} - 2 = F_\kappa$ might be fulfilled. Thus $F_{\ell/2}^2 - 2 = F_{\kappa-2}$. Recalling $\kappa = 2\ell$ we equivalently obtain

$$F_{\ell/2}^2 - 1 = F_{2\ell-2} + 1.$$ 

Both sides have decomposition described in Lemma 3 and Lemma 2, respectively, providing

$$F_{\ell-1}F_{\ell+1} = F_{\ell-2}L_\ell \quad \text{or} \quad F_{\ell-2}F_{\ell+2} = F_\ell L_{\ell-2}$$

if $\ell$ is odd or even, respectively. Firstly, $F_{\ell-2} \mid F_{\ell-1}F_{\ell+1}$, and then $F_{\ell-2} \mid F_{\ell+1}$ holds only for small $\ell$ values. Secondly, $F_{\ell} \mid F_{\ell-2}F_{\ell+2}$ contradicts to $\ell \geq 5$.

**Case II:** $\kappa$ is odd. Now we have $F_{\kappa-i}^2 \equiv -F_i^2 \pmod{F_\kappa}$. Indeed, Lemma 4(4) admits

$$F_{\kappa-i}^2 = (F_{\kappa}F_{-i+1} + F_{\kappa-1}F_{-i})^2 \equiv F_{\kappa-1}^2F_1^2 \pmod{F_\kappa},$$

and then Lemma 3 justifies the statement. It makes possible to split the proof into a few parts.

If $j = (\kappa - 1)/2$, then $F_{(\kappa-1)/2}^2 + F_{\kappa-1} - 2 < 2F_{\kappa-1} - 2 < 2F_\kappa$. Thus $F_{(\kappa-1)/2}^2 + F_{\kappa-1} - 2 = F_\kappa$ is the only one chance to fulfill (11). Then we apply Lemma 6 with $k_0 = 4 \leq (\kappa - 1)/2$ for $F_{\kappa-2} < F_{(\kappa-1)/2}^2$ to reach a contradiction.

If $j \leq (\kappa - 3)/2$, then

$$0 < F_j^2 + F_{\kappa-1} - 2 \leq F_{(\kappa-3)/2}^2 + F_{\kappa-1} - 2 \leq F_{\kappa-3} + F_{\kappa-1} - 2 < F_\kappa$$

holds since $F_{(\kappa-3)/2}^2 \leq F_{(\kappa-3)/2}L_{(\kappa-3)/2} = F_{\kappa-3}$.

Finally, if $(\kappa + 1)/2 \leq j \leq \kappa - 1$, then

$$F_j^2 + F_{\kappa-1} - 2 \equiv -F_{\kappa-j}^2 + F_{\kappa-1} - 2 \pmod{F_\kappa},$$

where $1 \leq j = \kappa - j \leq (\kappa - 1)/2$. Thus we need to check the equation $F_j^2 + 2 = F_{\kappa-1}$, since $-2 \leq -F_j^2 + F_{\kappa-1} - 2 < F_{\kappa-1} < F_\kappa$. When $J \leq (\kappa - 3)/2$ holds then $F_j^2 + 2 < F_{\kappa-3} + 2 < F_{\kappa-1}$. Lastly, $J = (\kappa - 1)/2$ leads to $F_{(\kappa-1)/2}^2 + 2 = F_{\kappa-1}$, and then to $2 = F_j(L_J - F_j)$. Obviously, it gives $J \leq 3$. Thus $\kappa \leq 7$, a contradiction.
Proof of Theorem 1

The statement for (8) is obviously true if \( k \leq 12 \). So we may assume \( k \geq 13 \). Let \( \tau \in \{0, 1\} \). The formula of the summation, together with Corollary 5 implies
\[
\alpha^{n-2} \leq F_n = F_k(kF_{k+1} - F_k) + \tau < kF_k F_{k+1} \leq \alpha^{k+1},
\]
and then
\[
n - k < k + 1 + \log_\alpha k < \frac{3}{2}k.
\]
Similarly,
\[
\alpha^{n-1} \geq F_n = F_k(kF_{k+1} - F_k) + \tau > F_k(kF_{k+1} - F_{k+1}) = (k - 1)F_k F_{k+1} > \alpha^{k+1},
\]
subsequently
\[
n - k > k - 2 + \log_\alpha (k - 1) > k + 3,
\]
that is \( n - k \geq k + 4 \). Putting together the two estimates it gives
\[
(12) \quad 2k + 4 \leq n < \frac{5}{2}k.
\]

Case I: \( k \) is even. Clearly, \( k + 2 < n - k - 1 \leq 3k/2 - 2 \) holds. By Lemma 1, we conclude
\[
F_n = F_{k+1} F_{n-k} + F_k F_{n-k-1} = F_k F_{k+1} - F_k^2,
\]
and equivalently
\[
F_k(F_k + F_{n-k-1}) = F_{k+1}(kF_k - F_{n-k}).
\]
Thus \( \gcd(F_k, F_{k+1}) = 1 \) admits \( F_{k+1} \mid F_k + F_{n-k-1} \). The periodicity of \( (F_u)_{u=0}^\infty \) modulo \( F_{k+1} \) guarantees, together with the bounds on \( n - k - 1 \) that
\[
F_k + F_{n-k-1} \equiv F_k + F_j F_k = F_k(F_j + 1) \pmod{F_{k+1}}
\]
holds for some \( j = 1, 2, \ldots, k/2 - 3 \). Consequently, \( F_{k+1} \mid F_j + 1 \), a contradiction.

Case II: \( k \) is odd. Again \( k + 2 < n - k - 1 \leq 3k/2 - 2 \) holds. Lemma 2 and Lemma 1 imply
\[
F_k(kF_{k+1} - F_k) = F_{n-\varepsilon}/2 L_{n+\varepsilon}/2 = (F_{k+1} F_{(n-\varepsilon)/2-k} + F_k F_{(n-\varepsilon)/2-k-1}) L_{(n+\varepsilon)/2},
\]
where \( \varepsilon \in \{\pm1, \pm2\} \) according to the modular property of \( n \). It leads to
\[
F_{k+1}(kF_k - F_{(n-\varepsilon)/2-k} L_{(n+\varepsilon)/2}) = F_k(F_k + 2 L_{(n+\varepsilon)/2}).
\]
Thus \( F_k \mid F_{(n-\varepsilon)/2-k} L_{(n+\varepsilon)/2} \).

By (12) it is obvious that
\[
k - k + 1 \leq \frac{n - 2}{2} \leq \frac{n + \varepsilon}{2} \leq \frac{n + 2}{2} \leq \frac{5}{4}k + 1 < 2k,
\]
which exclude \( \gcd(k, (n + \varepsilon)/2) = k \). Thus \( \gcd(k, (n + \varepsilon)/2) \leq k/3 \) since \( k \) is odd, subsequently \( \gcd(F_k, L_{(n+\varepsilon)/2}) \leq L_{k/3} \). On the other hand
\[
\frac{n - \varepsilon}{2} - k \leq \frac{n + 2}{2} - k \leq \frac{1}{4}k + 1,
\]
which implies \( \gcd(F_k, F_{(n-\varepsilon)/2-k}) \leq F_{k/4+1} \).
Thus, $F_k \mid F_{(n-\varepsilon)/2-k}L_{(n+\varepsilon)/2}$, together with the previous arguments entails $F_k \leq F_{k/4+1}L_{k/3}$, but it leads to a contradiction since $L_{k/3} = F_{k/3-1} + F_{k/3+1} < 2F_{k/3+1}$, and the application of Corollary 5 on $F_k \leq 2F_{k/4+1}F_{k/3+1}$ returns with $k < 9$.

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