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NORM CONTINUITY OF POINTWISE QUASI-CONTINUOUS MAPPINGS

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Abstract. Let $X$ be a Baire space, $Y$ be a compact Hausdorff space and $\varphi: X \to C_p(Y)$ be a quasi-continuous mapping. For a proximal subset $H$ of $Y \times Y$ we will use topological games $G_1(H)$ and $G_2(H)$ on $Y \times Y$ between two players to prove that if the first player has a winning strategy in these games, then $\varphi$ is norm continuous on a dense $G_\delta$ subset of $X$. It follows that if $Y$ is Valdivia compact, each quasi-continuous mapping from a Baire space $X$ to $C_p(Y)$ is norm continuous on a dense $G_\delta$ subset of $X$.

Keywords: function space; weak continuity; generalized continuity; quasi-continuous function; pointwise topology

MSC 2010: 54C35, 54C08, 54C05

1. Introduction

Let $X$ and $Z$ be topological spaces. A function $\varphi: X \to Z$ is called quasi-continuous at $x_0 \in X$ if for any neighborhood $U$ of $x_0$ in $X$ and any neighborhood $V$ of $z_0 = \varphi(x_0)$ in $Z$ there exists a nonempty open subset $G$ of $U$ such that $\varphi(G) \subset V$. The mapping $\varphi: X \to Z$ is called quasi-continuous if it is quasi-continuous at any point of $X$.

Let $Y$ be a compact space and $C(Y)$ be the space of all continuous real-valued functions on $Y$. We consider two topologies on $C(Y)$, the norm topology, which is the topology generated by the supremum norm $\|f\| = \sup_{y \in Y} |f(y)|$, $f \in C(Y)$, and the pointwise topology, which is the topology inherited from $\mathbb{R}^Y$ with product topology. The space $C(Y)$ equipped with the pointwise topology will be denoted by $C_p(Y)$.

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In 1974, Namioka [15] proved that every continuous mapping \( \varphi: X \to C_p(Y) \) is norm continuous at the points of a dense \( G_\delta \) subset of \( X \) provided that \( X \) is countably Čech-complete. Christensen [5] showed Namioka’s theorem is still valid when \( X \) is \( \sigma\)-\( \beta \)-unfavorable. It was expected that the result of Namioka remains true when \( X \) is an arbitrary Baire space. However, Talagrand [17] provided an example of a pointwise continuous mapping \( \varphi: X \to C_p(X) \), where \( X \) is on an \( \alpha \)-favorable space \( X \) which is nowhere norm continuous. The result of Talagrand raises the following question:

**What are compact spaces \( Y \) such that for every Baire space \( X \) and continuous (or quasi-continuous) mapping \( \varphi: X \to C_p(Y) \) must be norm continuous at each point of some dense \( G_\delta \) subset of \( X \) ?**

Several partial answers to the above question have been obtained by some authors (see e.g. [3], [6]–[14]). In particular, Bouziad [2] introduced two person games \( G_1(H) \) and \( G_2(H) \) on product \( Y \times Y \), where \( H \) is a proximal subset of \( Y \times Y \), to show that if the first player has winning strategies in both plays, then \( Y \) is a co-Namioka compact space.

In this paper, we will show that if in a compact space \( Y \) the second player in games \( G_1(H) \) and \( G_2(H) \) has no winning strategies, then every quasi-continuous mapping \( \varphi: X \to C_p(Y) \) is norm continuous on a dense \( G_\delta \) subset of \( X \).

### 2. Results

We start this section by introducing the following topological games. The first one is known as “Banach-Mazur game” (or “Choquet game”, see [4] or [16]).

The Banach-Mazur game \( BM(X) \): Two players \( \beta \) and \( \alpha \) select alternately non-empty open subsets of \( X \) as follows. Player \( \beta \) starts the game by selecting a nonempty open subset \( U_1 \) of \( X \). In return, \( \alpha \) replies by selecting some nonempty open subset \( V_1 \) of \( U_1 \). At the \( n \)-th stage of the game, \( n \geq 1 \), player \( \beta \) chooses a nonempty open subset \( U_n \subset V_{n-1} \) and \( \alpha \) answers by choosing a nonempty open subset \( V_n \) of \( U_n \). Proceeding in this fashion, the players generate a sequence \( (U_n, V_n)_{n=1}^{\infty} \) which is called a play. Player \( \alpha \) wins the play \( (U_n, V_n)_{n=1}^{\infty} \) if \( \bigcap_{n \geq 1} U_n = \bigcap_{n \geq 1} V_n \neq \emptyset \); otherwise player \( \beta \) wins this play. A partial play is a finite sequence of sets consisting of the first few moves of a play. A strategy for player \( \alpha \) is a rule by means of which the player makes his/her choices. An s-play is a play in which \( \alpha \) selects his/her moves according to the strategy \( s \). The strategy \( s \) for the player \( \alpha \) is said to be a winning strategy if every s-play is won by \( \alpha \). A space \( X \) is called \( \alpha \)-favorable if there exists a winning strategy for \( \alpha \) in \( BM(X) \).
It is easy to verify that every \( \alpha \)-favorable space \( X \) is a Baire space. There are examples of Baire spaces which are not \( \alpha \)-favorable (see for example [10]). It is known that \( X \) is a Baire space if and only if player \( \beta \) does not have a winning strategy in the game \( B.M(X) \).

Let \( Y \) be a compact Hausdorff space and \( \Delta \) denote the diagonal of \( Y \times Y \). Following [2], a subset \( H \) of \( Y \times Y \) is called proximal if it intersects every neighborhood of \( \Delta \). For a proximal set \( H \subset \Delta \) we consider the following two player topological games.

\[ G_1(H) : \text{At the } n\text{-th stage, } a \text{ selects a pair } (W_n, D_n), \text{ where } W_n \text{ is an open neighborhood of } \Delta \text{ and } D_n \cap H \text{ is a dense subset of } H. \text{ Then } b \text{ answers by taking a point } (y_n, y'_n) \in W_n \cap H \cap D_n. \text{ This play is won by } a \text{ if for every neighborhood } W \text{ of } \Delta \text{ there is some } n \in \mathbb{N} \text{ such that } (y_n, y'_n) \in W. \text{ Otherwise, } b \text{ wins the play. The space } Y \text{ is called } G_1(H)-b\text{-favorable if } b \text{ has a winning strategy in } G_1(H). \text{ Otherwise, } Y \text{ is called } G_1(H)-b\text{-unfavorable.} \]

\[ G_2(H) : \text{At the } n\text{-th stage, } a \text{ selects a pair } (W_n, D_n), \text{ where } W_n \text{ is an open neighborhood of } \Delta \text{ and } D_n \text{ is a dense subset of } W_n. \text{ Then the answer of } b \text{ will be a point } (y_n, y'_n) \in W_n \cap D_n. \text{ The play is won by } a \text{ if for every neighborhood } W \text{ of } \Delta \text{ containing } H \text{ there is some } n \in \mathbb{N} \text{ such that } (y_n, y'_n) \in W. \text{ Otherwise, } b \text{ wins the game. The space } Y \text{ is called } G_2(H)-b\text{-favorable if } b \text{ has a winning strategy in } G_2(H). \text{ Otherwise, } Y \text{ is called } G_2(H)-b\text{-unfavorable.} \]

Hereafter, we will assume that \( Y \) is a compact space and \( H \) is a proximal subset of \( Y \). In order to prove the main result of this paper, we need the following lemmas.

**Lemma 1.** Let \( A \subset C(Y) \) be such that for some \( \varepsilon > 0 \) there is a neighborhood \( W \) of \( \Delta \) such that \( |f(y) - f(y')| < \frac{1}{4}\varepsilon \) for each \( f \in A \) and \((y, y') \in W \). Then for every \( f \in A \) there is a relatively open, with respect to pointwise topology on \( A \), set \( B \subset A \) such that \( f \in B \) and \( \|f\| - \text{diam}(B) < \varepsilon \).

**Proof.** For each \( y \in Y \) let \( W_y = \{z : (y, z) \in W\} \). Then each \( W_y \) is open and \( |f(y) - f(z)| < \frac{1}{4}\varepsilon \) for each \( f \in A \) and \( z \in W_y \). Since \( Y \) is compact, there are points \( y_1, \ldots, y_n \in Y \) such that \( Y = \bigcup_{i=1}^{n} W_{y_i} \). Choose an element \( f_0 \in A \) and define

\[
B = \left\{ f \in A : |f(y_i) - f_0(y_i)| < \frac{\varepsilon}{8}, \ 1 \leq i \leq n \right\}.
\]

Then for each \( f, g \in B \) and \( y \in Y \) there is some \( 1 \leq i \leq n \) such that \( y \in W_{y_i} \). Therefore we have

\[
|f(y) - g(y)| \leq |f(y) - f(y_i)| + |f(y_i) - f_0(y_i)| + |f_0(y_i) - g(y_i)| + |g(y_i) - g(y)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.
\]

It follows that \( \|f - g\| < \varepsilon \). \( \square \)
Lemma 2. Let $X$ be a topological space and $\varphi: X \to C_p(Y)$ be a quasi-continuous mapping. If $X$ is $\alpha$-favorable and $a$ has no winning strategy in $G_1(H)$ or $X$ is Baire and $a$ has a winning strategy in $G_1(H)$, then for each $\epsilon > 0$ and a nonempty open subset $U$ of $X$ there are an open neighborhood $E$ of $\Delta$ and a nonempty open subset $O \subset U$ such that for each $f \in \varphi(O)$ and $(y, y') \in E \cap H$ we have $|f(y) - f(y')| < \epsilon$.

Proof. If the result of the lemma were not true, then there are some $\epsilon > 0$ and an open subset $U$ of $X$ such that for each open subset $O \subset U$ and open neighborhood $E$ of $\Delta$, $|f(y) - f(y')| \geq \epsilon$ for some $f \in \varphi(O)$ and $(y, y') \in E \cap H$. Let $U_1 = U$ be the first move of player $\beta$ in $BM(X)$ and $V_1 \subset U_1$ be the answer of $\alpha$ to this movement. Suppose that $(W_1, D_1)$ is the first move of $a$ in $G_1(H)$. By our assumption, there is some $f_1 \in \varphi(V_1)$ and $(y_1, y'_1) \in W_1 \cap D_1 \cap H$ such that $|f_1(y_1) - f_1(y'_1)| > \frac{1}{2}\epsilon$. Let $(y_1, y'_1)$ be the answer of $b$ to $(W_1, D_1)$. In step $n$, when $V_1, \ldots, V_n$ and $(W_1, D_1), \ldots, (W_n, D_n)$ are specified by $\alpha$ and $a$, respectively, we select some $f_n \in \varphi(V_n)$ and $(y_n, y'_n) \in W_n \cap D_n \cap H$ such that $|f_n(y_n) - f_n(y'_n)| > \frac{1}{2}\epsilon$. Let $\delta_n = |f_n(y_n) - f_n(y'_n)| - \frac{1}{2}\epsilon$ and define

$$B_n = \left\{ f: |f(y_n) - f_n(y_n)| < \frac{\delta_n}{2} \text{ and } |f(y'_n) - f_n(y'_n)| < \frac{\delta_n}{2} \right\}.$$  

If $f \in B_n$, we have

$$|f(y_n) - f(y'_n)| \geq |f_n(y_n) - f_n(y'_n)| - \{ |f(y_n) - f_n(y_n)| + |f(y'_n) - f_n(y'_n)| \} > |f_n(y_n) - f_n(y'_n)| - \delta_n = \frac{\epsilon}{2}.$$  

Thanks to the quasi-continuity of $\varphi$, there is some nonempty subset $U_{n+1}$ of $V_n$ such that $\varphi(U_{n+1}) \subset B_n$. Let $U_{n+1}$ be the answer of $\beta$ to the partial play $(U_1, V_1, \ldots, U_n, V_n)$ and $(y_n, y'_n)$ be the response of $b$ to $(W_1, D_1), \ldots, (W_n, D_n)$. In this way by induction on $n$, a strategy for $\beta$ in $BM(X)$ and a strategy for $b$ in $G_1(H)$ is defined. Under either every assumption of the lemma, there are related games $\{(W_n, D_n), (y_n, y'_n)\}$ and $\{(U_n, V_n)\}$ which are won by $a$ and $\alpha$, respectively. Let $z \in \bigcap_{n \geq 1} U_n$ and $f = \varphi(z)$. Define

$$W = \left\{ (y, y'): |f(y) - f(y')| < \frac{\epsilon}{3} \right\}.$$  

Then $W$ is a neighborhood of $\Delta$, hence there is some $n \in \mathbb{N}$ such that $(y_n, y'_n) \in W$. However, $f \in \varphi(U_{n+1}) \subset B_n$, hence by (2.1), $|f(y_n) - f(y'_n)| > \frac{1}{2}\epsilon$. This contradiction proves the lemma. 

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Lemma 3. Let $X$ and $\varphi$ satisfy the assumptions of Lemma 2 and let $Y$ be $b$-unfavorable for play $G_2(H)$. Then for every nonempty open subset $U$ of $X$ and every $\varepsilon > 0$ there is a nonempty open subset $O$ of $U$ and an open neighborhood $W$ of $\Delta$ such that $|f(y) - f(y')| < \varepsilon$ for each $f \in \varphi(O)$ and $(y, y') \in W$.

Proof. Suppose that the lemma is not true. Then there is some $\varepsilon > 0$ and a nonempty open subset $U$ of $X$ such that for every nonempty open subset $O$ of $U$ and every open neighborhood $E$ of $\Delta$ there are $f \in \varphi(O)$ and $(y, y') \in E$ such that $|f(y) - f(y')| \geq \varepsilon$. By Lemma 2, there is a nonempty open subset $O'$ of $U$ and an open neighborhood $E$ of $\Delta$ such that $|f(y) - f(y')| < \frac{1}{2}\varepsilon$ for each $(y, y') \in E \cap H$ and $f \in \varphi(O')$. Let $U_1 = O'$ be the first choice of $b$ in $BM(X)$ and $V_1 \subseteq U_1$ be the response of $\alpha$ to $U_1$. Let $E'$ be an open neighborhood of $\Delta$ such that $E' \subseteq E$. Let $(W_1, D_1)$ be the first choice of $a$ in the play $G_2(H)$. Then there is some $f \in \varphi(V_1)$ such that $|f(y_1) - f(y_1')| > \frac{1}{2}\varepsilon$ for some $(y_1, y_1') \in W_1 \cap E'$. Since $D_1 \cap E'$ is dense in $W_1 \cap E'$, we can assume that $(y_1, y_1') \in W_1 \cap E' \cap D_1$. Let $(y_1', y_1')$ be the answer of $b$ to $(W_1, D_1)$.

Let the partial plays $(U_1, \ldots, U_n, V_n)$ in $BM(X)$ and $((W_1, D_1), \ldots, (W_n, D_n))$ in $G_2(H)$ for some $n \in \mathbb{N}$ be specified. Then by our assumption, there is some $f_n \in \varphi(V_n)$ and $(y_n, y_n') \in W_n \cap E' \cap D_n$ such that $|f_n(y_n) - f_n(y_n')| > \frac{1}{2}\varepsilon$. Let $(y_n, y_n')$ be the answer of $b$ to $(W_1, D_1), \ldots, (W_n, D_n)$. Define $\delta_n = |f_n(y_n) - f_n(y_n')| - \frac{1}{2}\varepsilon$ and

$$B_n = \left\{ f : |f(y_n) - f_n(y_n)| < \frac{\delta_n}{2} \text{ and } |f(y_n') - f_n(y_n')| < \frac{\delta_n}{2} \right\}.$$

Then $B_n$ is a pointwise open subset of $C(Y)$ which contains $f_n \in \varphi(V_n)$. Thanks to quasi-continuity of $\varphi$, there is an open subset $U_{n+1} \subseteq V_n$ such that $\varphi(U_{n+1}) \subseteq B_n$. Let $U_{n+1}$ be the next move of player $\beta$. By (2.1), $|f(y_n) - f(y_n')| > \frac{1}{2}\varepsilon$ for each $f \in \varphi(U_{n+1})$. In this way, by induction on $n$ a strategy for $\beta$ in $BM(X)$ and a strategy for $b$ in $G_2(H)$ are determined. Since $b$ does not have a winning strategy, there is a play $\{(W_n, D_n), (y_n, y_n')\}_{n \geq 1}$ which is won by $a$. Let $\{(U_n, V_n)\}_{n \geq 1}$ be its corresponding $BM(X)$ game. Then $\bigcap_{n \geq 1} U_n \neq \emptyset$. Let $f = \varphi(z) \in \varphi\left(\bigcap_{n \geq 1} U_n\right)$ and define

$$W = \left\{ (y, y') : |f(y) - f(y')| < \frac{\varepsilon}{3}\right\} \cup (Y \times Y \setminus E').$$

Then $W$ is a neighborhood of $\Delta$ which contains $H$. Therefore, there is some $n$ such that $(y_n, y_n') \in W$. Since $(y_n, y_n') \in E'$, it follows that $|f(y_n) - f(y_n')| < \frac{1}{3}\varepsilon$. However, $f \in \varphi(U_n) \subseteq B_n$. This contradiction proves the lemma.

Now, we are ready to state the main result of this section.
Theorem 4. Let $X$ be a topological space and $\varphi: X \to C_p(Y)$ be a quasi-continuous mapping. Suppose that $X$ is $\alpha$-favorable and $b$ has no winning strategy in $G_1(H)$ or $X$ is Baire and $a$ has a winning strategy in $G_1(H)$. If $Y$ is $b$-unfavorable for play $G_2(H)$, there is a dense $G_\delta$ subset $D$ of $X$ such that $\varphi$ is norm continuous on $D$.

Proof. Let $\varphi: X \to C_p(Y)$ be a quasi-continuous mapping. Define

$$G_n = \bigcup \{O: O \text{ is open in } X \text{ and norm-diam}(\varphi(O)) < \frac{1}{n}\}.$$ 

Then each $G_n$ is open in $X$. Let $U$ be an arbitrary nonempty open subset of $X$. By Lemma 3, there is a nonempty open subset $O$ of $U$ and an open neighborhood $W$ of $\Delta$ such that $|f(y) - f(y')| < \frac{1}{n} - 1$ for each $f \in \varphi(O)$ and $(y, y') \in W$. In view of Lemma 1, there is a pointwise open set $B \subset C_p(Y)$ such that $B \cap \varphi(O) \neq \emptyset$ and norm-diam$(B \cap \varphi(O)) < n^{-1}$. Since $\varphi$ is quasi-continuous, the set $\varphi^{-1}(B) \cap O$ is semi-open and nonempty, and consequently, it contains a nonempty open set $V$. Thus $V \subset G_n \cap U$, hence $G_n$ is dense in $X$. Clearly $\varphi$ is norm continuous on $D = \bigcap_{n \geq 1} G_n$. □

Let $\Gamma$ be a set and

$$\sigma(\Gamma) = \{x \in [0, 1]^\Gamma: \{\gamma \in \Gamma: x(\gamma) \neq 0 \text{ is countable}\}\}.$$ 

A compact space $Y$ is called Corson compact if it can be embedded in some $\sigma(\Gamma)$. The space $Y$ is called Valdivia compact if it can be embedded in some subset $K$ of $[0, 1]^\Gamma$ such that $K \cap \sigma(\Gamma)$ is dense in $K$. It follows from the definition that every Corson compact space is Valdivia compact but the converse is not true in general (see [8]). Debs [6] proved that if $X$ is a Baire space and $Y$ is a Corson compact, then every continuous mapping $\varphi: X \to C_p(Y)$ is norm continuous at any point of a dense $G_\delta$ subset of $X$. Bouziad [2] improved this result by showing that $Y$ can be any $\alpha$-favorable space for the games $G_1(H)$ and $G_2(H)$, where $H$ is a proximal subset of $Y \times Y$. So the above result holds when $Y$ is Valdivia compact (see [1]).

Kendeov et al. [11], Corollaries 5 and 8, have shown that this result remains true if $X$ is $\alpha$-favorable, $Y$ is Valdivia compact and $\varphi$ is quasi-continuous. Theorem 4 enables us to give a simultaneous generalization of these results.

Corollary 5. Let $X$ be a Baire space and $Y$ be a Valdivia compact space. Then every quasi-continuous mapping $\varphi: X \to C_p(Y)$ is norm continuous at any point of a dense $G_\delta$ subset of $X$. 

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